ABSTRACT. In this paper some open problems in analysis are formulated. These problems were formulated and discussed by the author at ICMAA6.
1. **Injectivity of the Classical Radon Transform**

Consider the Radon transform:

\[
Rf := \int_{\ell_{\alpha,p}} f \, ds,
\]

where \(\ell_{\alpha,p}\) is a straight line \(\alpha \cdot x = p\) on the plane \(x = \{x_1, x_2\}\), \(\alpha\) is a unit vector, \(p\) is a real number, and \(ds\) is the element of the arclength of the straight line.

Assume that

\[
f \in L^1(\ell_{\alpha,p})
\]

for all \(p\) and \(\alpha\), that \(f\) is a continuous function, and that

\[
|f(x)| \leq c(1 + |x|^m),
\]

where \(c = \text{const} > 0\), and \(m \geq 0\) is a fixed number. Assume that

\[
Rf = 0
\]

for all \(p\) and \(\alpha\).

**Problem 1.** Does it follow from the assumptions (1.2) – (1.4) that \(f = 0\)?

There is a large amount of literature on the Radon transform (see, e.g., [3] and references therein). It is known (see, e.g., [1], [3]) that there are entire functions not vanishing identically, such that (1.2) and (1.4) hold.

The open problem is to understand what the weakest natural restriction on the growth of \(f\) at infinity is for the Radon transform to be injective. In other words, under what weakest growth restriction at infinity do the assumptions (1.2) – (1.4) imply \(f = 0\)?

It is known (see [3]) that if \(f \in L^1(\mathbb{R}^2, \frac{1}{1+|x|})\) and (1.4) holds, then \(f = 0\), i.e., the Radon transform is injective on \(L^1(\mathbb{R}^2, \frac{1}{1+|x|})\).

2. **A Uniqueness Problem**

Let \(L\) and \(M\) be elliptic, second order, selfadjoint, strictly positive Dirichlet operators in a bounded domain \(D \subset \mathbb{R}^n, n > 1\), with a smooth connected boundary \(S\), and the coefficients of \(L\) and \(M\) are real-valued functions, so that all the functions below are real-valued. Let \(a(x)\) and \(b(x)\) be strictly positive functions, smooth in the closure of \(D\). Let

\[
Lu + a(x)v = 0 \text{ in } D, \quad -b(x)u + Mv = 0 \text{ in } D, \quad u = v = 0 \text{ on } S,
\]

**Problem 2.** Does (2.1) imply

\[
u = v = 0 \text{ in } D?
\]

It is of no interest to give sufficient conditions for (2.2) to hold, such as, e.g., \(|b - a|\) is small, or \(L = M\), or some other conditions.

*What is of interest is to answer the question as stated, without any additional assumptions, by either proving (2.2) or constructing a counterexample.*

In the one-dimensional case the answer to the question (2.2) is yes (see [2]).
3. A PROBLEM IN OPERATOR THEORY

The question in two different forms is stated below as Problem 3 and Problem 4. These problems are closely related.

3.1. Let $D$ be a bounded domain in $\mathbb{R}^3$, $D$ can be a box or a ball, $f \in L^2(D)$ be a function, $f \not\equiv 0$. Define

$$F(z) := \int_D f(x) \exp(iz \cdot x) dx, \quad z \in \mathbb{C}^3.$$ 

The function $F(z)$ is an entire function of exponential type.

Let $L_j(z)$, $j = 1, 2$, be polynomials of degree not less than one, $\deg L_j(z) \geq 1$,

$$L_j := \{z : z \in \mathbb{C}^3, L_j(z) = 0\}$$

be the corresponding algebraic varieties.

