The Australian Journal of Mathematical Analysis and Applications

AJMAA

Volume 6, Issue 1, Article 2, pp. 1-9, 2009

LINEARLY TRANSFORMABLE MINIMAL SURFACES

HAROLD R. PARKS AND WALTER B. WOODS

Special Issue in Honor of the 100th Anniversary of S.M. Ulam

Received 25 March, 2009; accepted 25 April, 2009; published 4 September, 2009.

DEPARTMENT OF MATHEMATICS
OREGON STATE UNIVERSITY
CORVALLIS, OREGON 97331–4605, USA
parks@math.oregonstate.edu
URL: http://www.math.oregonstate.edu/people/view/parks/

ABSTRACT. We give a complete description of a nonplanar minimal surface in $\mathbb{R}^3$ with the surprising property that the surface remains minimal after mapping by a linear transformation that dilates by three distinct factors in three orthogonal directions. The surface is defined in closed form using Jacobi elliptic functions.

Key words and phrases: Minimal surface, Aeolotropic linear transformation, Jacobi elliptic function.

2000 *Mathematics Subject Classification.* Primary 53A10. Secondary 49Q05, 33E05.

ISSN (electronic): 1449-5910

1. INTRODUCTION

Minimal surfaces in $\mathbb{R}^3$ are so varied and plentiful that the solution to Plateau's problem requires but negligible hypotheses on the curve. Plateau's problem has inspired much important mathematics. It has also generated an immense literature from which it is appropriate here to mention [5].

The condition of minimality is also very restrictive: Adding one or two conditions in addition to minimality is often sufficient to determine a surface, if one even exists. Classical examples of minimal surfaces can be derived by exploiting this restrictiveness. For example, the catenoid is the only minimal surface that is also a surface of revolution, the helicoid, is the only ruled minimal surface, and Sherk's surface, is the only minimal surface of translation (see [3, p. 17 ff]).

With the preceding motivation in mind, we considered the question of whether a minimal surface could remain minimal after a nonsingular linear transformation. Since orthogonal transformations and homotheties map minimal surfaces to minimal surfaces, we should "mod out" by such transformations. Also, a helicoid is mapped to another helicoid by a linear transformation that leaves the axis fixed and is a homothety in the plane orthogonal to the axis, so we will want to consider linear transformations that do not have that structure. Thus, in light of the singular value decomposition, it is appropriate to suppose that T is represented by a diagonal matrix with three distinct positive numbers on the diagonal. It seems appropriate to describe such a linear transformation as being *aeolotropic*.

One might reasonably conjecture that there are no nonplanar minimal surfaces that remain minimal after an aeolotropic transformation, but that conjecture is false: Such surfaces do exist. In this paper, we describe one such surface, investigate its structure, and illustrate it.

The existence of minimal surfaces that remain minimal after an aeolotropic linear transformation was demonstrated in [4], but their existence was only a side issue in that paper. Also, the computer graphics available at that time were primitive compared to what is readily available today, so it was not possible to gain much global understanding of such surfaces. Thus it seems appropriate now to further explore such surfaces in their own right.

2. THE SURFACE

In this paper, we will consider the aeolotropic transformation given by

$$(x_1,\, x_2,\, x_3) \longrightarrow \left(\sqrt{2}\, x_1,\, x_2,\, \sqrt{3}\, x_3\right).$$

There is a well-known link between minimal surfaces and holomorphic functions. In fact, given any triple of holomorphic functions ϕ_j, $j = 1, 2, 3$, satisfying

$$\phi_1^2 + \phi_2^2 + \phi_3^2 = 0, \tag{2.1}$$

one defines an isothermal parametrization of a minimal surface in $\mathbb{R}^3$ by setting

$$x_j(z) = \Re\left\{\int_0^z \phi_j(\zeta)\, d\zeta\right\}, \qquad j = 1, 2, 3\,. \tag{2.2}$$

We will describe the surface as being *associated* with the triple of holomorphic functions. A reference for this material is [3], in particular, § 4 of [3].

