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PURELY UNRECTIFIABLE SETS WITH LARGE PROJECTIONS

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ABSTRACT. For $n \geq 2$, we give a construction of a compact subset of $\mathbb{R}^n$ that is dispersed enough that it is purely unrectifiable, but that nonetheless has an orthogonal projection that hits every point of an $(n-1)$-dimensional unit cube. Moreover, this subset has the additional surprising property that the orthogonal projection onto any straight line in $\mathbb{R}^n$ is a set of positive 1-dimensional Hausdorff measure.

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1. INTRODUCTION

Many of Stanisław Ulam's early papers addressed topics in measure theory. Indeed his paper [6] was the basis for Section 2.1.6 of Federer's treatise on geometric measure theory [1]. In this article, we address a more concrete and geometrical question in measure theory than the foundational issues considered [6].

The structure theorem for sets of finite Hausdorff measure (the theorem referred to here is 3.3.13 of [1]) alerts us to the fact that unrectifiable sets typically have small projections in most directions. But of course an unrectifiable set can have large projections in certain special directions. The question then is how extreme this sort of large projection behavior can be. In this article, we give a construction that yields a purely unrectifiable subset of $\mathbb{R}^n$, $n \geq 2$, that projects orthogonally onto a set with positive $(n-1)$-dimensional Lebesgue measure. Additionally, this subset has the property that the orthogonal projection onto any straight line in $\mathbb{R}^n$ is a set of positive 1-dimensional Hausdorff measure.

Our terminology here differs slightly from that of Federer in that what we call a purely unrectifiable set is called *purely* $(\mathcal{H}^1, 1)$ *unrectifiable* in [1]. The precise definition is the following:

Definition 1.1. We say that a set $E \subseteq \mathbb{R}^n$ is *purely unrectifiable* if, for every Lipschitz function $f : \mathbb{R} \to \mathbb{R}^n$, we have
$$\mathcal{H}^1 \left[f(\mathbb{R}) \cap E \right] = 0 \,,$$
where $\mathcal{H}^1$ denotes the 1-dimensional Hausdorff measure. (We will always mean Lipschitz of order 1 when we say a function is Lipschitz.)

It is immediate from the definition that a subset of $\mathbb{R}$ is purely unrectifiable if and only if it is of Lebesgue measure zero, so purely unrectifiable subsets of $\mathbb{R}$ will not interest us. For $n \geq 2$, Fubini's theorem gives us the following observation (which is also true for $n = 1$, as we have noted):

Remark 1.1. If $E \subseteq \mathbb{R}^n$ is Lebesgue measurable and purely unrectifiable, then $\mathcal{L}^n[E] = 0$.

It seems reasonable to make the following definition:

Definition 1.2. A purely unrectifiable set $E \subseteq \mathbb{R}^n$ *has a large projection* if there is an orthogonal projection $\Pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ such that $\mathcal{L}^{n-1}[\,\Pi(E)\,] > 0$.

Next, we summarize our main result.

Theorem 1.1. *For each integer $n \geq 2$, there exists a purely unrectifiable set $S \subseteq \mathbb{R}^n$ with a large projection. Additionally, we can require that*
$$\Pi(S) = [0,1]^{n-1} \,,$$
where $\Pi : \mathbb{R}^{n-1} \times \mathbb{R} \to \mathbb{R}^{n-1}$ is projection onto the first factor, and we can require that the orthogonal projection of S on any straight line contained in $\mathbb{R}^n$ has positive 1-dimensional Hausdorff measure.

Another recent paper that also looks at sets with surprisingly large projections is [5]. The goals, results, and constructions in that paper differ from those in this article.

2. A MOTIVATING EXAMPLE

One convenient way to construct purely unrectifiable sets is via a geometric iteration like those used to construct Cantor sets or the Sierpinski gasket. We will illustrate this approach with the following example from [2] that motivated the present article. The important feature of this geometric iteration is that there are two special directions in which the subsets involved line up perfectly so that the orthogonal projections also form a nested sequence of diminishing compact sets.

