FIXED POINTS AND STABILITY OF THE CAUCHY FUNCTIONAL EQUATION

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ABSTRACT. Using fixed point methods, we prove the generalized Hyers-Ulam stability of homomorphisms in Banach algebras and of derivations on Banach algebras for the Cauchy functional equation.

Key words and phrases: Cauchy functional equation, Fixed point, Homomorphism in Banach algebra, Generalized Hyers-Ulam stability, Derivation on Banach algebra.

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1. Introduction and Preliminaries

The stability problem of functional equations originated from a question of Ulam [32] concerning the stability of group homomorphisms: Let \((G_1, \ast)\) be a group and let \((G_2, \circ, d)\) be a metric group with the metric \(d(\cdot, \cdot)\). Given \(\epsilon > 0\), does there exist a \(\delta(\epsilon) > 0\) such that if a mapping \(h : G_1 \to G_2\) satisfies the inequality

\[
d(h(x \ast y), h(x) \circ h(y)) < \delta
\]

for all \(x, y \in G_1\), then there is a homomorphism \(H : G_1 \to G_2\) with

\[
d(h(x), H(x)) < \epsilon
\]

for all \(x \in G_1\)? If the answer is affirmative, we would say that the equation of homomorphism \(H(x \ast y) = H(x) \circ H(y)\) is stable. The concept of stability for a functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. Thus the stability question of functional equations is: "How do the solutions of the inequality differ from those of the given functional equation?"

Hyers [7] gave the first affirmative answer to the question of Ulam for Banach spaces. Let \(X\) and \(Y\) be Banach spaces. Assume that \(f : X \to Y\) satisfies

\[
\|f(x + y) - f(x) - f(y)\| \leq \epsilon
\]

for all \(x, y \in X\) and some \(\epsilon \geq 0\). Then there exists a unique additive mapping \(T : X \to Y\) such that

\[
\|f(x) - T(x)\| \leq \epsilon
\]

for all \(x \in X\).

Th.M. Rassias [27] provided a generalization of Hyers’ Theorem which allows the Cauchy difference to be unbounded.

**Theorem 1.1** (Th.M. Rassias). Let \(f : E \to E'\) be a mapping from a normed vector space \(E\) into a Banach space \(E'\) subject to the inequality

\[
\|f(x + y) - f(x) - f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p)
\]

for all \(x, y \in E\), where \(\epsilon\) and \(p\) are constants with \(\epsilon > 0\) and \(p < 1\). Then the limit

\[
L(x) = \lim_{n \to \infty} \frac{f(2^n x)}{2^n}
\]

exists for all \(x \in E\) and \(L : E \to E'\) is the unique additive mapping which satisfies

\[
\|f(x) - L(x)\| \leq \frac{2\epsilon}{2 - 2^p} \|x\|^p
\]

for all \(x \in E\). Also, if for each \(x \in E\) the function \(f(tx)\) is continuous in \(t \in \mathbb{R}\), then \(L\) is \(\mathbb{R}\)-linear.

The inequality (1.1) has been influential in the development of what is now known as the generalized Hyers-Ulam stability or Hyers-Ulam-Rassias stability of functional equations. Beginning around the year 1980, the topic of approximate homomorphisms, or the stability of the equation of homomorphisms, has been studied by a number of mathematicians. Gavruta [6], following Th.M. Rassias’ approach for the stability of the linear mapping between Banach spaces obtained a generalization of Th.M. Rassias’ Theorem. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [3], [4], [9], [10], [11], [13] – [25], [28] – [30]).

We recall two fundamental results in fixed point theory.
Theorem 1.2 ([12 26]). Let \((X, d)\) be a complete metric space and let \(J : X \to X\) be strictly contractive, i.e.,
\[
d(Jx, Jy) \leq Ld(x, y), \quad \forall x, y \in X
\]
for some Lipschitz constant \(L < 1\). Then
(1) the mapping \(J\) has a unique fixed point \(x^* = Jx^*\);
(2) the fixed point \(x^*\) is globally attractive, i.e.,
\[
\lim_{n \to \infty} J^n x = x^*
\]
for any starting point \(x \in X\);
(3) one has the following estimation inequalities:
\[
d(J^n x, x^*) \leq L^n d(x, x^*),
\]
\[
d(J^n x, x^*) \leq \frac{1}{1 - L} d(J^n x, J^{n+1} x),
\]
\[
d(x, x^*) \leq \frac{1}{1 - L} d(x, Jx)
\]
for all nonnegative integers \(n\) and all \(x \in X\).

