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CONTACT WITH ADHESION BETWEEN A DEFORMABLE BODY AND A FOUNDATION

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ABSTRACT. The aim of this work is study a dynamic contact problem between a deformable body and a foundation where the deformations are supposed to be small. The contact is with adhesion and normal compliance. The behavior of this body is modeled by a nonlinear elastic-visco-plastic law. The evolution of bonding field is described by a nonlinear differential equation. We derive a variational formulation of the contact problem and we prove the existence and uniqueness of its solution. The proof is based on the construction of three intermediate problems and then we construct a contraction mapping whose unique fixed point will be the weak solution of the mechanical problem.

Key words and phrases: Elastic-visco-plastic material, Contact, Adhesion, Weak solution, Variational inequality, Differential equation, Fixed point.

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1. INTRODUCTION

The phenomena of contact with or without friction are frequently met. The contact of the tires of a car with the ground, the shoe with disc of break are current examples. Because of the industrial importance of the physical processes that take place during contact, a considerable effort has been made in mathematical analysis, numerical approximation and numerical simulation of these problems.
Process of adhesion is important in many industrial setting where parts nonmetallic, are glued together. For this reason, adhesive contact between bodies when a glue is added to prevent the surfaces from relative motion, has recently received increased attention in the mathematical literature. In this work we introduce an internal variable of surface, known as bonding field and denoted in this paper by β, which describes the fractional density of active bonds on the contact surface. The problems of contact with adhesion were studied by several authors. Significant results on these problems can be found in [2], [3], [4], [6] and references therein.
Here, the novelty consists in the introduction of the bonding field into the contact between a body elastic-visco-plastic and a deformable foundation where the process is dynamic. The main contribution of this study lies in the proof of existence and unicity of the weak solution of the mechanical problem.
This work is organized as follows. In Section 2 we present some notations and preliminaries. In Section 3 we state the mechanical models of elastic-visco-plastic contact with adhesion, list the assumptions on the data of the mechanical problem and deduce its variational formulation. In Section 4 we state and prove the existence of a unique weak solution to mechanical problem; the proof is based on arguments of evolutionary equations and Banach fixed point.

2. NOTATIONS AND PRELIMINARIES

In this section, we specify the notations standards used and we remind some definitions and results necessary for the study of this mechanical problem.
We denote by S^N the space of second order symmetric tensors on $\mathbb{R}^N$ ($N = 1, 2, 3$) while "$\cdot$" and $\|\cdot\|$ represent the inner product and the Euclidean norm on $\mathbb{R}^N$ and S^N, respectively. Thus, for every $u, v \in \mathbb{R}^N$ and $\sigma, \tau \in S^N$ we have :

$$u \cdot v = u_i v_i, \quad \|u\| = (u \cdot u)^{\frac{1}{2}}, \quad \sigma \cdot \tau = \sigma_{ij}\tau_{ij}, \quad \|\sigma\| = (\sigma \cdot \sigma)^{\frac{1}{2}}$$

Here and below, the indices i, j run between 1, N and the summation convention over repeated indices is adopted.
Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a Lipschitz boundary Γ and let ν denote the unit outer normal on Γ. Moreover, we use also the spaces :

$$H = \left\{u = (u_i)\,/u_i \in L^2(\Omega)\right\}, \quad Q = \left\{\sigma = (\sigma_{ij})\,/\sigma_{ij} = \sigma_{ji} \in L^2(\Omega) \quad \right\}$$

$$H_1 = \{u \in H/\varepsilon(u) \in Q\}, \quad Q_1 = \{\sigma \in Q/Div\sigma \in H\}$$

Where $\varepsilon : H_1 \longrightarrow Q$, $Div : Q \longrightarrow H$ are the deformation and the divergence operators, respectively, defined by :

$$\varepsilon(u) = (\varepsilon_{ij}(u)), \quad \varepsilon_{ij}(u) = \frac{1}{2}(\partial_j u_i + \partial_i u_j), \quad Div\sigma = (\partial_j \sigma_{ij})$$

The spaces H, Q, H_1 and Q_1 are real Hilbert spaces endowed with the canonical inner products given by :

$$\langle u,v\rangle_H = \int_\Omega u_i v_i dx \quad \forall u,v \in H$$

$$\langle \sigma,\tau\rangle_Q = \int_\Omega \sigma_{ij}\tau_{ij} dx \quad \forall \sigma,\tau \in Q$$

$$\langle u,v\rangle_{H_1} = \langle u,v\rangle_H + \langle \varepsilon(u), \varepsilon(v)\rangle_Q \quad \forall u,v \in H$$

$$\langle \sigma,\tau\rangle_{Q_1} = \langle \sigma,\tau\rangle_Q + \langle Div\sigma, Div\tau\rangle_H \quad \forall \sigma,\tau \in Q$$