Consider four closed discs of radius $1/4$ inside the unit circle as shown in Figure 1. Two of the discs are centered on the horizontal diameter of the unit circle and are internally tangent to the circle. The other two discs are tangent to the vertical diameter of the unit circle, on opposite sides, and are also internally tangent to the circle. In fact, the centers of those four discs are at $q_1 = (-3/4, 0)$, $q_2 = (-1/4, -1/\sqrt{2})$, $q_3 = (1/4, 1/\sqrt{2})$, and $q_4 = (3/4, 0)$.

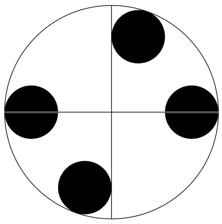

Figure 1: The four discs of radius $1/4$ that form the basic figure inside the unit circle.

If each disc is itself replaced by the four discs inside a scaled down circle, we obtain a collection of sixteen discs, and if that replacement operation is iterated, we obtain a nested sequence of compact sets P_j. To be precise, let us agree that P_1, which we call the "basic figure," is the union of the four discs of radius $1/4$ illustrated in Figure 1. The basic figure and two stages of the iteration are shown in Figure 2. The intersection of the full nested sequence will be denoted by P.

To see that P is purely unrectifiable, we argue as follows: Let ℓ_+ be the line in $\mathbb{R}^2$ that passes through $q_2 = (-1/4, -1/\sqrt{2})$ and $q_4 = (3/4, 0)$, the centers of two of the discs in Figure 1. Let ℓ_- be the line that passes through $q_4 = (3/4, 0)$ and $q_3 = (1/4, 1/\sqrt{2})$, also the centers of two of the discs in Figure 1. Note that the lines ℓ_+ and ℓ_- are orthogonal to each other (see Figure 3). Consider the orthogonal projection from $\mathbb{R}^2$ to ℓ_+. Because the discs are aligned with ℓ_- in pairs, the orthogonal projections of the P_j on ℓ_+ form a nested sequence of compact sets with lengths decreasing by a factor of 2 each time j is incremented. Thus we see that the orthogonal projection of P on ℓ_+ is a set of Lebesgue measure zero. Similarly, we see that the orthogonal projection of P on ℓ_- is also a set of Lebesgue measure zero. Now, suppose $f : \mathbb{R} \to \mathbb{R}^2$ is a Lipschitz function and suppose A is a measurable set with $f(A) \subseteq P$. Then because f followed by an orthogonal projection onto ℓ_+ maps A to a set of Lebesgue measure zero, $f'(t)$ must be orthogonal to the direction of ℓ_+ for almost every $t \in A$. Similarly, $f'(t)$ must be orthogonal to the direction of ℓ_- for almost every $t \in A$. Accordingly, $f'(t) = 0$ holds for almost every $t \in A$, and we have $\mathcal{H}^1[f(A)] = 0$.

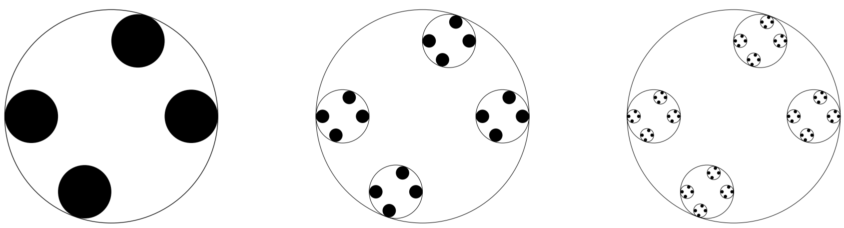

Figure 2: The basic figure and the next two stages of the construction of P.

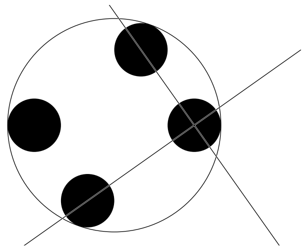

Figure 3: The lines ℓ_+ and ℓ_-.

What is interesting about P, in addition to its being purely unrectifiable, is that it has a large projection. Observe that, for any j, the orthogonal projection of P_j on the horizontal axis is the interval $[-1, 1]$. We conclude that the orthogonal projection of P on the horizontal axis is also the interval $[-1, 1]$.