Our particular choice for the triple of holomorphic functions is

$$\phi_1(z) = \sqrt{2}\, i\ \mathrm{cn}[z]\,, \tag{2.3}$$

$$\phi_2(z) = \sqrt{2}\,\mathrm{dn}[z]\,, \tag{2.4}$$

$$\phi_3(z) = \ i\ \mathrm{sn}[z]\,, \tag{2.5}$$

where $\mathrm{sn}[\cdot]$, $\mathrm{cn}[\cdot]$, and $\mathrm{dn}[\cdot]$ are the Jacobi elliptic functions with modulus

$$k = 1 \Big/ \sqrt{2}\ . \tag{2.6}$$

One reference for elliptic functions is [1]; a more concise compilation of facts is found in [2, § 8.1].

The Jacobi elliptic functions with modulus k satisfy the following identities

$$\mathrm{sn}^2[z] + \mathrm{cn}^2[z] = 1\,, \quad \mathrm{dn}^2[z] + k^2\,\mathrm{sn}^2[z] = 1\,, \quad \mathrm{dn}^2[w] - k^2\,\mathrm{cn}^2[w] = k'^2, \tag{2.7}$$

where k' is the *complementary modulus* defined by $k' = \sqrt{1-k^2}$. In our case, the modulus and complementary modulus are equal; a fact that simplifies calculations. Using these identities, it is easy to verify that (2.1) holds.

The associated minimal surface is given by

$$\begin{aligned} x_1(z) &= \Re\left\{\int_0^z \phi_1(\zeta)\,d\zeta\right\} \\ &= -2\,\Re\left(\log\left[\,\mathrm{dn}\left[\sqrt{i}\ z\right] - \tfrac{i}{\sqrt{2}}\,\mathrm{sn}\left[\sqrt{i}\ z\right]\right]\right)\,, \end{aligned} \tag{2.8}$$

$$\begin{aligned} x_2(z) &= \Re\left\{\int_0^z \phi_2(\zeta)\,d\zeta\right\} \\ &= -\sqrt{2}\,\Im\left(\,\log\left[\,\mathrm{cn}\left[\sqrt{i}\ z\right] - \ i\ \mathrm{sn}\left[\sqrt{i}\ z\right]\right]\right)\,, \end{aligned} \tag{2.9}$$

$$\begin{aligned} x_3(z) &= \Re\left\{\int_0^z \phi_3(\zeta)\,d\zeta\right\} \\ &= -\sqrt{2}\,\Im\left(\,\log\left[\,\mathrm{dn}\left[\sqrt{i}\ z\right] - \tfrac{1}{\sqrt{2}}\,\mathrm{cn}\left[\sqrt{i}\ z\right]\right]\right)\,. \end{aligned} \tag{2.10}$$

The surface is illustrated in Figure 1. It is clearly nonplanar, but this fact also will be demonstrated independently in the next section using the defining formulas.

The image of our surface under the aeolotropic transformation is the surface associated with the following triple of holomorphic functions:

$$\psi_1(z) = \sqrt{2}\,\phi_1(z) \ = \ 2\ i\ \mathrm{cn}\left[\sqrt{i}\ z\right], \tag{2.11}$$

$$\psi_2(z) = \phi_2(z) \ = \ \sqrt{2}\,\mathrm{dn}\left[\sqrt{i}\ z\right]\,, \tag{2.12}$$

$$\psi_3(z) = \sqrt{3}\,\phi_3(z) \ = \ \sqrt{3}\ i\ \mathrm{sn}\left[\sqrt{i}\ z\right]. \tag{2.13}$$

