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FEKETE-SZEGÖ PROBLEM FOR UNIVALENT FUNCTIONS WITH RESPECT TO k-SYMMETRIC POINTS

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ABSTRACT. In the present investigation, sharp upper bounds of $|a_3 - \mu a_2^2|$ for functions $f(z) = z + a_2z^2 + a_3z^3 + ...$ belonging to certain subclasses of starlike and convex functions with respect to k-symmetric points are obtained. Also certain applications of the main results for subclasses of functions defined by convolution with a normalized analytic function are given. In particular, Fekete- Szegö inequalities for certain classes of functions defined through fractional derivatives are obtained.

Key words and phrases: Analytic functions, Starlike functions, Convex functions, Subordination, Coefficient problem, Fekete-Szegö inequality, k-Symmetric points.

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1. INTRODUCTION

Let $\mathcal{A}$ denote the class of all *analytic* functions $f(z)$ of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (z \in \mathbf{U} := \{z \in \mathbb{C} : |z| < 1\}), \tag{1.1}$$

and let $\mathcal{S}$ denotes the subclass of $\mathcal{A}$ consisting of all functions which are univalent in **U**. For two functions $f, g \in \mathcal{A}$, we say that the function $f(z)$ is *subordinate* to $g(z)$ in **U** and write $f \prec g$ or $f(z) \prec g(z)(z \in \mathbf{U})$, if there exists an analytic function $w(z)$ with $w(0) = 0$ and $|w(z)| < 1(z \in \mathbf{U})$, such that $f(z) = g(w(z))(z \in \mathbf{U})$. In particular, if the function g is univalent in **U**, the above subordination is equivalent to $f(0) = g(0)$ and $f(\mathbf{U}) \subset g(\mathbf{U})$.
Let P denote the class of functions of the form $p(z) = 1 + p_1 z + p_2 z^2 + \cdots (z \in \mathbf{U})$ analytic in **U** which satisfy the condition $\Re\{p(z)\} > 0$.

For $\phi \in P$, let $S^*(\phi)$ be the class of function $f \in \mathcal{S}$ for which $\frac{zf'(z)}{f(z)} \prec \phi(z), (z \in \mathbf{U})$ and $C(\phi)$ be the class of functions $f \in \mathcal{S}$ for which $1 + \frac{zf''(z)}{f'(z)} \prec \phi(z), (z \in \mathbf{U})$. These classes were introduced and studied by Ma and Minda [4]. When $\phi(z) = (1 + Az)/(1 + Bz), (-1 \leq B < A \leq 1)$, the class $S^*(\phi)$ reduces to the class $S^*[A, B]$ studied by Janowski [3]. See also Silverman and Silvia [10].

Sakaguchi [8] once introduced a class S_S^* of functions starlike with respect to symmetric points, which consists of functions $f \in \mathcal{S}$ satisfying the inequality

$$\Re\left\{\frac{zf'(z)}{f(z) - f(-z)}\right\} > 0, \qquad z \in \mathbf{U}, \tag{1.2}$$

Many different authors following Sakaguchi [8] and discussed this class and its subclasses. In 1979 Chand and Singh [1] introduced a class $S_S^{(k)}$ of functions starlike with respect to k-symmetric points, which consists of functions $f \in \mathcal{S}$ satisfying the inequality

$$\Re\left\{\frac{zf'(z)}{f_k(z)}\right\} > 0, \qquad z \in \mathbf{U}, \tag{1.3}$$

where

$$f_k(z) = \frac{1}{k} \sum_{\nu=0}^{k-1} \varepsilon^{-\nu} f(\varepsilon^{\nu} z), \qquad (k \in \mathbb{N}; \varepsilon^k = 1). \tag{1.4}$$

Definition 1.1. [2] Let $\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$ be a univalent starlike function with respect to 1 which maps the unit disk **U** onto a region in the right half plane which is symmetric with respect to the real axis, and let $B_1 > 0$. The function $f \in \mathcal{A}$ is in the class $S_s^{(k)}(\phi)$ if

$$\frac{zf'(z)}{f_k(z)} \prec \phi(z) \qquad (z \in \mathbf{U}),$$

where $\phi(z) \in P$ and k is a fixed positive integer and $f_k(z)$ is defined by (1.4).