We find the above example aesthetically pleasing, but it does not generalize readily to higher dimensions. The difficulty is that spheres do not fit together well, so the higher dimensional construction will use cubes instead of spheres.

3. A CRITERION FOR PURE UNRECTIFIABILITY

The next theorem generalizes the method used above to show that P is purely unrectifiable. What the theorem tells us is that, for a subset E of $\mathbb{R}^n$, if there are n linearly independent real-valued functions each of which maps E to a set of Lebesgue measure zero, then E is purely unrectifiable.

Theorem 3.1. *Suppose $E \subseteq \mathbb{R}^n$ is a Borel set. If there exist linearly independent functions $g_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, 2, \ldots, n$, such that $\mathcal{L}^1[g_i(E)] = 0$ holds for each i, then E is purely unrectifiable.*

Proof. Suppose $f : \mathbb{R} \to \mathbb{R}^n$ is Lipschitz. Set $A = f^{-1}(E)$. For $i = 1, 2, \ldots, n$, we have $\mathcal{L}^1[g_i \circ f(A)] = 0$, so we see that

$$0 = (g_i \circ f)'(t) = g_i[f'(t)]$$

holds for $\mathcal{L}^1$-almost every $t \in A$. The only vector in $\mathbb{R}^n$ that is mapped to 0 by all n of the independent functions $g_1, g_2, \ldots, g_n$ is the zero vector. Accordingly, we conclude that $f'(t) = 0$ holds for $\mathcal{L}^1$-almost every $t \in A$ and consequently we have $\mathcal{H}^1[f(\mathbb{R}) \cap E] = 0$. ∎

The next lemma allows us to build larger examples of purely unrectifiable sets from smaller ones.

Lemma 3.2. *The collection of purely unrectifiable sets is closed under countable unions.*

Proof. Suppose each set $E_i \subseteq \mathbb{R}^n$ is purely unrectifiable. Set $E = \cup_i E_i$. If $A \subseteq E$ is the Lipschitz image of a subset of $\mathbb{R}$, then we have

$$\mathcal{H}^1(A) = \mathcal{H}^1(A \cap E) = \mathcal{H}^1\left[\bigcup_i (A \cap E_i)\right] \leq \sum_i \mathcal{H}^1(A \cap E_i) = 0\,.$$

∎

4. NOTATION

We will use the following notation for the construction in $\mathbb{R}^n$, $n \geq 2$:

(1) Let $p_1, p_2, p_3, \ldots$ denote the prime numbers in increasing order.

(2) Set

$$m = \prod_{i=1}^{n} p_i \,.$$

(3) All sets we construct will be compact subsets of the closed unit n-cube $S_0 = [0,1]^n$.

(4) The unit cube will be divided into m^n congruent closed subcubes (m as above) denoted by

$$C[i_1, i_2, \ldots, i_n] = \{\, (x_1, x_2, \ldots, x_n) : i_j/m \leq x_j \leq (i_j + 1)/m \,,\ j = 1, 2, \ldots, n \,\} \;,$$

where $0 \leq i_j \leq m-1$ for $j = 1, 2, \ldots, n$.

(5) For $[i_1, \ldots, i_n]$ with $0 \leq i_j \leq m-1$ for $j = 1, 2, \ldots, n$, we let $\eta[i_1, \ldots, i_n]$ denote the composition of a translation and a homothety which maps $[0,1]^n$ to $C[i_1, \ldots, i_n]$. Specifically, $\eta[i_1, \ldots, i_n]$ is given by

$$(x_1, x_2, \ldots, x_n) \longmapsto \Big((x_1 + i_1)/m, (x_2 + i_2)/m, \ldots, (x_n + i_n)/m \Big) \,.$$