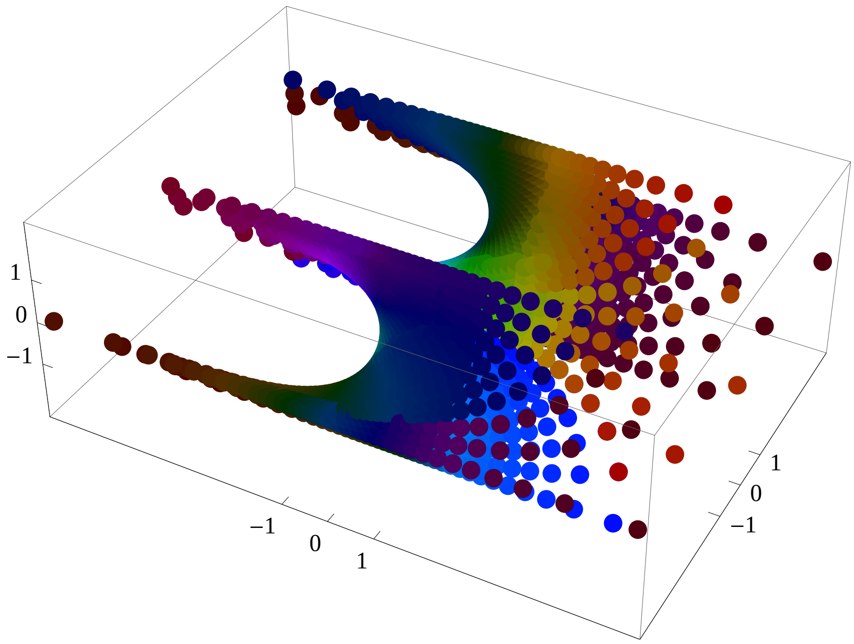

Figure 1: The surface.

We have

$$\widehat{x}_1(z) = \Re\left\{\int_0^z \sqrt{2}\,\phi_1(\zeta)\,d\zeta\right\} \;=\; \sqrt{2}\,x_1(z)\,,$$
$$\widehat{x}_2(z) = \Re\left\{\int_0^z \phi_2(\zeta)\,d\zeta\right\} \;=\; x_2(z)\,,$$
$$\widehat{x}_3(z) = \Re\left\{\int_0^z \sqrt{3}\,\phi_3(\zeta)\,d\zeta\right\} \;=\; \sqrt{3}\,x_3(z)\,.$$

In Section 4, we will show that $\big(\widehat{x}_1(z),\,\widehat{x}_2(z),\,\widehat{x}_3(z)\big)$ is also a minimal surface.

3. THE STRUCTURE OF THE SURFACE

To discern the global behavior of the surface $\big(x_1(z),\,x_2(z),\,x_3(z)\big)$ associated with the triple of functions ϕ_j, $j = 1, 2, 3$, it will be convenient to introduce the following auxiliary functions:

$$F(z) = \mathrm{dn}[z] - \tfrac{i}{\sqrt{2}}\,\mathrm{sn}[z]\,, \tag{3.1}$$

$$G(z) = \mathrm{cn}[z] - i\;\mathrm{sn}[z]\,, \tag{3.2}$$

$$H(z) = \mathrm{dn}[z] - \tfrac{1}{\sqrt{2}}\,\mathrm{cn}[z]\,. \tag{3.3}$$

Recall that we have chosen $k = 1\big/\sqrt{2}$, so the modulus and complementary modulus are equal. Also of significance for the Jacobi elliptic functions are the *complete elliptic integrals*

$$K = \int_0^1 \frac{dt}{\sqrt{1-t^2}\,\sqrt{1-k^2\,t^2}}$$

and

$$K' = \int_0^1 \frac{dt}{\sqrt{(1-t^2)\,(1-k'^2\,t^2)}}\,.$$

In our case, we have $K = K' \approx 1.8541$. The figures in this paper are in units of K. We will often use k and k' interchangeably, and likewise K and K'.

The next lemma tells us that the surface contains a lattice of lines in the (x_2, x_3)-plane.

Lemma 3.1. *Suppose $u, v \in \mathbb{R}$ and $m, n \in \mathbb{Z}$. Then we have*
(1) $|\,F(u + 2nK\,i\,)\,| = 1$, *and* $|\,F((1 + 2m)\,K + \,i\,v)\,| = 1$,
(2) $H(u) > 0$, *and* $H(u + 2\,K\,i\,) < 0$,
(3) $-\,i\,G(K + \,i\,v) < 0$, *and* $-\,i\,G(K + \,i\,v) > 0$.