The associated norms are denoted by $\|.\|_H$, $\|.\|_Q$, $\|.\|_{H_1}$et $\|.\|_{Q_1}$ respectively.
Since the boundary Γ is Lipschitz continuous, the unit outward normal vector ν on the boundary is defined almost everywhere for every vector field $u \in H_1$, we also use the notation u for the trace of u on Γ and we denote by u_ν and u_τ the normal and tangential components of u on the boundary Γ, given by :

$$u_\nu = u \cdot \nu, \quad u_\tau = u - u_\nu \nu$$

For a regularly (say C^1) stress field σ, the application of its trace on the boundary to ν is the Cauchy stress vector $\sigma\nu$. We define, similarly, the normal and tangential components of the stress on the boundary Γ, by :

$$\sigma_\nu = (\sigma\nu)\cdot\nu, \quad \sigma_\tau = \sigma\nu - \sigma_\nu\nu$$

And we recall that the following Green's formula holds :

$$\langle \sigma, \varepsilon(u)\rangle_Q + \langle Div\sigma, u\rangle_H = \int_\Gamma \sigma\nu u ds \quad \forall u \in H_1$$

Let Γ_1 be a measurable part of Γ such that $meas\,(\Gamma_1) > 0$ and let V be the closed subspace of H_1 defined by :

$$V = \{v \in H_1 / v = 0 \quad \text{on } \Gamma_1\}$$

Since $meas\,(\Gamma_1) > 0$, the following Korn's inequality holds :

$$\|\varepsilon(u)\|_Q \geq c\,\|u\|_{H_1} \quad \forall u \in V$$

Where $c > 0$ is a constant depending only on Ω and Γ_1.
Over space V we consider the inner product defined by :

$$\langle u,v\rangle_V = \langle \varepsilon(u), \varepsilon(v)\rangle_Q \quad \forall u,v \in V$$

It follows from Korn's inequality that $\|.\|_V$ and $\|.\|_{H_1}$are equivalent norms on V. Therefore $(V, \|.\|_V)$ is a real Hilbert space. Moreover, by the Sobolev trace theorem, there exists a positive constant $c > 0$, depending only on Ω, Γ_1 et Γ_3 such that :

$$\|v\|_{L^2(\Gamma_3)^N} \leq c\,\|v\|_V \quad \forall v \in V$$

Finally, we shall use the notation $\mathcal{Q}$ for the set defined by :

$$\mathcal{Q} = \left\{\beta : [0,T] \longrightarrow L^2(\Gamma_3) : 0 \leq \beta(t) \leq 1 \quad \forall t \in [0,T]\,, \text{a.e. on } \Gamma_3\right\}$$

For the convenience of the reader, we recall the following abstract result which may be found in [1] (p.140).

Theorem 2.1. *Let $V \subset H \subset V'$ be a Guelfand triple. Assume that $A : V \longrightarrow V'$ is a hemicontinuous and monotone operator which satisfies :*

$$\langle Av, v\rangle_{V'\times V} \geq \omega \|v\|_V^2 + \alpha \quad \forall v \in V$$

$$\|Av\|_{V'} \leq c\left(\|v\|_V + 1\right) \quad \forall v \in V$$

where ω, c are two strictly positive constants and $\alpha \in \mathbb{R}$. Then, given $u_0 \in H$ and $f \in L^2\left([0,T]; V'\right)$, there exists a unique u which satisfies :

$$u \in L^2\left([0,T]; V\right) \cap C\left([0,T]; H\right), \quad \dot{u} \in L^2\left([0,T]; V'\right)$$

$$\dot{u}(t) + Au(t) = f(t) \quad \textit{a.e. } t \in]0,T[$$

$$u(0) = u_0.$$

We end this preliminary with the following version of the classical theorem of Cauchy-Lipschitz which can be found in [5] (p. 60).