Note that if $[\overline{\imath_h}] = [i_{h,1}, i_{h,2}, \ldots, i_{h,n}]$, for $h = 1, 2, \ldots, k$, then

$$\eta[\overline{\imath_1}] \circ \cdots \circ \eta[\overline{\imath_k}]$$

is given by

$$(x_1, x_2, \ldots, x_n)$$
$$\longmapsto \left(\sum_{r=1}^{k} i_{r,1} m^{-r} + x_1 m^{-k}, \sum_{r=1}^{k} i_{r,2} m^{-r} + x_2 m^{-k}, \ldots, \sum_{r=1}^{k} i_{r,n} m^{-r} + x_n m^{-k} \right) \,.$$

(6) The *basis of the geometric iteration* is the set

$$S_1 = \bigcup_{[i_1, \ldots, i_n] \in \mathcal{I}} C[i_1, \ldots, i_n]$$

derived from the nonempty collection $\mathcal{I}$ of indices $[i_1, \ldots, i_n]$ where $0 \leq i_j \leq m-1$ for $j = 1, 2, \ldots, n$.

(7) The *kth stage of the geometric iteration* is defined inductively by setting

$$S_k = \bigcup_{[i_1, \ldots, i_n] \in \mathcal{I}} \eta[i_1, \ldots, i_n]\,(S_{k-1}) \,.$$

Of course, this amounts to replacing each cube in S_1 by a scaled copy of S_{k-1}. Alternatively, we can look at each stage of the geometric iteration as being accomplished by replacing each cube in S_{k-1} by a scaled copy of S_1. We can also write

$$S_k = \bigcup_{\overline{\imath_1}, \ldots, \overline{\imath_k} \in \mathcal{I}} \eta[\overline{\imath_1}] \circ \cdots \circ \eta[\overline{\imath_k}](S_0) \,.$$

(8) The *set resulting from the geometric iteration* is

$$S = \bigcap_{k} S_k \,.$$

5. INDEPENDENT FUNCTIONS WITH IMAGES OF LENGTH ZERO

Definition 5.1. For $H \in \{1, 2, \ldots, n-1\}$, we let $\mathcal{I}_H$ denote the set of indices of the form $[i_1, i_2, \ldots, i_{n-1}, i_n]$, where $0 \le i_j \le m-1$ for $j = 1, 2, \ldots, n$, that satisfy

$$i_n \equiv i_H \bmod (p_{H+1})\,. \tag{5.1}$$

We also let $g_H : \mathbb{R}^n \to \mathbb{R}$ denote the linear map given by

$$g_H(x) = (\mathbf{e}_H - \mathbf{e}_n) \cdot x\,. \tag{5.2}$$

Lemma 5.1. *If S is the set resulting from the geometric iteration with basis S_1 derived from $\mathcal{I}_H$, then $\mathcal{L}^1\left[\, g_H(S)\,\right] = 0$.*

Proof. If $x \in C[i_1, i_2, \ldots, i_{n-1}, i_n]$, with $[i_1, i_2, \ldots, i_{n-1}, i_n] \in \mathcal{I}_H$, then

$$\frac{i_H - i_n - 1}{m} \le g_H(x) \le \frac{i_H - i_n + 1}{m}\,,$$

where

$$0 \le i_H \le m-1\,, \qquad 0 \le i_n \le m-1\,, \qquad i_H - i_n \equiv 0 \bmod (p_{H+1})\,.$$

Set

$$q_H = m/p_{H+1}\,.$$

We see that $g_H(S_1)$ consists of the $2q_H - 1$ disjoint intervals

$$[\,(\ell-1)/m,\ (\ell+1)/m\,],\ \ \ell = -q_H+1, \ldots, 0, \ldots, q_H - 1$$

each of length $2/m$.