Proof. (1) We have ([2, p. 914]),

$$\mathrm{dn}[u + 2nK\,i\,] = (-1)^n\,\mathrm{dn}[u]\,, \quad \mathrm{sn}[u + 2nK\,i\,] = \mathrm{sn}[u]\,.$$

Since $\mathrm{dn}[u]$ and $\mathrm{sn}[u]$ are real-valued, we have

$$|\mathrm{dn}[u] - \tfrac{i}{\sqrt{2}}\,\mathrm{sn}[u]|^2 = \mathrm{dn}^2[u] + k^2\,\mathrm{sn}^2[u] = 1\,.$$

Likewise, we have ([2, p. 914]),

$$\mathrm{dn}[(1 + 2m)\,K + \,i\,v] = \mathrm{dn}[K + \,i\,v]\,, \quad \mathrm{sn}[(1 + 2m)\,K + \,i\,v] = (-1)^m\,\mathrm{sn}[K + \,i\,v]\,.$$

Also, $\mathrm{dn}[K + \,i\,v]$ and $\mathrm{sn}[K + \,i\,v]$ are real-valued ([1, p. 38]), so we have

$$|\mathrm{dn}[K + \,i\,v] - \tfrac{i}{\sqrt{2}}\,\mathrm{sn}[K + \,i\,v]|^2 = \mathrm{dn}^2[K + \,i\,v] + k^2\,\mathrm{sn}^2[K + \,i\,v] = 1\,.$$

(2) Since $\mathrm{sn}[u] \geq k$ and $|\mathrm{cn}[u]| \leq 1$ hold for real arguments and equality does not occur simultaneously, we have the first inequality. The second follows because both $\mathrm{dn}[z]$ and $\mathrm{cn}[z]$ change sign when the argument is incremented by $2\,K'\,i$.
(3) We see that

$$\mathrm{cn}[K + \,i\,v] - \,i\,\mathrm{sn}[K + \,i\,v] = \,i\,\mathrm{dn}[v]\Big(k\,\mathrm{sn}[v] - 1\Big)\Big/\Big(1 - (1/2)\mathrm{sn}[v]\Big)$$

from which the first inequality follows. The second follows because $\mathrm{sn}[z]$ and $\mathrm{cn}[z]$ both change sign when the argument is incremented by $2\,K$. ∎

The elliptic functions are doubly periodic with periods $4\,K$ and $2\,K'\,i$, $4\,K$ and $2\,K{+}2\,K'\,i$, $2\,K$ and $4\,K'\,i$, for $\mathrm{sn}[z]$, $\mathrm{cn}[z]$, $\mathrm{dn}[z]$, respectively. The values $4\,K$ and $4\,K'i$ are periods for all three functions. The elliptic functions have simple poles at $K'\,i$ and $2\,K + \,K'\,i$, $K'\,i$ and $2\,K{+}\,K'\,i$, $K'\,i$ and $-K'\,i$, for $\mathrm{sn}[z]$, $\mathrm{cn}[z]$, $\mathrm{dn}[z]$, respectively. Thus we see that the auxiliary functions F, G, H are holomorphic except possibly at the points

$$i\,K\,, \quad 2\,K + \,i\,K\,, \quad -\,i\,K\,, \quad 2\,K - \,i\,K$$

and their translates by $4\,m\,K + 4\,n\,K\,i$. Also note that the identities in (2.7) tell us that none of F, G, H can vanish away from this set of exceptional points.

Using the Maclaurin series for the elliptic functions (see [2]), we have the following approximations at the exceptional points:

$z\ =$	$iK' + u$	$2K + iK' + u$	$-iK' + u$	$2K - iK'$
$F(z)\ =$	$-2iu^{-1} + \mathrm{O}(u)$	$\mathrm{O}(u)$	$\mathrm{O}(u)$	$2iu^{-1} + \mathrm{O}(u)$
$G(z)\ =$	$-2ik^{-1}u^{-1} + \mathrm{O}(u)$	$2ik^{-1}u^{-1} + \mathrm{O}(u)$	$\mathrm{O}(u)$	$\mathrm{O}(u)$
$H(z)\ =$	$\mathrm{O}(u)$	$-2iu^{-1} + \mathrm{O}(u)$	$\mathrm{O}(u)$	$2iu^{-1} + \mathrm{O}(u)$

We see that there is interesting behavior for all three functions at all four exceptional points. In particular, we see that F has poles and zeros, so x_1 takes all real values.