Let \(X\) be a set. A function \(d : X \times X \to [0, \infty]\) is called a generalized metric on \(X\) if \(d\) satisfies
(1) \(d(x, y) = 0\) if and only if \(x = y\);
(2) \(d(x, y) = d(y, x)\) for all \(x, y \in X\);
(3) \(d(x, z) \leq d(x, y) + d(y, z)\) for all \(x, y, z \in X\).

Theorem 1.3 ([5]). Let \((X, d)\) be a complete generalized metric space and let \(J : X \to X\) be a strictly contractive mapping with a Lipschitz constant \(L < 1\). Then for each given element \(x \in X\), either
\[
d(\text{J}^n x, \text{J}^{n+1} x) = \infty
\]
for all nonnegative integers \(n\) or there exists a positive integer \(n_0\) such that
(1) \(d(\text{J}^n x, \text{J}^{n+1} x) < \infty\), \quad \forall n \geq n_0;
(2) the sequence \(\{\text{J}^n x\}\) converges to a fixed point \(y^*\) of \(J\);
(3) \(y^*\) is the unique fixed point of \(J\) in the set \(Y = \{y \in X \mid d(\text{J}^{n_0} x, y) < \infty\}\);
(4) \(d(y, y^*) \leq \frac{1}{1 - L} d(y, Jy)\) for all \(y \in Y\).

This paper is organized as follows: In Section 2 using the fixed point method, we prove the generalized Hyers-Ulam stability of homomorphisms in Banach algebras for the Cauchy functional equation.

In Section 3 using the fixed point method, we prove the generalized Hyers-Ulam stability of derivations on Banach algebras for the Cauchy functional equation.

Throughout this section, assume that \(A\) is a complex Banach algebra with norm \(\| \cdot \|_A\) and that \(B\) is a complex Banach algebra with norm \(\| \cdot \|_B\).

2. Stability of Homomorphisms in Banach Algebras

For a given mapping \(f : A \to B\), we define
\[
D_\mu f(x, y) := \mu f(x + y) - f(\mu x) - f(\mu y)
\]
for all \(\mu \in \mathbb{T}^1 := \{\nu \in \mathbb{C} : |\nu| = 1\}\) and all \(x, y \in A\).

Note that a \(\mathbb{C}\)-linear mapping \(H : A \to B\) is called a homomorphism in Banach algebras if \(H\) satisfies \(H(xy) = H(x)H(y)\) for all \(x, y \in A\).
We prove the generalized Hyers-Ulam stability of homomorphisms in Banach algebras for the functional equation $D_\mu f(x, y) = 0$.

**Theorem 2.1.** Let $f : A \to B$ be a mapping for which there exists a function $\varphi : A^2 \to [0, \infty)$ such that

\begin{align}
\text{(2.1)} & \quad \lim_{j \to \infty} 2^{-j} \varphi(2^j x, 2^j y) = 0, \\
\text{(2.2)} & \quad \|D_\mu f(x, y)\|_B \leq \varphi(x, y), \\
\text{(2.3)} & \quad \|f(xy) - f(x)f(y)\|_B \leq \varphi(x, y)
\end{align}

for all $\mu \in T^1$ and all $x, y \in A$. If there exists an $L < 1$ such that $\varphi(x, x) \leq 2L\varphi(\frac{x}{2}, \frac{x}{2})$ for all $x \in A$, then there exists a unique homomorphism $H : A \to B$ such that

\begin{align}
\text{(2.4)} & \quad \|f(x) - H(x)\|_B \leq \frac{1}{2 - 2L}\varphi(x, x)
\end{align}

for all $x \in A$.