Theorem 2.2. *Assume that $(X, \|.\|_X)$ is a real Banach space. Let $F(t,.) : X \longrightarrow X$ be an operator defined almost everywhere on $]0,T[$, satisfying the following conditions :*
1) $\|F(t,u) - F(t,v)\|_X \leq L_F \|u - v\|_X \quad \forall u, v \in X$ a.e. $t \in]0,T[$, for some L_F
2) $t \mapsto F(t,u) \in L^p([0,T]; X) \quad \forall u \in X$, and some $p \geq 1$.
Then, for every $u_0 \in X$, there exists a unique function $u \in W^{1,p}([0,T]; X)$ such that

$$\begin{cases} \dot{u}(t) = F(t, u(t)) \quad \textit{a.e. } t \in]0,T[\\ u(0) = u_0. \end{cases}$$

These two theorems will be used in Section 4, to prove the theorem of existence and uniqueness of weak solution of mechanical problem.

3. MECHANICAL PROBLEM AND VARIATIONAL FORMULATION

We consider an elastic-visco-plastic body which occupies a bounded domain $\Omega \subset \mathbb{R}^N$ ($N = 2, 3$) and assume that its boundary Γ is regular and partitioned into three disjoint measurable parts Γ_1, Γ_2 and Γ_3 such that $meas(\Gamma_1) > 0$. Let $[0,T]$, denote the time interval of interest. The body is clamped on $\Gamma_1 \times [0,T]$, therefore, the displacement field vanished there. A volume force of density f_0 acts in $\Omega \times [0,T]$ and surface traction of density f_2 acts on $\Gamma_2 \times [0,T]$. The body is in adhesive contact on $\Gamma_3 \times [0,T]$ with a foundation, the contact is frictionless and modeled with normal compliance. Moreover, the process is dynamic. Under these conditions the formulation of the mechanical problem is the following.
ProblemP. Find a displacement field $u : \Omega \times [0,T] \longrightarrow \mathbb{R}^N$, a stress field $\sigma : \Omega \times [0,T] \longrightarrow S^N$ and a bonding field $\beta : \Gamma_3 \times [0,T] \longrightarrow [0,1]$ such that

$$\sigma(t) = \mathcal{A}\varepsilon(\dot{u}(t)) + \mathcal{E}\varepsilon(u(t)) + \int_0^t \mathcal{G}\left(\sigma(s) - \mathcal{A}\varepsilon(\dot{u}(s)), \varepsilon(u(s))\right) ds \quad \text{in } \Omega \times]0,T[\tag{3.1}$$

$$Div\sigma(t) + f_0(t) = \rho\ddot{u}(t) \quad \text{in } \Omega \times]0,T[\tag{3.2}$$

$$u = 0 \quad \text{on } \Gamma_1 \times]0,T[\tag{3.3}$$

$$\sigma\nu = f_2 \quad \text{on } \Gamma_2 \times]0, T[\tag{3.4}$$

$$-\sigma_\nu = p_\nu(u_\nu) - \gamma_\nu \beta^2 \widetilde{R}(u_\nu) \quad \text{on } \Gamma_3 \times]0, T[\tag{3.5}$$

$$-\sigma_\tau = p_\tau(\beta) \overset{*}{R}(u_\tau) \quad \text{on } \Gamma_3 \times]0, T[\tag{3.6}$$

$$\dot{\beta} = -\left(\gamma_\nu \beta \widetilde{R}(u_\nu)^2 - \epsilon_a\right)_+ \quad \text{on } \Gamma_3 \times]0, T[\tag{3.7}$$

$$u(0) = u_0, \quad \dot{u}(0) = v_0 \quad \text{in } \Omega \tag{3.8}$$

$$\beta(0) = \beta_0 \quad \text{on } \Gamma_3. \tag{3.9}$$

Here (3.1) is the elastic-visco-plastic constitutive law. (3.2) represents the equation of motion in which ρ denotes the density of mass, (3.3) and (3.4) are the displacement-traction boundary conditions.
We now dscribe briefly the conditions (3.5)-(3.7) on the contact surface Γ_3. (3.5) represents the normal compliance condition with adhesion where p_ν is given function; also $\widetilde{R}(u_\nu) = (-R(u_\nu))_+$, $\widetilde{R}(u_\nu)^2 = \left[\widetilde{R}(u_\nu)\right]^2$ and R is the truncation operator. (3.6) is the tangential boundary condition in which $p_\tau(\beta)$ is a given function and $\overset{*}{R}$ is a truncation operator :

$$\overset{*}{R}(s) = \overset{*}{R}_L(s) = \begin{cases} s & \text{if } \|s\| \leq L \\ L\frac{s}{\|s\|} & \text{if } \|s\| > L \end{cases}$$

$L > 0$ being the characteristic length of the bond. Equation (3.7) describes the evolution of the bonding field with given material parameters γ_ν and ϵ_a. Also, the data u_0, v_0 and β_0 in (3.8) and (3.9) are the given initial displacement, velocity and bonding fields respectively.