We can further understand the auxiliary functions by examining their behavior on the line segments connecting the exceptional points. This information is captured in the following table. The line segments are those connecting the points named in the cells above and below or connecting the points named in the cells to the left and to the right.

$3\,i\,K'$ zero of F	line segment $-i\,G<0$ $-i\,H<0$	$2\,K+3\,i\,K'$ pole of F	line segment $-i\,G>0$ $-i\,H>0$	$4\,K+3\,i\,K'$ zero of F
line segment $G<0$ $H<0$		line segment $G>0$ $H<0$		line segment $G<0$ $H<0$
$i\,K'$ pole of F	line segment $-i\,G<0$ $-i\,H>0$	$2\,K+i\,K'$ zero of F	line segment $-i\,G>0$ $-i\,H<0$	$4\,K+i\,K'$ pole of F
line segment $G>0$ $H>0$		line segment $G<0$ $H>0$		line segment $G>0$ $H>0$
$-i\,K'$ zero of F	line segment $-i\,G<0$ $-i\,H<0$	$2\,K-i\,K'$ pole of F	line segment $-i\,G>0$ $-i\,H>0$	$4\,K-i\,K'$ zero of F

From the above table, we conclude that the surface contains lines parallel to the x_1 axis which pass through the points in a lattice in the (x_2, x_3)-plane, the same lattice as the lattice of points through which the lines parallel to the x_1-direction pass. This structure of lines in the surface might be described as the "spine" of the surface. The spine contains lines in three mutually orthogonal directions; in particular, it is nonplanar. The spine is shown in Figure 2. Another visualization of the surface that includes more of the spine is shown in Figure 3.

4. THE AEOLOTROPICALLY MAPPED SURFACE

Recall that the aeolotropically deformed surface is associated to the triple of holomorphic functions:

$$\psi_1(z) = \sqrt{2}\,\phi_1(z) \;=\; 2\;i\;\mathrm{cn}\left[\sqrt{i}\;z\right]\,, \tag{4.1}$$

$$\psi_2(z) = \phi_2(z) \;=\; \sqrt{2}\,\mathrm{dn}\left[\sqrt{i}\;z\right]\,, \tag{4.2}$$

$$\psi_3(z) = \sqrt{3}\,\phi_3(z) \;=\; \sqrt{3}\;i\;\mathrm{sn}\left[\sqrt{i}\;z\right]\,. \tag{4.3}$$

The sum of squares of the ψ_j, $j = 1, 2, 3$, is not zero, but we do have

$$\psi_1^2 + \psi_2^2 + \psi_3^2 = -2\,. \tag{4.4}$$

Even though $\sum_j \psi_j^2$ does not equal zero, the associated surface $(\widehat{x}_1(z),\, \widehat{x}_2(z),\, \widehat{x}_3(z))$ is well-defined and harmonic (these facts rely only on the Cauchy–Riemann equations holding for the ψ_j). The main consequence of $\sum_j \psi_j^2$ not equaling zero is that the parametrization is not