**Proof.** Consider the set

\[ X := \{g : A \to B\} \]

and introduce the generalized metric on $X$:

\[ d(g, h) = \inf\{C \in \mathbb{R}_+ : \|g(x) - h(x)\|_B \leq C\varphi(x, x), \ \forall x \in A\} \]

It is easy to show that $(X, d)$ is complete.

Now we consider the linear mapping $J : X \to X$ such that

\[ Jg(x) := \frac{1}{2}g(2x) \]

for all $x \in A$.

By Theorem 3.1 of [1],

\[ d(Jg, Jh) \leq Ld(g, h) \]

for all $g, h \in X$.

Letting $\mu = 1$ and $y = x$ in (2.2), we get

\begin{align}
\text{(2.5)} & \quad \|f(2x) - 2f(x)\|_B \leq \varphi(x, x)
\end{align}

for all $x \in A$. So

\[ \|f(x) - \frac{1}{2}f(2x)\|_B \leq \frac{1}{2}\varphi(x, x) \]

for all $x \in A$. Hence $d(f, Jf) \leq \frac{1}{2}$.

By Theorem 1.3 there exists a mapping $H : A \to B$ such that

1. $H$ is a fixed point of $J$, i.e.,

\[ H(2x) = 2H(x) \]

for all $x \in A$. The mapping $H$ is a unique fixed point of $J$ in the set

\[ Y = \{g \in X : d(f, g) < \infty\} \]

This implies that $H$ is a unique mapping satisfying (2.6) such that there exists $C \in (0, \infty)$ satisfying

\[ \|H(x) - f(x)\|_B \leq C\varphi(x, x) \]

for all $x \in A$.

2. $d(J^n f, H) \to 0$ as $n \to \infty$. This implies the equality

\[ \lim_{n \to \infty} \frac{f(2^n x)}{2^n} = H(x) \]
for all $x \in A$.

(3) $d(f, H) \leq \frac{1}{1-L}d(f, Jf)$, which implies,

$$d(f, H) \leq \frac{1}{2-2L}.$$  

This implies that the inequality (2.4) holds.

It follows from (2.1), (2.2) and (2.7) that

$$\|H(x + y) - H(x) - H(y)\|_B$$

$$= \lim_{n \to \infty} \frac{1}{2^n} \|f(2^n(x + y)) - f(2^n x) - f(2^n y)\|_B$$

$$\leq \lim_{n \to \infty} \frac{1}{2^n} \varphi(2^n x, 2^n y) = 0$$

for all $x, y \in A$. So

(2.8) $H(x + y) = H(x) + H(y)$

for all $x, y \in A$.

Letting $y = x$ in (2.2), we get

$$\mu f(2x) = f(\mu 2x)$$

for all $\mu \in T^1$ and all $x \in A$. By a similar method to that above, we obtain

$$\mu H(2x) = H(2\mu x)$$

for all $\mu \in T^1$ and all $x \in A$. Thus one can show that the mapping $H : A \to B$ is $C$-linear.

It follows from (2.3) that

$$\|H(xy) - H(x)H(y)\|_B = \lim_{n \to \infty} \frac{1}{4^n} \|f(4^n(xy)) - f(4^n x)f(4^n y)\|_B$$

$$\leq \lim_{n \to \infty} \frac{1}{4^n} \varphi(2^n x, 2^n y)$$

$$\leq \lim_{n \to \infty} \frac{1}{2^n} \varphi(2^n x, 2^n y) = 0$$

for all $x, y \in A$. So

$$H(xy) = H(x)H(y)$$

for all $x, y \in A$.

Thus $H : A \to B$ is a homomorphism satisfying (2.4), as desired. ■

**Corollary 2.2.** Let $r < \frac{1}{2}$ and $\theta$ be nonnegative real numbers, and let $f : A \to B$ be a mapping such that

(2.9) $\|D_\mu f(x, y)\|_B \leq \theta \cdot \|x\|_A^r \cdot \|y\|_A^r$,

(2.10) $\|f(xy) - f(x)f(y)\|_B \leq \theta \cdot \|x\|_A^r \cdot \|y\|_A^r$

for all $\mu \in T^1$ and all $x, y \in A$. Then there exists a unique homomorphism $H : A \to B$ such that

$$\|f(x) - H(x)\|_B \leq \frac{\theta}{2 - 4r} \|x\|_A^{2r}$$

for all $x \in A$.