The total length of $g_H(S_1)$ is

$$(2q_H - 1)\,(2/m) = 4/p_{H+1} - 2/m \le 4/p_{H+1} \le 4/3\,.$$

Setting $I_0 = [-1, 1] = g_H(S_0)$ and $I_1 = g_H(S_1)$, we conclude that

$$\frac{\mathcal{L}^1(I_1)}{\mathcal{L}^1(I_0)} \le 2/3\,. \tag{5.3}$$

Observe that $\gamma[\ell] : \mathbb{R} \to \mathbb{R}$ given by

$$\gamma[\ell](x) = (\ell + x)/m$$

is the composition of a translation and a homothety that maps $[-1, 1]$ to $[(\ell-1)/m, (\ell+1)/m]$. So we have

$$I_1 = \bigcup_{\ell=-q_H+1}^{q_H+1} \gamma[\ell]\,(I_0)\ .$$

For $k = 2, 3, \ldots$, set

$$I_k \ = \bigcup_{\ell=-q_H+1}^{q_H+1} \gamma[\ell]\,(I_{k-1}) \ = \bigcup_{\ell_1=-q_H+1}^{q_H+1} \cdots \bigcup_{\ell_k=-q_H+1}^{q_H+1} \gamma[\ell_1] \circ \cdots \circ \gamma[\ell_k](I_0)\,.$$

From (5.3), we conclude that $\mathcal{L}^1\left(\bigcap_{k=1}^{\infty} I_k\right) = 0$.

It is immediate that

$$g_H \circ \eta[i_1, i_2, \ldots, i_{n-1}, i_n] = \gamma[i_H - i_n] \circ g_H$$

holds, so it follows that

$$g_H \circ \eta[\overline{\imath_1}] \circ \cdots \circ \eta[\overline{\imath_k}] = \gamma[i_{1,H} - i_{1,n}] \circ \cdots \circ \gamma[i_{k,H} - i_{k,n}] \circ g_H\,,$$

where $[\overline{\imath_h}] = [i_{h,1}, i_{h,2}, \ldots, i_{h,n}]$, for $h = 1, 2, \ldots, k$.

We have

$$\begin{aligned} g_H(S_k) &= \bigcup_{\overline{\imath_1},\ldots,\overline{\imath_k}\in\mathcal{I}} g_H \circ \eta[\overline{\imath_1}] \circ \cdots \circ \eta[\overline{\imath_k}](S_0) \\ &= \bigcup_{\ell_1=-q_H+1}^{q_H+1} \cdots \bigcup_{\ell_k=-q_H+1}^{q_H+1} \gamma[\ell_1] \circ \cdots \circ \gamma[\ell_k] \circ g_H(S_0) \\ &= \bigcup_{\ell_1=-q_H+1}^{q_H+1} \cdots \bigcup_{\ell_k=-q_H+1}^{q_H+1} \gamma[\ell_1] \circ \cdots \circ \gamma[\ell_k](I_0) = I_k\,. \end{aligned}$$

Notice that because g_H is continuous and because $S = \bigcap_{k=1}^{\infty} S_k$ is formed by the nested intersection of compact sets, the mapping and the intersection may be interchanged. (We emphasize this point because of the history involved: It was by inappropriately interchanging a mapping and an intersection that Lebesgue was led to falsely assert that the projection of a Borel set is a Borel set (see [4, p. 191]).) Thus we have $g_H(S) = \bigcap_{k=1}^{\infty} g_H(S_k) = \bigcap_{k=1}^{\infty} I_k$, from which the result follows. ∎

The preceding lemma gives us $n-1$ independent linear functions that map various sets to sets of 1-dimensional Lebesgue measure zero. To apply Theorem 3.1, we will need an nth independent linear function. The required function is provided by the next lemma.

Definition 5.2. Define $\mathcal{I}_n$ to be the set of indices of the form $[i_1, i_2, \ldots, i_{n-1}, i_n]$ where $0 \leq i_j \leq m-1$ (j $= 1, 2, \ldots,$ n), that satisfy

$$i_n \neq m/2\,,$$

Also, let $g_n : \mathbb{R}^n \to \mathbb{R}$ denote the linear function given by

$$g_n(x) = \mathbf{e}_n \cdot x\,.$$

Lemma 5.2. *If S is the set resulting from the geometric iteration with basis S_1 derived from $\mathcal{I}_n$, then $\mathcal{L}^1\left[g_n(S)\right] = 0$.*

Proof. We see that $g_n(S_1)$ consists of the $m-1$ disjoint intervals

$$[\ell/m, (\ell+1)/m],\;\; \ell = 0, 1, \ldots, (m/2)-1, (m/2)+1, (m/2)+2, \ldots, m-1,$$

each of length $1/m$.