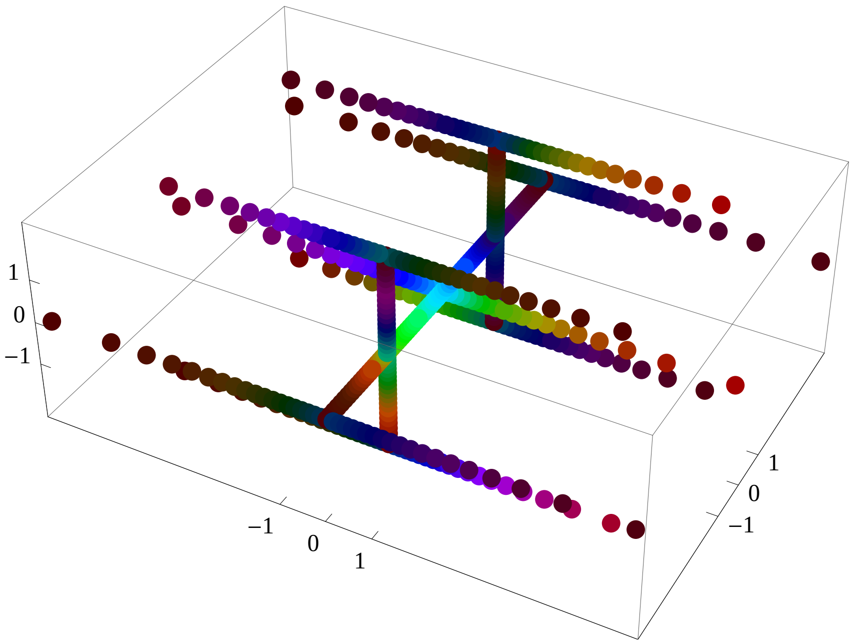

Figure 2: The spine of the surface.

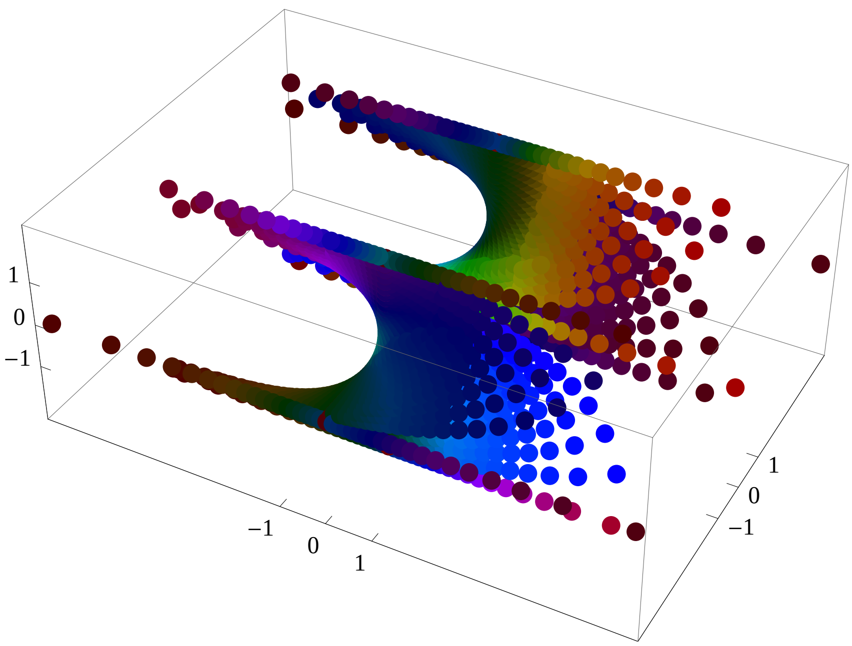

Figure 3: The surface showing some of its spine.

isothermal. The surface is nonetheless minimal, but to show its minimality we must examine the mean curvature of the surface more closely.

Recall the following differential geometric definitions:

Definition 4.1. For a surface in $\mathbb{R}^3$ parametrized by $\vec{r}(\xi_1, \xi_2)$, the components of the *first* and *second fundamental forms* are given by

$$g_{jk} = \vec{r}_j \cdot \vec{r}_k \,,$$
$$b_{jk} = \frac{\vec{r}_{jk}\vec{r}_1\vec{r}_2}{\sqrt{g}} \,,$$

where the subscripts on $\vec{r}$ denote partial derivatives and where

$$g = g_{11}\, g_{22} - (g_{12})^2 \,.$$

Definition 4.2. The *mean curvature* H of a surface in $\mathbb{R}^3$ is given by

$$H = \frac{g_{22}\, b_{11} + g_{11}\, b_{22} - 2g_{12}\, b_{12}}{2g} \,. \tag{4.5}$$

