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ON SOME MAPPING PROPERTIES OF MÖBIUS TRANSFORMATIONS

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ABSTRACT. We consider spheres corresponding to any norm function on the complex plane and their images under the Möbius transformations. We see that the sphere preserving property is not an invariant characteristic property of Möbius transformations except in the Euclidean case.

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1. INTRODUCTION

Möbius transformations are the automorphisms of the extended complex plane $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$, that is, the meromorphic bijections $T : \mathbb{C}_\infty \to \mathbb{C}_\infty$. A Möbius transformation T has the form

$$T(z) = \frac{az+b}{cz+d}; \qquad a,b,c,d \in \mathbb{C} \quad \text{and} \quad ad-bc \neq 0. \tag{1.1}$$

The set of all Möbius transformations is a group under composition. The Möbius transformations with $c = 0$ form the subgroup of *similarities*. Such transformations have the form

$$S(z) = \alpha z + \beta; \quad \alpha, \beta \in \mathbb{C}, \alpha \neq 0.$$

The transformation $J(z) = \frac{1}{z}$ is called an *inversion*. Every Möbius transformation T of the form (1.1) is a composition of finitely many similarities and inversions (see [2], [9], [10] and [11]).

It is well-known that the image of a line or a circle under a Möbius transformation is another line or circle and the principle of circle transformation is an invariant characteristic property of Möbius transformations. For example, the following results are known.

Theorem A ([1]). *If $f : \widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \to \widehat{\mathbb{C}}$ is a circle-preserving map, then f is a Möbius transformation if and only if f is a bijection.*

Theorem B ([4]). *If $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a circle-preserving map, then f is a Möbius transformation if and only if f is a non constant meromorphic function.*

Also it is well-known that all norms on $\mathbb{C}$ are equivalent. It seems natural, then, to consider the images of a sphere corresponding to any norm function on $\mathbb{C}$ under the Möbius transformations.

Throughout the paper, we consider the real linear space structure of the complex plane $\mathbb{C}$. We investigate the images of spheres, corresponding to any norm function on $\mathbb{C}$, under the Möbius transformations. In Section 2, we see that the sphere preserving property is not an invariant characteristic property of Möbius transformations except in the Euclidean case. In Section 3, we consider the relationships between the notion of "*Apollonius quadrilateral*" which was introduced by H. Haruki and Th.M. Rassias [7] and the spheres corresponding to any norm function on $\mathbb{C}$.

2. MÖBIUS TRANSFORMATIONS AND SPHERES CORRESPONDING TO ANY NORM FUNCTION ON $\mathbb{C}$

Let $\|\cdot\|$ be any norm function on $\mathbb{C}$. A sphere whose center is at z_0 and is of radius r is denoted by $S_r(z_0)$ and defined by $S_r(z_0) = \{z \in \mathbb{C} : \|z - z_0\| = r\}$. In general, we note that Möbius transformations do not map spheres to spheres corresponding to any norm function on $\mathbb{C}$. For example, if we consider the norm function

$$\|z\| = \sqrt{\frac{x^2}{9} + 4y^2} \tag{2.1}$$

on $\mathbb{C}$, the spheres corresponding to this norm are ellipses. Indeed, for any sphere in this norm we have

$$S_r(z_0) = \{z \in \mathbb{C} : \|z - z_0\| = r\} = \left\{ z \in \mathbb{C} : \frac{(x-x_0)^2}{9} + 4(y-y_0)^2 = r^2 \right\},$$

which is an ellipse with foci $\left(\frac{\sqrt{35}}{2}+x_0, y_0\right)$, $\left(-\frac{\sqrt{35}}{2}+x_0, y_0\right)$. However, from [3], we know that the only Möbius transformations which map ellipses to ellipses are the similarity transformations.