Define Hilbert spaces $H_j := L^2(L_j, dm_j)$, where $dm_j(z)$ are smooth, rapidly decaying, strictly positive measures on $L_j$, such that any exponential $\exp(iz \cdot x)$ with any $x \in \mathbb{R}^3$ belongs to $H_j$. Define a linear operator $T$ from $H_1$ into $H_2$ by the formula:

$$Th := \int_{L_1} dm_1(u_1)h(u_1)F(u_1 + u_2) := g(u_2),$$

where $u_j \in L_j$, $h \in H_1$, $g \in H_2$. We assume that the measures $dm_j$ decay so rapidly that for any $h \in H_1$ the function $g = Th$ belongs to $H_2$, $Th \in H_2$. For example, this happens if the measures decay as $e^{-|z|^2}$.

Assume that $L_1$ and $L_2$ are transversal, which by definition means that there exist two points, one in $L_1$ and one in $L_2$, such that the union of the bases of the tangent spaces to $L_1$ and to $L_2$ at these points form a basis in $\mathbb{C}^3$. The same setting is of interest in dimension $n > 3$ as well.

Problem 3. Is it true that $T$ is not a finite-rank operator?

In other words, is it true that the dimension of the range of $T$ is infinite?

Remark 3.1. The assumption that $f(x)$ is in $L^2(D)$ is important. If, for example, $f(x)$ is a delta-function, then the answer to the question of Problem 3 is no, the dimension of the range of $T$ in this case is equal to 1 if the delta-function is supported at one point.

3.2. In the notations of Problem 3 choose points $p_m \in L_2$, $m = 1, 2, \ldots, M$, where $M$ is an arbitrary large fixed integer. Consider the set $S$ of $M$ functions $F(z + p_m), m = 1, 2, \ldots, M$, where $z \in L_1$, and $F(z)$ is defined above. It is the Fourier transform of a compactly supported $L^2(D)$ function, where $D$ is a bounded domain in $\mathbb{R}^n$, $n > 1$.

Problem 4. Can one choose $p_m \in L_2$ such that the above set $S$ of $M$ functions is linearly independent?

In other words, can one choose $p_m \in L_2$, $m = 1, 2, \ldots, M$, such that the relation:

$$\sum_{m=1}^M c_m F(z + p_m) = 0 \quad \forall z \in L_1$$

implies $c_m = 0$ for all $m = 1, 2, \ldots, M$? Here $c_m$ are constants.

These questions arise in the study of Property C (see [4, p. 298]).
4. A Problem Related to the Pompeiu Problem

Let $D \subset \mathbb{R}^3$ be a bounded domain homeomorphic to a ball, with a real analytic boundary $S$. Let $u_j = u_j(x)$, $j = 1, 2, 3$, solve the problem:

\begin{equation}
\Delta u_j + k^2 u_j = 0 \quad \text{in} \quad D, \quad u_j|_S = 0,
\end{equation}

where $k^2 > 0$ is a constant. Let $N = N_s$ be the unit normal to the surface $S$ at the point $s \in S$, pointing out of $D$. Define the following vector-function:

\begin{equation}
u(x) = \sum_{j=1}^{3} u_j(x)e_j,
\end{equation}

where $\{e_j\}_{j=1}^{3}$ is the standard Euclidean orthonormal basis of $\mathbb{R}^3$. Let $[a, b]$ denote the cross product of two vectors $a$ and $b$ in $\mathbb{R}^3$.

Assume that

\begin{equation}
u_N = [s, N_s] \quad \forall \ s \in S,
\end{equation}

where $\nu = \nu(x)$ is defined in (4.2) and $u_j(x)$ solve problem (4.1).

**Problem 5.** Does (4.3) imply $[s, N_s] = 0$ on $S$?

**Conjecture 4.1.** Assumptions (4.1) and (4.3) imply

\begin{equation}
[s, N_s] = 0 \quad \forall \ s \in S.
\end{equation}

It is pointed out in [4, p. 416], that if (4.4) holds, then $S$ is a sphere.

A proof of the above conjecture implies a positive solution to the Pompeiu problem, see [4, Chapter 11] and [5, 6].

**References**