Lemma 4.1. *If a surface* $\big(x_1(z),\, x_2(z),\, x_3(z)\big)$ *is associated with a triple of holomorphic functions* $\theta_j(z)$, $j = 1, 2, 3$, *then*

$$\sum_{j=1}^{3} \theta_j^2 = g_{11} - g_{22} + 2\, i\, g_{12} \,.$$

Proof. This result follows from the computation on page 29 of [3]. ∎

The preceding lemma shows us why the condition $\sum_j \theta_j^2 = 0$ implies that the parametrization of the associated surface is isothermal. It also shows why the parametrization of the surface $(\widehat{x}_1(z),\, \widehat{x}_2(z),\, \widehat{x}_3(z))$ associated to the ψ_j, $j = 1, 2, 3$, is not isothermal. On the other hand, because $\sum_j \psi_j^2$ takes only real values, the equation $g_{12} = 0$ holds for $(\widehat{x}_1(z),\, \widehat{x}_2(z),\, \widehat{x}_3(z))$.

Proposition 4.2. *The surface* $(\widehat{x}_1(z),\, \widehat{x}_2(z),\, \widehat{x}_3(z))$ *associated to the functions* $\psi_j(z)$, $j = 1, 2, 3$, *satisfies*

$$g_{12} = 0 \,, \quad \textit{and} \quad b_{11} + b_{22} = 0 \,.$$

The surface is minimal if $b_{11} = -b_{22} = 0$.

Proof. The equation $g_{12} = 0$ is a consequence of (4.4) and Lemma 4.1. The equation $b_{11} + b_{22} = 0$ is a consequence of the fact that the parametrization is harmonic. The final conclusion follows from the first two and the equation (4.5) that defines the mean curvature. ∎

To see that the surface associated to the functions ψ_j, $j = 1, 2, 3$, is minimal, we will show that $b_{11} \equiv 0$.

Theorem 4.3. *If* θ_j, $j = 1, 2, 3$, *is any triple of holomorphic functions, then*

$$\begin{aligned} \tfrac{1}{2} \sum_{r,s,t=1}^{3} \epsilon_{r\,s\,t}\, \theta_r'\, \theta_s\, \overline{\theta_t} &= \vec{r}_{12}\, \vec{r}_1\, \vec{r}_2 + \, i\, \vec{r}_{11}\, \vec{r}_1\, \vec{r}_2 \\ &= \sqrt{g}\, (\, b_{12} + \, i\, b_{11}\,) \\ &= \sqrt{g}\, (\, b_{12} - \, i\, b_{22}\,) \end{aligned}$$

holds for the associated surface, where $\epsilon_{r\,s\,t}$ *is the sign of the permutation* (r, s, t) *of* $(1, 2, 3)$.

Proof. The proof is a calculation. Using

$$\theta_j = x_{j,1} - \, i\, x_{j,2} \;\text{ and }\; \theta_j' = x_{j,11} - \, i\, x_{j,12}$$

one computes

$$\begin{aligned}\sum_{r,s,t=1}^{3} \epsilon_{r\,s\,t}\,\theta'_r\,\theta_s\,\overline{\theta_t} = &+(x_{1,11} - i\,x_{1,12})(x_{2,1} - i\,x_{2,2})(x_{3,1} + i\,x_{3,2}) \\ &- (x_{1,11} - i\,x_{1,12})(x_{3,1} - i\,x_{3,2})(x_{2,1} + i\,x_{2,2}) \\ &+ (x_{3,11} - i\,x_{3,12})(x_{1,1} - i\,x_{1,2})(x_{2,1} + i\,x_{2,2}) \\ &- (x_{3,11} - i\,x_{3,12})(x_{2,1} - i\,x_{2,2})(x_{1,1} + i\,x_{1,2}) \\ &+ (x_{2,11} - i\,x_{2,12})(x_{3,1} - i\,x_{3,2})(x_{1,1} + i\,x_{1,2}) \\ &- (x_{2,11} - i\,x_{2,12})(x_{1,1} - i\,x_{1,2})(x_{3,1} + i\,x_{3,2})\,.\end{aligned}$$