On the other hand, under the rotation map $z \to e^{i\theta}z$, the image of the ellipse with foci $\left(\frac{\sqrt{35}}{2}+x_0, y_0\right)$, $\left(-\frac{\sqrt{35}}{2}+x_0, y_0\right)$ is also an ellipse but it is not a sphere in the norm $\|\cdot\|$ unless $\theta = k\pi, k \in \mathbb{Z}$, since its foci are

$$\left(\left(\frac{\sqrt{35}}{2}+x_0\right)\cos\theta - y_0\sin\theta,\ \left(\frac{\sqrt{35}}{2}+x_0\right)\sin\theta + y_0\cos\theta\right)$$

and

$$\left(\left(-\frac{\sqrt{35}}{2}+x_0\right)\cos\theta - y_0\sin\theta,\ \left(-\frac{\sqrt{35}}{2}+x_0\right)\sin\theta + y_0\cos\theta\right).$$

Also notice that $\|i\| = 2 \neq \|1\| = \frac{1}{3}$ for the standard basis $\{1, i\}$ of $\mathbb{C}$.

The following lemma can be easily justified.

Lemma 2.1. *Let $\|\cdot\|$ be any norm function on the complex plane. Then for every $\phi \in \mathbb{R}$, the following function defines a norm on the complex plane:*

$$\|z\|_\phi = \left\|e^{-i\phi}z\right\|. \tag{2.2}$$

Remark 2.1. Notice that for the Euclidean norm, all the norm functions $\|\cdot\|_\phi$ are equal to the Euclidean norm. For any other norm function we have $\|\cdot\|_{k\pi} = \|\cdot\|$, where $k \in \mathbb{Z}$.

Lemma 2.2. *Let $\|\cdot\|$ be any norm on $\mathbb{C}$. Then the similarity transformations of the form*

$$T(z) = \alpha z + \beta; \quad \alpha \neq 0, \alpha \in \mathbb{R}, \tag{2.3}$$

map spheres to spheres corresponding to this norm function.

Proof. Let $\|\cdot\|$ be any norm and $S_r(z_0)$ be a sphere of radius r and with center z_0 corresponding to this norm. If $T(z)$ is a similarity transformation of the form $T(z) = \alpha z + \beta$; $\alpha \in \mathbb{R}$, $\alpha \neq 0$, then the image of $S_r(z_0)$ under T is the sphere of radius $|\alpha| r$ with center $T(z_0)$. Indeed, we have

$$\|T(z) - T(z_0)\| = \|\alpha(z - z_0)\| = |\alpha| \cdot \|z - z_0\| = |\alpha| r.$$

□

Theorem 2.3. *Let $w = T(z) = \alpha z + \beta$; $\alpha \neq 0$, α, $\beta \in \mathbb{C}$. Then for every sphere $S_r(z_0)$ corresponding to any norm function $\|\cdot\|$ on $\mathbb{C}$, $T(S_r(z_0))$ is a sphere corresponding to the same norm function or corresponding to the norm function $\|z\|_\phi = \left\|e^{-i\phi}\cdot z\right\|$, where $\phi = \arg(\alpha)$.*

Proof. Let $T(z) = \alpha z + \beta$; $\alpha \neq 0$, α, $\beta \in \mathbb{C}$. If $S_r(z_0)$ is an Euclidean sphere, then $T(S_r(z_0))$ is again an Euclidean sphere. Suppose that $S_r(z_0)$ is not an Euclidean sphere. Let us write $T(z) = |\alpha| e^{i\phi}z + \beta$; $\alpha \neq 0$, $\phi = \arg(\alpha)$ and let $T_1(z) = e^{i\phi}z$, $T_2(z) = |\alpha| z + e^{-i\phi}\beta$. We have $T(z) = (T_1 \circ T_2)(z)$.

Then by Lemma 2.2, the transformation $T_2(z)$ maps spheres to spheres corresponding to this norm function. Let $w = T_1(z) = e^{i\phi}z$, $\phi \neq k\pi$, $k \in \mathbb{Z}$ and write $w_0 = e^{i\phi}z_0$. Now we consider the norm function $\|\cdot\|_\phi$ given in Lemma 2.1. We get

$$\|w - w_0\|_\phi = \left\|e^{i\phi}(z - z_0)\right\|_\phi = \left\|e^{-i\phi}\left[e^{i\phi}(z - z_0)\right]\right\| = \|z - z_0\| = r.$$

This shows that the image of the sphere $S_r(z_0)$ under the transformation $w = T_1(z) = e^{i\phi}z$, ($\phi \neq k\pi$, $k \in \mathbb{Z}$) is the sphere centered at $w_0 = e^{i\phi}z_0$ of radius r corresponding to the norm function $\|\cdot\|_\phi$ given in (2.1). □