The total length of $g_n(S_1)$ is $(m-1)/m$. Setting $I_0 = [0,1] = g_n(S_0)$ and $I_1 = g_n(S_1)$, we conclude that

$$\frac{\mathcal{L}^1(I_1)}{\mathcal{L}^1(I_0)} = (m-1)/m\,.$$

The argument then proceeds as in the proof of Lemma 5.1. ∎

6. MAIN RESULTS

Each basis for a geometric iteration, $\mathcal{I}_H$, $H = 1, 2, \ldots, n$, defined in the preceding section is designed so that the set resulting from the geometric iteration will map to a set of 1-dimensional Lebesgue measure zero when a particular linear function is applied. To complete the construction, we need to show that if we use as our basis the intersection of all the $\mathcal{I}_H$, $H = 1, 2, \ldots, n$, then the resulting basis is large enough such that the set resulting from the geometric iteration will have the desired large projection. To accomplish this goal we need the next theorem which is an immediate application of the Chinese remainder theorem (see [3, p. 63]).

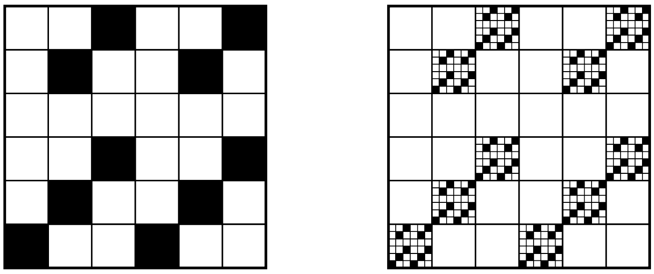

Figure 4: The sets S_1 and S_2 for the construction in $\mathbb{R}^2$

Theorem 6.1. *For $i_1, i_2, \ldots, i_{n-1}$ with $0 \leq i_j \leq m-1$ $(j = 1, 2, \ldots, n-1)$, there exists a unique*

$$0 \leq \widehat{\imath}_n < m/2$$

satisfying

$$\begin{aligned}
\widehat{\imath}_n &\equiv i_1 \bmod (p_2)\,,\\
\widehat{\imath}_n &\equiv i_2 \bmod (p_3)\,,\\
&\vdots\\
\widehat{\imath}_n &\equiv i_{n-1} \bmod (p_n)\,.
\end{aligned}$$

Theorem 6.2. *Let $\mathcal{I}$ consist of the indices $[i_1, i_2, \ldots, i_{n-1}, i_n]$ such that*

$$0 \leq i_j \leq m-1\,, \text{ for } j = 1, 2, \ldots, n\,, \tag{6.1}$$

$$i_n \equiv i_k \bmod (p_{k+1})\,, \text{ for } k = 1, 2, \ldots, n-1\,, \tag{6.2}$$

$$i_n \neq m/2\,, \tag{6.3}$$

and let S be the set resulting from the geometric iteration with basis S_1 derived from $\mathcal{I}$. Then

(1) *for $i_1, i_2, \ldots, i_{n-1}$ with $0 \leq i_j \leq m-1$ $(j = 1, 2, \ldots, n-1)$, there exists an i_n such that $[i_1, i_2, \ldots, i_{n-1}, i_n] \in \mathcal{I}$,*
(2) *$\Pi(S) = [0,1]^{n-1}$ where $\Pi : \mathbb{R}^{n-1} \times \mathbb{R} \to \mathbb{R}^{n-1}$ is a projection onto the first factor,*
(3) *S is purely unrectifiable,*
(4) *in the topology of the unit sphere $\mathbb{S}^{n-1} \subseteq \mathbb{R}^n$, there is an open neighborhood U of*

$$[(n-1)/n]^{1/2}\,\mathbf{e}_n + \sum_{i=1}^{n-1} [n(n-1)]^{-1/2}\,\mathbf{e}_i \tag{6.4}$$

such that, for any straight line ℓ having its direction in U, the orthogonal projection of S on ℓ has positive 1-dimensional Hausdorff measure.