**Proof.** The proof follows from Theorem 2.1 by taking

$$\varphi(x, y) := \theta \cdot \|x\|_A^r \cdot \|y\|_A^r$$

for all $x, y \in A$. Then $L = 2^{2r-1}$ and we get the desired result. ■
Theorem 2.3. Let $f : A \rightarrow B$ be a mapping for which there exists a function $\varphi : A^2 \rightarrow [0, \infty)$ satisfying (2.2) and (2.3) such that

$$(2.11) \quad \lim_{j \to \infty} 4^j \varphi \left( \frac{x}{2^j}, \frac{y}{2^j} \right) = 0$$

for all $x, y \in A$. If there exists an $L < 1$ such that $\varphi(x, x) \leq \frac{1}{2} L \varphi(2x, 2x)$ for all $x \in A$, then there exists a unique homomorphism $H : A \rightarrow B$ such that

$$(2.12) \quad \| f(x) - H(x) \|_B \leq \frac{L}{2 - 2L} \varphi(x, x)$$

for all $x \in A$.

Proof. We consider the linear mapping $J : X \rightarrow X$ such that

$$Jg(x) := 2g \left( \frac{x}{2} \right)$$

for all $x \in A$.

It follows from (2.5) that

$$\left\| f(x) - 2f \left( \frac{x}{2} \right) \right\|_B \leq \varphi \left( \frac{x}{2}, \frac{x}{2} \right) \leq \frac{L}{2} \varphi(x, x)$$

for all $x \in A$. Hence $d(f, Jf) \leq \frac{L}{2}$.

By Theorem 1.3 there exists a mapping $H : A \rightarrow B$ such that:

(1) $H$ is a fixed point of $J$, i.e.,

$$(2.13) \quad H(2x) = 2H(x)$$

for all $x \in A$. The mapping $H$ is a unique fixed point of $J$ in the set

$$Y = \{ g \in X : d(f, g) < \infty \}.$$  

This implies that $H$ is a unique mapping satisfying (2.13) such that there exists $C \in (0, \infty)$ satisfying

$$\| H(x) - f(x) \|_B \leq C \varphi(x, x)$$

for all $x \in A$.

(2) $d(J^n f, H) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n \to \infty} 2^n f \left( \frac{x}{2^n} \right) = H(x)$$

for all $x \in A$.

(3) $d(f, H) \leq \frac{1}{1-L} d(f, Jf)$, which implies,

$$d(f, H) \leq \frac{L}{2 - 2L},$$

which implies that the inequality (2.12) holds.

The rest of the proof is similar to the proof of Theorem 2.1.

Corollary 2.4. Let $r > 1$ and $\theta$ be nonnegative real numbers, and let $f : A \rightarrow B$ be a mapping satisfying (2.9) and (2.10). Then there exists a unique homomorphism $H : A \rightarrow B$ such that

$$\| f(x) - H(x) \|_B \leq \frac{\theta}{4^r - 2} \| x \|_A^{2r}$$

for all $x \in A$.  

Proof. The proof follows from Theorem 2.3 by taking
\[ \varphi(x, y) := \theta \cdot \|x\|^r_A \cdot \|y\|^r_A \]
for all \( x, y \in A \). Then \( L = 2^{1-2r} \) and we get the desired result. \( \square \)

3. Stability of Derivations on Banach Algebras

We prove the generalized Hyers-Ulam stability of derivations on Banach algebras for the functional equation \( D_\mu f(x, y) = 0 \).