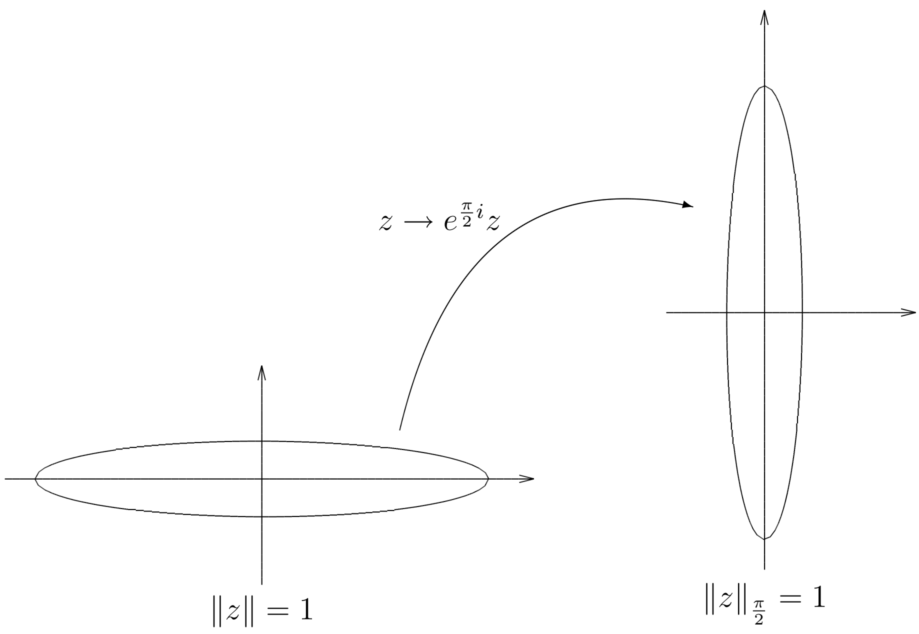

Figure 1:

Remark 2.2. If $\|\cdot\|_\phi = \|\cdot\|$, then the transformation $T_1(z) = e^{i\phi}z$ maps spheres to spheres corresponding to this norm function. However, in general $T_1(z) = e^{i\phi}z$ does not map spheres to spheres corresponding to the same norm function. For example, let $\|\cdot\|$ be any norm with $\|1\| \neq \|i\|$ and $\phi = \frac{\pi}{2}$. Assume that $\|z\|_{\frac{\pi}{2}} = \|z\|$ for all $z \in \mathbb{C}$. For $z = 1$ we have $\|i\| = \|1\|$, which is a contradiction. Therefore the transformation $z \to e^{\frac{\pi}{2}i}z$ maps spheres corresponding to the norm function $\|\cdot\|$ to spheres corresponding to the norm function $\|\cdot\|_{\frac{\pi}{2}}$. Thus sphere preserving property is not an invariant characteristic property of Möbius transformations except in the Euclidean case.

Example 2.1. *Let us consider the norm function* $\|\cdot\|$ *given in* (2.1). $S_1(0)$ *is the ellipse* $\frac{x^2}{9} + 4y^2 = 1$. *The image of* $S_1(0)$ *under the transformation* $z \to e^{\frac{\pi}{2}i}z$ *is the ellipse* $4x^2 + \frac{y^2}{9} = 1$. *This image ellipse is not a sphere corresponding to the norm* $\|\cdot\|$ *but it is the unit sphere corresponding to the norm function* $\|z\|_{\frac{\pi}{2}} = \sqrt{4x^2 + \frac{y^2}{9}}$, (*see Figure* 1).

Example 2.2. *Let us consider the norm function* $\|z\| = \max\{|x|, |y|\}$ *and the sphere* $S_1(0)$ *corresponding to this norm function. The image of* $S_1(0)$ *under the transformation* $T_1(z) = e^{\frac{\pi}{4}i}z + 1$ *is not a sphere corresponding to the same norm function.* $T_1(S_1(0))$ *is the sphere* $S_1(1)$ *corresponding to the norm function* $\|z\|_{\frac{\pi}{4}} = \frac{1}{\sqrt{2}} \max\{|x+y|, |y-x|\}$, (*see Figure* 2).