The function $f \in \mathcal{A}$ is in the class $C_s^{(k)}(\phi)$ if

$$\frac{(zf'(z))'}{f_k'(z)} \prec \phi(z) \qquad (z \in \mathbf{U}),$$

where $\phi(z) \in P$ and k is a fixed positive integer and $f_k(z)$ is defined by (1.4).

When $\phi(z) = (1 + Az)/(1 + Bz)$, $(-1 \leq B < A \leq 1)$, we denote the subclasses $S_s^{(k)}(\phi)$ and $C_s^{(k)}(\phi)$ by $S_s^{(k)}[A, B]$ and $C_s^{(k)}[A, B]$ respectively. For $0 \leq \alpha < 1$, let $S_s^{(k)}(\alpha) := S_s^{(k)}[1 - 2\alpha, -1]$ and $C_s^{(k)}(\alpha) := C_s^{(k)}[1 - 2\alpha, -1]$.

Note that if $k = 1$, then the classes $S_s^{(k)}(\phi)$ and $C_s^{(k)}(\phi)$ reduces to the classes $S^*(\phi)$ and $C(\phi)$, respectively [4]. Also that if $k = 2$ and $\phi(z) = (1 + z)/(1 - z)$, then the classes $S_s^{(k)}(\phi)$ and $C_s^{(k)}(\phi)$ reduces to the classes S_s^* and C_s, respectively, which were also introduced and investigated recently by Ravichandran [7].

In the present paper, we obtain the Fekete-Szegö inequality for functions in the subclasses $S_s^{(k)}(\phi)$ and $C_s^{(k)}(\phi)$. Also we give applications of our results to certain functions defined through Hadamard product (or convolution) and in particular we consider classes $S_s^{(k),\lambda}(\phi)$ and $C_s^{(k),\lambda}(\phi)$ defined by fractional derivatives.

To prove our main result, we need the following:

Lemma 1.1. *If $p_1(z) = 1 + c_1 z + c_2 z^2 + \cdots$ is an analytic function with positive real part in **U**, then*

$$|c_2 - \upsilon c_1^2| \leq \begin{cases} -4\upsilon + 2 & if\ \upsilon \leq 0, \\ 2 & if\ 0 \leq \upsilon \leq 1, \\ 4\upsilon - 2 & if\ \upsilon \geq 1. \end{cases}$$

When $\upsilon < 0$ or $\upsilon > 1$, the equality holds if and only if $p_1(z)$ is $(1 + z)/(1 - z)$ or one of its rotations. If $0 < \upsilon < 1$, then equality holds if and only if $p_1(z)$ is $(1 + z^2)/(1 - z^2)$ or one of its rotations. If $\upsilon = 0$, the equality holds if and only if

$$p_1(z) = \left(\frac{1}{2} + \frac{1}{2}\lambda\right)\frac{1 + z}{1 - z} + \left(\frac{1}{2} - \frac{1}{2}\lambda\right)\frac{1 - z}{1 + z} \quad (0 \leq \lambda \leq 1)$$

or one of its rotations. If $\upsilon = 1$, the equality holds if and only if $p_1(z)$ is the reciprocal of one of the functions such that the equality holds in the case of $\upsilon = 0$. Also the above upper bound is sharp, it can be improved as follows when $0 < \upsilon < 1$:

$$|c_2 - \upsilon c_1^2| + \upsilon|c_1|^2 \leq 2 \qquad (0 < \upsilon \leq 1/2)$$

and

$$|c_2 - \upsilon c_1^2| + (1 - \upsilon)|c_1|^2 \leq 2 \qquad (1/2 < \upsilon \leq 1).$$