Theorem 3.1. Let \( f : A \to A \) be a mapping for which there exists a function \( \varphi : A^2 \to [0, \infty) \) satisfying (2.7) such that
\begin{align}
(3.1) & \quad \|D_\mu f(x, y)\|_A \leq \varphi(x, y), \\
(3.2) & \quad \|f(xy) - f(x)y - xf(y)\|_A \leq \varphi(x, y)
\end{align}
for all \( \mu \in \mathbb{T}^1 \) and all \( x, y \in A \). If there exists an \( L < 1 \) such that \( \varphi(x, x) \leq 2L\varphi \left( \frac{x}{2}, \frac{x}{2} \right) \) for all \( x \in A \), then there exists a unique derivation \( \delta : A \to A \) such that
\[ \|f(x) - \delta(x)\|_A \leq \frac{1}{2 - 2L} \varphi(x, x) \]
for all \( x \in A \).

Proof. By the same reasoning as the proof of Theorem 2.1 there exists a unique \( \mathbb{C} \)-linear mapping \( \delta : A \to A \) satisfying (3.3). The mapping \( \delta : A \to A \) is given by
\[ \delta(x) = \lim_{n \to \infty} \frac{f(2^n x)}{2^n} \]
for all \( x \in A \).

It follows from (3.2) that
\[ \|\delta(xy) - \delta(x)y - x\delta(y)\|_A \]
\[ \leq \lim_{n \to \infty} \frac{1}{4^n} \|f(4^n xy) - f(2^n x) \cdot 2^n y - 2^n x f(2^n y)\|_A \]
\[ \leq \lim_{n \to \infty} \frac{1}{4^n} \varphi(2^n x, 2^n y) \leq \lim_{n \to \infty} \frac{1}{2^n} \varphi(2^n x, 2^n y) = 0 \]
for all \( x, y \in A \). So
\[ \delta(xy) = \delta(x)y + x\delta(y) \]
for all \( x, y \in A \). Thus \( \delta : A \to A \) is a derivation satisfying (3.3). \( \square \)

Corollary 3.2. Let \( r < \frac{1}{2} \) and \( \theta \) be nonnegative real numbers, and let \( f : A \to A \) be a mapping such that
\begin{align}
(3.4) & \quad \|D_\mu f(x, y)\|_A \leq \theta \cdot \|x\|^r_A \cdot \|y\|^r_A, \\
(3.5) & \quad \|f(xy) - f(x)y - xf(y)\|_A \leq \theta \cdot \|x\|^r_A \cdot \|y\|^r_A
\end{align}
for all \( \mu \in \mathbb{T}^1 \) and all \( x, y \in A \). Then there exists a unique derivation \( \delta : A \to A \) such that
\[ \|f(x) - \delta(x)\|_A \leq \frac{\theta}{2 - 4^r} \|x\|^{2r} \]
for all \( x \in A \).
**Proof.** The proof follows from Theorem 3.1 by taking
\[ \varphi(x, y) := \theta \cdot \|x\|_A^r \cdot \|y\|_A^r \]
for all \( x, y \in A \). Then \( L = 2^{2r-1} \) and we get the desired result. 

**Theorem 3.3.** Let \( f : A \to A \) be a mapping for which there exists a function \( \varphi : A^2 \to [0, \infty) \) satisfying (2.11), (3.1) and (3.2). If there exists an \( L < 1 \) such that
\[ \varphi(x, x) \leq \frac{1}{2} L \varphi(2x, 2x) \]
for all \( x \in A \), then there exists a unique derivation \( \delta : A \to A \) such that
\[ \|f(x) - \delta(x)\|_A \leq \frac{L}{2 - 2L} \varphi(x, x) \]
for all \( x \in A \).

**Proof.** The proof is similar to the proofs of Theorems 2.3 and 3.1.

**Corollary 3.4.** Let \( r > 1 \) and \( \theta \) be nonnegative real numbers, and let \( f : A \to A \) be a mapping satisfying (3.4) and (3.5). Then there exists a unique derivation \( \delta : A \to A \) such that
\[ \|f(x) - \delta(x)\|_A \leq \frac{\theta}{4r - 2} \|x\|_A^{2r} \]
for all \( x \in A \).

**Proof.** The proof follows from Theorem 3.3 by taking
\[ \varphi(x, y) := \theta \cdot \|x\|_A^r \cdot \|y\|_A^r \]
for all \( x, y \in A \). Then \( L = 2^{1-2r} \) and we get the desired result.

**References**