3. APOLLONIUS QUADRILATERALS AND SPHERES CORRESPONDING TO ANY NORM FUNCTION ON $\mathbb{C}$

In 1998, H. Haruki and Th.M. Rassias [7] introduced the concept of an "*Apollonius quadrilateral*" to give a new characterization of Möbius transformations. We recall this definition from [7].

Definition 3.1. Let $ABCD$ be an arbitrary quadrilateral (not necessarily simple) on the complex plane. If $\overline{AB} \cdot \overline{CD} = \overline{BC} \cdot \overline{DA}$ holds, then $ABCD$ is said to be an Apollonius quadrilateral

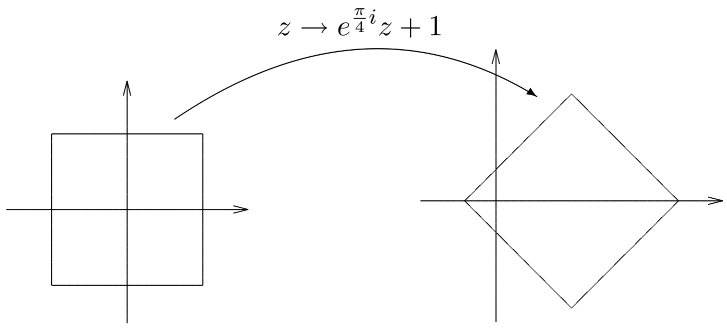

Figure 2:

($\overline{AB} = |z_1 - z_2|$ where z_1, z_2 are the complex numbers corresponding to the points A, B, respectively).

Haruki and Rassias proved that any univalent analytic function in the domain $R \subset \mathbb{C}$ is linear-fractional iff the images of the points A, B, C, D for any Apollonius quadrilateral $ABCD$ contained in R also form an Apollonius quadrilateral, (see [7] for more details as well as [5], [6] and [8] for further results in the subject).

We see that this notion of an "*Apollonius quadrilateral*" is important when we consider the images of spheres, corresponding to any norm function on $\mathbb{C}$, under the Möbius transformations.

Before formulating our main results, we prove the following lemma.

Lemma 3.1. *Let $\|\cdot\|$ be any norm on $\mathbb{C}$. Then we have*

(1) $\|i\| = \|1\|$ *if and only if the sphere $S_1(0)$ cuts the coordinate axes at the points $\pm\frac{1}{\|1\|}$ and $\pm\frac{1}{\|1\|}i$,*
and
(2) $\|i\| \neq \|1\|$ *if and only if the sphere $S_1(0)$ cuts the coordinate axes at the points $\pm\frac{1}{\|1\|}$ and $\pm\frac{1}{\|i\|}i$.*

Proof. Let $\|i\| = \|1\|$. If we take $t = \frac{1}{\|1\|}$, clearly we have $\|t\| = 1$ and $\|ti\| = t \cdot \|i\| = t\,\|1\| = \|t\| = 1$. Similarly, we have $\|-ti\| = \|-t\| = 1$. Therefore the sphere $S_1(0)$ cuts the coordinate axes at the points $\pm\frac{1}{\|1\|}$ and $\pm\frac{1}{\|1\|}i$. Conversely, let $S_1(0)$ cut the coordinate axes at the points $\pm\frac{1}{\|1\|}$ and $\pm\frac{1}{\|1\|}i$. Then from the equation $\left\|\frac{1}{\|1\|}i\right\| = 1$, we have $\left\|\frac{1}{\|1\|}i\right\| = \frac{1}{\|1\|}\|i\| = 1$ and so $\|i\| = \|1\|$.