Assumptions.

For the study variational of the mechanical problem, we assume that the operators $\mathcal{A}$, $\mathcal{E}$ and $\mathcal{G}$ satisfy the following conditions :

$$\begin{cases} \text{(a) } \mathcal{A} : \Omega \times S^N \longrightarrow S^N \text{ such that} \\ \text{(b) } \exists m_{\mathcal{A}} > 0 \text{ such that } (\mathcal{A}(x,\varepsilon_1) - \mathcal{A}(x,\varepsilon_2)) \cdot (\varepsilon_1 - \varepsilon_2) \geq m_{\mathcal{A}} \|\varepsilon_1 - \varepsilon_2\|^2 \\ \text{a.e. } x \in \Omega \quad \forall \varepsilon_1, \varepsilon_2 \in S^N \\ \text{(c) } \exists L_{\mathcal{A}} > 0 \text{ such that } \|\mathcal{A}(x,\varepsilon_1) - \mathcal{A}(x,\varepsilon_2)\| \leq L_{\mathcal{A}} \|\varepsilon_1 - \varepsilon_2\| \\ \text{a.e. } x \in \Omega \quad \forall \varepsilon_1, \varepsilon_2 \in S^N \\ \text{(d) The mapping } x \mapsto \mathcal{A}(x,\varepsilon) \text{ is Lebesgue measurable} \\ \text{a.e. } x \in \Omega, \quad \forall \varepsilon \in S^N \\ \text{(e) The mapping } x \mapsto \mathcal{A}(x,0) \in Q \end{cases} \tag{3.10}$$

$$(3.11)\qquad \begin{cases} \text{(a) } \mathcal{E} : \Omega \times S^N \longrightarrow S^N \text{ such that} \\ \text{(b) } \exists L_{\mathcal{E}} > 0 \text{ such that } \|\mathcal{E}(x, \varepsilon_1) - \mathcal{E}(x, \varepsilon_2)\| \leq L_{\mathcal{E}} \|\varepsilon_1 - \varepsilon_2\| \\ \forall \varepsilon_1, \varepsilon_2 \in S^N, \text{ a.e. } x \in \Omega \\ \text{(c) The mapping } x \mapsto \mathcal{E}(x, \varepsilon) \text{ is Lebesgue measurable} \\ \text{a.e. } x \in \Omega \quad \forall \varepsilon \in S^N \\ \text{(d) The mapping } x \mapsto \mathcal{E}(x, 0) \in Q \end{cases}$$

$$(3.12)\qquad \begin{cases} \text{(a) } \mathcal{G} : \Omega \times S^N \times S^N \longrightarrow S^N \text{ such that} \\ \text{(b) } \exists L_{\mathcal{G}} > 0 \text{ such that } \|\mathcal{G}(x, \sigma_1, \varepsilon_1) - \mathcal{G}(x, \sigma_2, \varepsilon_2)\| \\ \leq L_{\mathcal{G}} (\|\sigma_1 - \sigma_2\| + \|\varepsilon_1 - \varepsilon_2\|) \\ \forall \sigma_1, \sigma_2, \varepsilon_1, \varepsilon_2 \in S^N, \text{ a.e. } x \in \Omega. \end{cases}$$

The normal compliance function p_ν and the tangential function p_τ satisfy the assumptions :

$$(3.13)\qquad \begin{cases} \text{(a) } p_\nu : \Gamma_3 \times \mathbb{R} \longrightarrow \mathbb{R}_+ \text{ such that} \\ \text{(b) } \exists L_\nu > 0 \text{ such that } |p_\nu(x, r_1) - p_\nu(x, r_2)| \leq L_\nu |r_1 - r_2| \\ \forall r_1, r_2 \in \mathbb{R}, \text{ a.e. } x \in \Gamma_3 \\ \text{(c) } (p_\nu(x, r_1) - p_\nu(x, r_2))(r_1 - r_2) \geq 0 \\ \forall r_1, r_2 \in \mathbb{R}, \text{ a.e. } x \in \Gamma_3 \\ \text{(d) For any } r \in \mathbb{R},\ x \mapsto p_\nu(x, r) \text{ is measurable on } \Gamma_3 \\ \text{(e) } p_\nu(x, r) = 0 \text{ for all } r \leq 0 \end{cases}$$