Figures 4 and 5 illustrate the sets used in the geometric iteration when $n = 2$ and when $n = 3$.

Proof.
(a) Part (a) is an immediate consequence of Theorem 6.1.

(b) By part (a), we see that $\Pi(S_1) = [0,1]^{n-1}$. The definition of S_k from S_{k-1} guarantees that $\Pi(S_k) = [0,1]^{n-1}$ holds for all k. Finally, because $S_1 \supseteq S_2 \supseteq \cdots$ and $S = \cap_{k=1}^{\infty} S_k$, we conclude that $\Pi(S) = [0,1]^{n-1}$.

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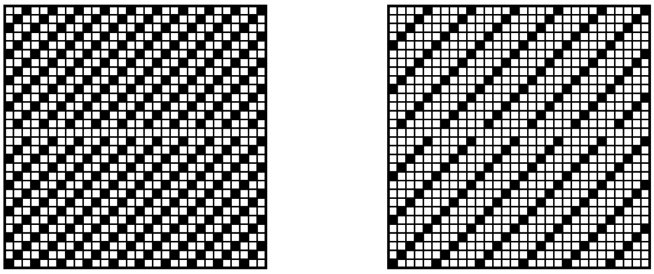

Figure 5: The set S_1 for the construction in $\mathbb{R}^3$. The set has been projected onto the (x, z)-plane in the figure on the left and onto the (y, z)-plane in the figure on the right.

(c) Because $\mathcal{I} \subseteq \mathcal{I}_H$, for $H = 1, 2, \ldots, n$, we see that $\mathcal{L}^1[\, g_H(S)\,] = 0$, for $H = 1, 2, \ldots, n$. Since the linear functions g_H, $H = 1, 2, \ldots, n$, are linearly independent, we conclude from Theorem 3.1 that S is purely unrectifiable.

(d) Consider $\mathbf{v} = \sum_{i=1}^{n} \alpha_i \mathbf{e}_i$ where

$$\frac{2}{m-2}\,\alpha_n < \sum_{i=1}^{n-1} \alpha_i < \frac{m-2}{2}\,\alpha_n\,. \tag{6.5}$$

Define the linear function $h_{\mathbf{v}} : \mathbb{R}^n \to \mathbb{R}$ by setting

$$h_{\mathbf{v}}(x) = \mathbf{v} \cdot x\,.$$

If $\mathbf{v}$ is a unit vector, then $h_{\mathbf{v}}$ can be isometrically identified with orthogonal projection onto a straight line having direction $\mathbf{v}$.

We have

$$h_{\mathbf{v}}(S_0) = \left[0,\ \sum_{i=1}^{n} \alpha_i\right] .$$

Also,

$$h_{\mathbf{v}}(\, C[i, i, \ldots, i]\,) = \left[\ \frac{i}{m}\sum_{i=1}^{n} \alpha_i,\ \ \frac{i+1}{m}\sum_{i=1}^{n} \alpha_i\ \right] .$$

So

$$h_{\mathbf{v}}(D) = \left[0,\ \sum_{i=1}^{n} \alpha_i\right] \Big\backslash \left(2^{-1}\sum_{i=1}^{n} \alpha_i,\ (2^{-1} + m^{-1})\sum_{i=1}^{n} \alpha_i\right) ,$$

where

$$D = \left(\bigcup_{i=0}^{m/2-1} C[i, i, \ldots, i]\right) \bigcup \left(\bigcup_{i=m/2+1}^{m-1} C[i, i, \ldots, i]\right)$$

can be considered as the *main diagonal* in S_1.