If $\|i\| \neq \|1\|$, the proof follows similarly. □

Corollary 3.2. *Let $\|\cdot\|$ be any norm on $\mathbb{C}$. Then we have*

(1) $\|i\| = \|1\|$ *if and only if any sphere $S_r(z_0)$ of radius r with center z_0 passes through the points $z_0 \pm \frac{1}{\|1\|}r$, $z_0 \pm i\frac{1}{\|1\|}r$,*
and
(2) $\|i\| \neq \|1\|$ *if and only if any sphere $S_r(z_0)$ passes through the points $z_0 \pm \frac{1}{\|1\|}r$, $z_0 \pm i\frac{1}{\|i\|}r$.*

Proof. It is well-known that any sphere $S_r(z_0)$ can be represented as

$$S_r(z_0) = z_0 + rS_1(0).$$

Then the proof follows from Lemma 3.1. □

Now we can give the following theorem.

Theorem 3.3. *Let $\|\cdot\|$ be any norm on $\mathbb{C}$ and $\overline{S_r(z_0)} = \{z \in \mathbb{C} : \|z - z_0\| \leq r\}$ be any closed ball corresponding to this norm. Then an Apollonius quadrilateral can be drawn inside $\overline{S_r(z_0)}$ whose vertices lie on $S_r(z_0)$.*

Proof. If the norm function satisfies the property $\|i\| = \|1\|$, then from Corollary 3.2, it follows that any sphere $S_r(z_0)$ will pass through the points $A = z_0 + \frac{1}{\|1\|}r$, $B = z_0 + i\frac{1}{\|1\|}r$, $C = z_0 - \frac{1}{\|1\|}r$ and $D = z_0 - i\frac{1}{\|1\|}r$. Now we have

$$\overline{AB}.\overline{CD} = \left(\frac{r}{\|1\|}\right)^2 \cdot |1 - i|^2 = 2\left(\frac{r}{\|1\|}\right)^2$$

and

$$\overline{BC}.\overline{DA} = \left(\frac{r}{\|1\|}\right)^2 \cdot |1 + i|^2 = 2\left(\frac{r}{\|1\|}\right)^2,$$

that is, we get

$$\overline{AB} \cdot \overline{CD} = \overline{BC} \cdot \overline{DA}.$$

This shows that the points A, B, C and D are the vertices of an Apollonius quadrilateral. Since $\overline{S_r(z_0)}$ is a convex set, this Apollonius quadrilateral lies inside $\overline{S_r(z_0)}$.

Let $\|i\| \neq \|1\|$. Then from Corollary 3.2, we know that any sphere $S_r(z_0)$ passes through the points $A = z_0 + \frac{1}{\|1\|}r$, $B = z_0 + i\frac{1}{\|i\|}r$, $C = z_0 - \frac{1}{\|1\|}r$ and $D = z_0 - i\frac{1}{\|i\|}r$. We have

$$\overline{AB} \cdot \overline{CD} = r^2 \cdot \left|\frac{1}{\|1\|} - \frac{1}{\|i\|}i\right|^2$$

and

$$\overline{BC} \cdot \overline{DA} = r^2 \cdot \left|\frac{1}{\|1\|} + \frac{1}{\|i\|}i\right|^2.$$

The last two equations show that the points A, B, C and D are the vertices of an Apollonius quadrilateral. Similarly, this Apollonius quadrilateral lies inside $\overline{S_r(z_0)}$. □

In the case where the norm function satisfies the property $\|z\| = \|\overline{z}\|$ for all $z \in \mathbb{C}$, the following property holds:

Theorem 3.4. *Let $\|\cdot\|$ be any norm on $\mathbb{C}$ with the property $\|z\| = \|\overline{z}\|$ and let $w = T(z)$ be a Möbius transformation. Then the points of any sphere $S_r(z_0)$ are inverse with respect to the two circles of Apollonius U and V; the points of the image $T(S_r(z_0))$ are also inverse with respect to the image circles $U' = T(U)$ and $V' = T(V)$.*

Proof. Let $S_r(z_0)$ be an arbitrary sphere. If $\|i\| = \|1\|$, then by Corollary 3.2, $S_r(z_0)$ passes through the points

$$A = z_0 + \frac{1}{\|1\|}r, \quad B = z_0 + i\frac{1}{\|1\|}r, \quad C = z_0 - \frac{1}{\|1\|}r, \quad D = z_0 - i\frac{1}{\|1\|}r.$$