After separating into real and imaginary parts, one identifies the real part as equaling $2\,\vec{r}_{12}\,\vec{r}_1\,\vec{r}_2$ and the imaginary part as equaling $2\,\vec{r}_{11}\,\vec{r}_1\,\vec{r}_2$. ∎

Corollary 4.4. *The surface associated to the holomorphic functions ψ_j, $j = 1, 2, 3$, is minimal.*

Proof. We compute

$$\begin{aligned}\sum_{r,s,t=1}^{3} \epsilon_{r\,s\,t}\,\psi'_r\,\psi_s\,\overline{\psi_t} &= \sqrt{6}\sum_{r,s,t=1}^{3} \epsilon_{r\,s\,t}\,\phi'_r\,\phi_s\,\overline{\phi_t} \\ &= \sqrt{6}\begin{vmatrix} \sqrt{2}\,i\,\mathrm{cn}'[z] & \sqrt{2}\,\mathrm{dn}'[z] & i\,\mathrm{sn}'[z] \\ \sqrt{2}\,i\,\mathrm{cn}[z] & \sqrt{2}\,\mathrm{dn}[z] & i\,\mathrm{sn}[z] \\ -\sqrt{2}\,i\,\overline{\mathrm{cn}[z]} & \sqrt{2}\,\overline{\mathrm{dn}[z]} & -i\,\overline{\mathrm{sn}[z]} \end{vmatrix} \\ &= -2\sqrt{6}\begin{vmatrix} -\mathrm{sn}[z]\,\mathrm{dn}[z] & -\frac{1}{2}\,\mathrm{sn}[z]\,\mathrm{cn}[z] & \mathrm{cn}[z]\,\mathrm{dn}[z] \\ \mathrm{cn}[z] & \mathrm{dn}[z] & \mathrm{sn}[z] \\ -\overline{\mathrm{cn}[z]} & \overline{\mathrm{dn}[z]} & -\overline{\mathrm{sn}[z]} \end{vmatrix} \\ &= -2\sqrt{6}\Big[(1 - \tfrac{1}{2}\,\mathrm{sn}^2)\,|\mathrm{sn}|^2 + \tfrac{1}{2}\,\mathrm{sn}^2\,|\mathrm{cn}|^2 + (1 - \mathrm{sn}^2)\,|\mathrm{dn}|^2 \\ &\qquad + (1 - \tfrac{1}{2}\,\mathrm{sn}^2)\,|\mathrm{cn}|^2 + \mathrm{sn}^2\,|\mathrm{dn}|^2 - \tfrac{1}{2}\,(1 - \mathrm{sn}^2)\,|\mathrm{sn}|^2\Big] \\ &= -2\sqrt{6}\,\left[\tfrac{1}{2}|\mathrm{sn}|^2 + |\mathrm{cn}|^2 + |\mathrm{dn}|^2\right]\,.\end{aligned}$$

Since

$$\sum_{r,s,t=1}^{3} \epsilon_{r\,s\,t}\,\psi'_r\,\psi_s\,\overline{\psi_t}$$

takes only real values, we see from Theorem 4.3 that $b_{11} = -b_{22} = 0$ holds for the surface $(\widehat{x}_1(z),\,\widehat{x}_2(z),\,\widehat{x}_3(z))$. Thus the surface is minimal by Proposition 4.2. ∎

REFERENCES

[1] F. BOWMAN, *Introduction to Elliptic Functions with Applications,* Dover Publications, N.Y., 1961.

[2] I.S. GRADSHTEYN and I.M. RYZHIK, *Table of Integrals, Series, and Products,* Acad. Press, San Diego, 1980.

[3] R. OSSERMAN, *A Survey of Minimal Surfaces,* Dover Publications, New York, 1986.

[4] H. PARKS, Elliptic isoperimetric problems, *Indiana Univ. Math. J.,* **30** (1981), 937–958.

[5] Th.M. RASSIAS (Ed.), *The Problem of Plateau,* World Scientific Publishing Co., Singapore, New Jersey, London, 1992.