The *secondary diagonal* in S_1 is the set

$$D' = \bigcup_{i=1}^{m/2-1} C[i, i, \ldots, i, m/2 + i]\,.$$

We see that

$$h_{\mathbf{v}}(D') = \left[\ 2^{-1}\alpha_n + m^{-1}\sum_{i=1}^{n} \alpha_i,\ 2^{-1}\sum_{i=1}^{n} \alpha_i + 2^{-1}\alpha_n\ \right] .$$

Observe that (6.5) implies

$$2^{-1}\alpha_n + m^{-1}\sum_{i=1}^{n}\alpha_i < 2^{-1}\sum_{i=1}^{n}\alpha_i$$

and

$$(2^{-1}+m^{-1})\sum_{i=1}^{n}\alpha_i < 2^{-1}\sum_{i=1}^{n}\alpha_i + 2^{-1}\alpha_n\,,$$

so we have

$$h_{\mathbf{v}}(D\cup D') = h_{\mathbf{v}}(S_0)\,.$$

Since $D\cup D'\subseteq S_1$, we conclude that $h_{\mathbf{v}}(S_1)=h_{\mathbf{v}}(S_0)$.

When the compostion of a translation and a homothety is followed by a linear map, the resulting composition can be rewritten as the same linear map followed by the composition of a translation and a homothety. Thus $h_{\mathbf{v}}\circ\eta[\bar{\imath}]=\gamma[\bar{\imath}]\circ h_{\mathbf{v}}$ holds, where $\gamma[\bar{\imath}]$ is also a composition of a translation and a homothety. Thus, we may make the following inductive argument: If $h_{\mathbf{v}}(S_k)=h_{\mathbf{v}}(S_{k-1})$ holds, then we have

$$\begin{aligned} h_{\mathbf{v}}(S_{k+1}) = h_{\mathbf{v}}\left(\bigcup_{\bar{\imath}\in\mathcal{I}}\eta[\bar{\imath}](S_k)\right) &= \bigcup_{\bar{\imath}\in\mathcal{I}} h_{\mathbf{v}}\circ\eta[\bar{\imath}](S_k)\\ =\bigcup_{\bar{\imath}\in\mathcal{I}}\gamma[\bar{\imath}]\circ h_{\mathbf{v}}(S_k) &= \bigcup_{\bar{\imath}\in\mathcal{I}}\gamma[\bar{\imath}]\circ h_{\mathbf{v}}(S_{k-1})\\ =\bigcup_{\bar{\imath}\in\mathcal{I}} h_{\mathbf{v}}\circ\eta[\bar{\imath}](S_{k-1}) &= h_{\mathbf{v}}\left(\bigcup_{\bar{\imath}\in\mathcal{I}}\eta[\bar{\imath}](S_{k-1})\right) = h_{\mathbf{v}}(S_k)\,. \end{aligned}$$

Since $h_{\mathbf{v}}(S_1)=h_{\mathbf{v}}(S_0)$ holds, we have

$$h_{\mathbf{v}}(S)=\bigcap_{k=1}^{\infty} h_{\mathbf{v}}(S_k) = h_{\mathbf{v}}(S_0)\,,$$

and so $h_{\mathbf{v}}(S)$ has positive 1-dimensional Lebesgue measure.

Finally, observe that if we set

$$\alpha_1=\alpha_2=\cdots=\alpha_{n-1}=n^{-1/2}\,(n-1)^{-1/2}\,,\ \text{ and }\ \alpha_n = n^{-1/2}\,(n-1)^{1/2}\,,$$

then the condition (6.5) is satisfied and $\mathbf{v}=\sum_{i=1}^{n}\alpha_i\mathbf{e}_i$ is the vector in (6.4). Consequently there is an open set of unit vectors of the form $\mathbf{v}=\sum_{i=1}^{n}\alpha_i\mathbf{e}_i$ for which condition (6.5) is satisfied. ∎

Proof of Theorem 1.1. The set S in Theorem 6.2 satisfies the requirements that it be purely unrectifiable and that it project onto $[0,1]^{n-1}$. Because $\mathbb{S}^{n-1}$ is compact, part (d) of Theorem 6.2 tells us that the union of S and finitely many rotated copies of S will form a set whose orthogonal projection on any line ℓ in $\mathbb{R}^n$ has positive 1-dimensional Hausdorff measure. Lemma 3.2 tells us that the set is purely unrectifiable. Thus all the requirements of the main theorem have been met. ∎

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