From the proof of Theorem 3.3, we know that the points A, B, C and D form the vertices of an Apollonius quadrilateral. By definition, we have

$$\overline{AB} \cdot \overline{CD} = \overline{BC} \cdot \overline{DA} \tag{3.1}$$

and hence

$$\frac{\overline{BA}}{\overline{BC}} = \frac{\overline{DA}}{\overline{DC}} = \mu.$$

We know that the locus of a point, which moves so that the ratio of its distances from two fixed points A, C is constant, is a circle with respect to which A, C are inverse. We denote this circle of Apollonius by U. Clearly, the points B and D lie on U. Now we find the equation of U. From the equation

$$\frac{|z-A|}{|z-C|}=\mu$$

we get

$$(1-\mu^2)z\overline{z}+\left(-\overline{A}+\mu^2\overline{C}\right)z+\left(-A+\mu^2C\right)\overline{z}+|A|^2-\mu^2|C|^2=0.$$

In our case $\mu=1$ and therefore the equation of U is

$$\left(-\overline{A}+\overline{C}\right)z+(-A+C)\,\overline{z}+|A|^2-|C|^2=0,$$

which is a straight line that passes through the points B and D. The inversion map with respect to U is

$$I_U(z)=-\frac{(C-A)\,\overline{z}+|A|^2-|C|^2}{\overline{C}-\overline{A}}=-\overline{z}+2x_0,$$

where $x_0=\mathrm{Re}(z_0)$. Let $w=I_U(z)$. We have

$$I_U(z_0)=-\overline{z_0}+2x_0=z_0$$

and

$$\begin{aligned}\|w-z_0\|&=\|-\overline{z}+2x_0-z_0\|=\|\overline{z}-(x_0-iy_0)\|\\&=\|\overline{z}-\overline{z_0}\|=\|\overline{z-z_0}\|=\|z-z_0\|=r,\end{aligned}$$

since the norm function has the property $\|z\|=\|\overline{z}\|$ for all $z\in\mathbb{C}$. This shows that the points of $S_r(z_0)$ are inverse with respect to the line U while B and D are fixed by $I_U(z)$. Since the points of $S_r(z_0)$ are inverse with respect to the line U, applying the reflection principle it follows that the points of $T(S_r(z_0))$ are also inverse with respect to the circle $U'=T(U)$. Because of the fact that T is a Möbius transformation, the image U' is a circle or a straight line. Similarly, from equation (3.1) we can write

$$\frac{\overline{AB}}{\overline{AD}}=\frac{\overline{CB}}{\overline{CD}}=\lambda.$$

Then a similar argument shows that the points of $S_r(z_0)$ are inverse with respect to the line V passing through the points A and C while these points are fixed by $I_V(z)$, and the points of $T(S_r(z_0))$ are also inverse with respect to the circle $V'=T(V)$.

If $\|1\|\neq\|i\|$, the proof follows similarly as in the case $\|1\|=\|i\|$. □

In the proof of Theorem 3.4, let $A'=T(A)$, $B'=T(B)$, $C'=T(C)$ and $D'=T(D)$. Since $w=T(z)$ is a univalent function it follows that A', B', C' and D' are different points. However the points A, B, C and D form the vertices of an Apollonius quadrilateral, thus the image points A', B', C' and D' also form the vertices of an Apollonius quadrilateral (see [7]).

Example 3.1. *Let us consider the norm function $\|z\|=|x|+|y|$ and the image of $S_1(0)$ under the transformation $T(z)=\frac{-1}{z+2}$. The images of the points $A=1, B=i, C=-1$ and $D=-i$ under the transformation T form an Apollonius quadrilateral,* (*see Figure* 3). *The points of $T(S_1(0))$ are inverse with respect to the two circles of Apollonius $\left|z+\frac{1}{4}\right|=\frac{1}{4}$ and the u-axis.*

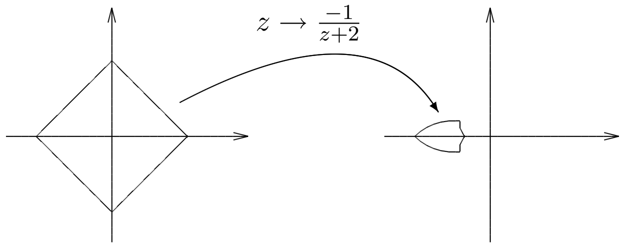

Figure 3:

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