$$(3.14)\qquad \begin{cases} \text{(a) } p_\tau : \Gamma_3 \times \mathbb{R} \longrightarrow \mathbb{R}_+ \text{ such that} \\ \text{(b) } \exists L_\tau > 0 \text{ such that } |p_\tau(x, \beta_1) - p_\tau(x, \beta_2)| \leq L_\tau |\beta_1 - \beta_2| \\ \forall \beta_1, \beta_2 \in \mathbb{R}, \text{ a.e. } x \in \Gamma_3 \\ \text{(c) } \exists M_\tau > 0 \text{ such that } |p_\tau(x, \beta)| \leq M_\tau \\ \forall \beta \in \mathbb{R}, \text{ a.e. } x \in \Gamma_3 \\ \text{(d) For any } \beta \in \mathbb{R},\ x \mapsto p_\tau(x, \beta) \text{ is measurable on } \Gamma_3 \\ \text{(e) The mapping } x \mapsto p_\tau(x, 0) \in L^2(\Gamma_3) \end{cases}$$

We suppose that the adhesion coefficients satisfy :

$$(3.15)\qquad \gamma_\nu \in L^\infty(\Gamma_3), \quad \epsilon_a \in L^2(\Gamma_3), \quad \gamma_\nu, \epsilon_a \geq 0 \quad \text{a.e. on } \Gamma_3$$

We suppose that the mass density satisfies :

$$(3.16)\qquad \rho \in L^\infty(\Omega), \text{ there exists } \rho^* > 0 \text{ such that } \rho(x) \geq \rho^* \text{ a.e. } x \in \Omega$$

And the body forces and surface traction have the regularity :

$$(3.17)\qquad f_0 \in L^2([0,T]; H), \quad f_2 \in L^2\left([0,T]; L^2(\Gamma_2)^N\right)$$

The initial data satisfy :

$$(3.18)\qquad u_0 \in V, \quad v_0 \in H, \quad \beta_0 \in L^2(\Gamma_3), \quad 0 \leq \beta_0 \leq 1 \quad \text{a.e. on } \Gamma_3$$

We will use a modified inner product on H, given by :

$$(3.19)\qquad ((u, v))_H = \langle \rho u, v \rangle_H \quad \forall u, v \in H$$

that is, it is weighted with ρ, and we let $|||.|||_H$ by the associated norm :

$$|||v|||_H = \langle \rho v, v\rangle_H^{\frac{1}{2}} \quad \forall v \in H \tag{3.20}$$

We denote by $V^{'}$ the dual space of V. Identifying H with its own dual, we can write the Guelfand triple :

$$V \subset H \subset V^{'}$$

We use the notation $\langle .,.\rangle_{V'\times V}$ to represent the duality pairing, between $V^{'}$ and V. We have :

$$\langle u, v\rangle_{V'\times V} = ((u,v))_H \quad \forall u \in H, v \in V \tag{3.21}$$

From the assumption made on the body forces and surface traction and from Riesz-Frechet's theorem, it results the existence of unique element $f(t) \in V'$ such that :

$$\langle f(t), v\rangle_{V'\times V} = \langle f_0(t), v\rangle_H + \langle f_2(t), v\rangle_{L^2(\Gamma_2)^N} \quad \forall v \in V, t \in]0,T[\tag{3.22}$$

Moreover, we have :

$$f \in L^2\left([0,T]; V^{'}\right) \tag{3.23}$$

Finally, we define the function of contact with adhesion $j : L^\infty(\Gamma_3) \times V \times V \longrightarrow \mathbb{R}$ by :

$$j(\beta, u, v) = \int_{\Gamma_3} p_\nu(u_\nu) v_\nu ds - \int_{\Gamma_3} \gamma_\nu \beta^2 \widetilde{R}(u_\nu) v_\nu ds + \int_{\Gamma_3} p_\tau(\beta) \overset{*}{R}(u_\tau) v_\tau ds. \tag{3.24}$$

3.1. **Variational formulation.** By applying Green's formula, and using the equation of motion and the boundary conditions,we easily deduce the following variational formulation of Prob-lemP:

ProblemPV : Find a displacement $u : [0,T] \longrightarrow V$, a stress field $\sigma : [0,T] \longrightarrow Q$ and a bonding field $\beta : [0,T] \longrightarrow L^\infty(\Gamma_3)$ such that :

$$\sigma(t) = \mathcal{A}\varepsilon(\dot{u}(t)) + \mathcal{E}\varepsilon(u(t)) + \int_0^t \mathcal{G}(\sigma(s) - \mathcal{A}\varepsilon(\dot{u}(s)), \varepsilon(u(s))) ds \quad \forall t \in]0,T[\tag{3.25}$$

$$\begin{aligned} & \langle \ddot{u}(t), w\rangle_{V'\times V} + \langle \sigma(t), \varepsilon(w)\rangle_Q + j(\beta(t), u(t), w) \\ = \; & \langle f(t), w\rangle_{V'\times V} \quad \forall w \in V, \forall t \in]0,T[\end{aligned} \tag{3.26}$$

$$\dot{\beta}(t) = -\left(\gamma_\nu \beta(t) \widetilde{R}(u_\nu(t))^2 - \epsilon_a\right)_+ \quad \text{a.e.} \quad t \in]0,T[\tag{3.27}$$

$$u(0) = u_0, \quad \dot{u}(0) = v_0, \quad \beta(0) = \beta_0 \tag{3.28}$$

4. EXISTENCE AND UNIQUENESS OF SOLUTION

Theorem 4.1. *Assume that (3.10)-(3.18) hold. Then there exists a unique solution* (u, σ, β)*, to ProblemPV and it satisfies :*

$$u \in H^1([0,T];V) \cap C^1([0,T];H), \quad \ddot{u} \in L^2\left([0,T];V^{'}\right) \tag{4.1}$$

$$\sigma \in L^2([0,T];Q), \quad Div\sigma \in L^2\left([0,T];V^{'}\right) \tag{4.2}$$

$$\beta \in W^{1,\infty}\left([0,T];L^2(\Gamma_3)\right) \cap \mathcal{Q} \tag{4.3}$$

We conclude that under the stated assumptions, mechanical problem has a unique weak solution.
The proof of this theorem will be carried out in several steps. In the first step we consider the following variational problem in which $\eta \in L^2\left([0,T];V^{'}\right)$, is given.
Problem1. Find a displacement field $u_\eta : [0,T] \longrightarrow V$ such that :

$$\begin{aligned} & \langle \ddot{u}_\eta(t), w\rangle_{V'\times V} + \left\langle \mathcal{A}\varepsilon\left(\dot{u_\eta}(t)\right), \varepsilon(w)\right\rangle_Q + \langle \eta(t), w\rangle_V \\ = \; & \langle f(t), w\rangle_V \quad \forall w \in V, t \in]0,T[\end{aligned} \tag{4.4}$$

$$u_\eta(0) = u_0, \quad \dot{u}_\eta(0) = v_0 \tag{4.5}$$

Lemma 4.2. *There exits a unique solution* u_η *to Problem1 and satisfies (4.1).*

Proof. Let $A : V \longrightarrow V^{'}$, an operator defined by :

$$\langle Av, w\rangle_V = \langle \mathcal{A}\varepsilon(v), \varepsilon(w)\rangle_Q \quad \forall v, w \in V \tag{4.6}$$

The operator A thus defined is hemicontinuous, monotone and satisfies the conditions of the Theorem 2.1. Then there exists a unique function v_η which satisfies :

$$v_\eta \in L^2([0,T];V) \cap C([[0,T];H]), \quad \dot{v_\eta} \in L^2\left([0,T];V^{'}\right)$$

$$\dot{v}_\eta(t) + Av_\eta(t) + \eta(t) = f(t) \quad \forall t \in]0,T[$$

$$v_\eta(0) = v_0$$

□

Now, we define the function $u_\eta : [0,T] \longrightarrow V$ by :

$$u_\eta(t) = \int_0^t v_\eta(s)\, ds + u_0 \quad \forall t \in [0,T] \tag{4.7}$$

From the definitions of the function u_η and the operator A, we deduce that the Problem1 has a unique solution u_η which satisfies (4.1). Moreover, if u_i are solutions of Problem1 for $\eta_i \in L^2\left([0,T];V^{'}\right)$, $i = 1, 2$, Then there exists a constant $c > 0$ such that :

$$\int_0^t \|\dot{u}_1(s) - \dot{u}_2(s)\|_V^2\, ds \le c \int_0^t \|\eta_1(s) - \eta_2(s)\|_{V'}^2\, ds \quad \forall t \in]0,T[\tag{4.8}$$

In the second step, we use the solution u_η of the Problem1 to formulate the second following auxiliary problem :
Problem2. Find a bonding field $\beta_\eta : [0,T] \longrightarrow L^2(\Gamma_3)$ such that :

(4.9)
$$\dot{\beta_\eta}(t) = -\left(\gamma_\nu \beta_\eta(t)\widetilde{R}(u_{\eta\nu}(t))^2 - \epsilon_a\right)_+ \quad \text{a.e.} \quad t \in [0,T]$$

(4.10)
$$\beta_\eta(0) = \beta_0$$

Where $u_{\eta\nu}$ represents the normal component of the function $u_\eta \in H^1([0,T];V)$.

Lemma 4.3. *There exits a unique solution β_η to Problem2 and satisfies (4.3).*

Proof. Let $F_\eta : [0,T] \times L^2(\Gamma_3) \longrightarrow L^2(\Gamma_3)$ be a mapping defined by :
$$F_\eta(t,\beta_\eta) = -\left(\gamma_\nu \beta_\eta(t)\widetilde{R}(u_{\eta\nu}(t))^2 - \epsilon_a\right)_+$$
It follows that F_η is a Lipschitz continuous with respect to second argument β_η, uniformly in time t. Moreover, for any $\beta_\eta \in L^2(\Gamma_3)$, the mapping $t \mapsto F_\eta(t,\beta_\eta)$ belongs to $L^\infty([0,T];L^2(\Gamma_3))$. Then from Theorem 2.2, we deduce the existence of a unique function $\beta_\eta \in W^{1,\infty}([0,T];L^2(\Gamma_3))$, which satisfies (4.9)-(4.10). The regularity $\beta_\eta \in \mathcal{Q}$, follows from (4.9)-(4.10) and assumption $0 \le \beta_0 \le 1$ a.e. on Γ_3. Moreover if u_i are solutions of Problem1 and β_i are solutions of Problem2 for $\eta_i \in L^2\left([0,T];V'\right)$, $i = 1,2$, then it exists a constant $c > 0$ such that :

(4.11)
$$\|\beta_1(t) - \beta_2(t)\|_{L^2(\Gamma_3)} \le c\int_0^t \|u_1(s) - u_2(s)\|_V\, ds \quad \forall t \in [0,T]$$

□

We use again the solution u_η of the Problem1 to formulate the third following auxiliary problem :
Problem3. Find a stress field $\sigma_\eta : [0,T] \longrightarrow Q$ such that :

(4.12)
$$\sigma_\eta(t) = \mathcal{E}\varepsilon(u_\eta(t)) + \int_0^t \mathcal{G}(\sigma_\eta(s), \varepsilon(u_\eta(s)))\, ds \quad \forall t \in [0,T]$$

Lemma 4.4. *The Problem3 has a unique solution $\sigma_\eta \in H^1([0,T];Q)$.*

Proof. We use the Banach fixed point theorem to prove Lemma 4.4. Moreover, if u_i and σ_i represents the solutions of Problem1 and Problem3, respectively, for $\eta_i \in L^2\left([0,T];V'\right)$, $i = 1,2$, there exists $c > 0$ such that :

(4.13)
$$\begin{aligned} &\|\sigma_1(t) - \sigma_2(t)\|_Q \\ \le\ & c\left(\|u_1(t) - u_2(t)\|_V + \int_0^t \|u_1(s) - u_2(s)\|_V\, ds\right) \quad \forall t \in [0,T] \end{aligned}$$

□

In the fourth step for every $\eta \in L^2\left([0,T];V'\right)$, we note by u_η the solution of Problem1, β_η the solution of Problem2 and σ_η the solution of Problem3.
Moreover we define the operator $\Lambda : L^2\left([0,T];V'\right) \longrightarrow L^2\left([0,T];V'\right)$ by :

(4.14)
$$\begin{aligned} &\langle \Lambda\eta(t), w\rangle_{V'\times V} \\ =\ & \left\{ \begin{array}{c} \langle \mathcal{E}\varepsilon(u_\eta(t)), \varepsilon(w)\rangle_Q + \left\langle \int_0^t \mathcal{G}(\sigma_\eta(s), \varepsilon(u_\eta(s)))\, ds, \varepsilon(w)\right\rangle_Q + \\ +j(\beta_\eta(t), u_\eta(t), w) \quad \forall w \in V, t \in [0,T] \end{array}\right. \end{aligned}$$

Lemma 4.5. *The operator Λ has a unique fixed point η^*.*

Proof. Let $\eta_1, \eta_2 \in L^2\left([0,T];V'\right)$ and $t \in [0,T]$. For simplicity we note : $u_i = u_{\eta_i}$, $v_i = \dot{u}_{\eta_i}$, $\beta_i = \beta_{\eta_i}$ for $i = 1, 2$. By integrating the differential equation (4.9), and using the definitions of R, $\widetilde{R}$ and a Gronwall's Lemma also the fact that operator $\mathcal{A}$ is strongly monotonous and the estimates (4.8), (4.11), (4.14) we deduce that :

$$\|\Lambda\eta_1(t) - \Lambda\eta_2(t)\|_{V'}^2 \leq c\int_0^t \|\eta_1(s) - \eta_2(s)\|_{V'}^2\, ds \quad \forall t \in [0,T]$$

Reiterating this inequality for m given times in $[0,T]$, we obtain :

$$\|\Lambda^m\eta_1 - \Lambda^m\eta_2\|_{L^2\left(0,T;V'\right)} \leq \frac{c^mT^m}{m!}\|\eta_1 - \eta_2\|_{L^2\left(0,T;V'\right)}$$

Which implies that for m sufficiently large a power Λ^m of Λ is a contraction in the Banach space $L^2\left([0,T];V'\right)$. Then, Λ has a unique fixed point $\eta^* \in L^2\left([0,T];V'\right)$. □

Now, the proof of the Theorem 4.1 is a consequence of the preceding lemmas.

Proof. **Existence.** Let $\eta^* \in L^2\left([0,T];V'\right)$ be the fixed point of the operator Λ and let u be the solution of Problem1, for $\eta = \eta^*$, i.e $u = u_{\eta^*}$. We denote by σ the function given by (3.25) and by β the solution of Problem2 for $\eta = \eta^*$, i.e $\beta = \beta_{\eta^*}$. Clearly, equalities (3.27) and (3.28) hold from (4.5) and (4.9), (4.10). Moreover, since $\eta^* = \Lambda\eta^*$, it follows from (4.5) and (4.15) that (3.27) holds, too. The regularity of the solution expressed in (4.1) follows from Lemma 4.2. Since $u \in H^1([0,T];V)$, it follows from (3.25), assumptions (3.10) and (3.11) that $\sigma \in L^2([0,T];Q)$. Choosing now $w = \varphi$ in (3.27), $\varphi \in D(\Omega)^N$, and using the definitions of f, j given respectively by (3.22), (3.24) we obtain :

$$Div\sigma(t) + f_0(t) = \rho\ddot{u}(t) \quad \text{a.e } t \in]0,T[\tag{4.15}$$

Now, assumptions (3.16), (3.17), the fact that $\ddot{u} \in L^2\left([0,T];V'\right)$ and (4.15) imply that $Div\sigma \in L^2\left([0,T];V'\right)$.
Recall also that the regularity of the bonding field $\beta \in W^{1,\infty}([0,T];L^2(\Gamma_3)) \cap \mathcal{Q}$ follows from Lemma 4.3. We conclude that (u,σ,β) is a solution of ProblemPV and it satisfies (4.1)-(4.3). □

Proof. **Uniqueness.** The uniqueness of the solution follows from the uniqueness of the fixed point of the operator Λ and from the unique solvability of Problem1, Problem2 and Problem3. Indeed; let (u,σ,β) be a solution of ProblemPV which satisfies (4.1)-(4.3) and denote by $\eta \in L^2\left([0,T];V'\right)$, the function defined by :

$$\langle\eta(t), w\rangle_V = \begin{cases} \langle\mathcal{E}\varepsilon(u(t), \varepsilon(w))\rangle_Q + \left\langle\int_0^t \mathcal{G}(\sigma(s), \varepsilon(u(s)))\, ds, \varepsilon(w)\right\rangle_Q + \\ \qquad +j(\beta(t), u(t), w) \quad \forall w \in V, t \in [0,T] \end{cases} \tag{4.16}$$

Equalities (3.25), (3.27) and (4.16) associated with the condition initial $u(0) = u_0$, $\dot{u}(0) = v_0$ imply that u is a solution of Problem1 and, since it follows from Lemma 4.2 that this problem has a unique solution denoted u_η, we conclude that :

$$u = u_\eta \tag{4.17}$$

Next, (3.27) and the condition initial $\beta(0) = \beta_0$ imply that β is a solution of Problem2 and, since it follows from Lemma 4.3 that this problem has unique solution denoted β_η, we conclude that :

$$\beta = \beta_\eta \tag{4.18}$$

Using now (4.15) and (4.16)-(4.18) we obtain that $\Lambda\eta = \eta$ and by the uniqueness of the fixed point of the operator Λ, guaranteed by Lemma 4.5, it follows that :

$$\eta = \eta^* \tag{4.19}$$

□

The uniqueness of the solution is now a consequence of (4.17)-(4.19) combined with (3.25).

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