2. FEKETE-SZEGÖ PROBLEM

Our main result is the following:

Theorem 2.1. *Let $\phi(z) = 1 + B_1 z + B_2 z^2 + \ldots$. If $f(z)$ given by (1.1) belongs to $S_s^{(k)}(\phi)$, then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{B_2}{(3-\psi_3)} - \frac{\mu B_1^2}{(2-\psi_2)^2} + \frac{\psi_2 B_1^2}{(3-\psi_3)(2-\psi_2)} & if\ \ \mu \leq \sigma_1\ ; \\ \frac{B_1}{3-\psi_3} & if\ \ \sigma_1 \leq \mu \leq \sigma_2\ ; \\ -\frac{B_2}{(3-\psi_3)} + \frac{\mu B_1^2}{(2-\psi_2)^2} - \frac{\psi_2 B_1^2}{(3-\psi_3)(2-\psi_2)} & if\ \ \mu \geq \sigma_2, \end{cases} \tag{2.1}$$

where

$$\sigma_1 := \frac{(2-\psi_2)\{(B_2-B_1)(2-\psi_2)+\psi_2 B_1^2\}}{(3-\psi_3)B_1^2},$$
$$\sigma_2 := \frac{(2-\psi_2)\{(B_2+B_1)(2-\psi_2)+\psi_2 B_1^2\}}{(3-\psi_3)B_1^2}.$$

The result is sharp.

Proof. For $f(z) \in S_s^{(k)}(\phi)$, let

$$p(z) := \frac{zf'(z)}{f_k(z)} = 1 + b_1 z + b_2 z^2 + \dots \,. \tag{2.2}$$

From the condition we know

$$f_k(z) = \frac{1}{k}\sum_{\nu=0}^{k-1} \varepsilon^{-\nu} f(\varepsilon^\nu z) = \frac{1}{k}\sum_{\nu=0}^{k-1} \varepsilon^{-\nu}[\varepsilon^\nu z + \sum_{n=2}^{\infty} a_n(\varepsilon^{-\nu} z)^n]$$
$$= z + \sum_{n=2}^{\infty} a_n \psi_n z^n$$

where

$$\psi_n = \frac{1}{k}\sum_{\nu=0}^{k-1} \varepsilon^{(n-1)\nu}, \qquad (n \geq 2,\, \varepsilon^k = 1)$$

From (2.2), we obtain

$$2a_2 = b_1 + a_2\psi_2 \quad \text{and} \quad 3a_3 = b_2 + b_1 a_2 \psi_2 + a_3\psi_3.$$

Since $\phi(z)$ is univalent and $p \prec \phi$, the function

$$p_1(z) = \frac{1+\phi^{-1}(p(z))}{1-\phi^{-1}(p(z))} = 1 + c_1 z + c_2 z^2 + \cdots,$$

is analytic and has a positive real part in **U**. Also we have

$$p(z) = \phi\Big(\frac{p_1(z)-1}{p_1(z)+1}\Big),$$

and from this equation ,

$$1 + b_1 z + b_2 z^2 + \dots = \phi\Big(\frac{c_1 z + c_2 z^2 + \dots}{2 + c_1 z + c_2 z^2 + \dots}\Big)$$
$$= \phi\Big[\frac{1}{2}c_1 z + \frac{1}{2}(c_2 - \frac{1}{2}c_1^2)z^2 + \dots\Big]$$
$$= 1 + B_1\frac{1}{2}c_1 z + B_1\frac{1}{2}(c_2 - \frac{1}{2}c_1^2)z^2 + \dots + B_2\frac{1}{4}c_1^2 z^2 + \dots$$

we obtain

$$b_1 = \frac{1}{2}B_1 c_1 \text{ and } b_2 = \frac{1}{2}B_1(c_2 - \frac{1}{2}c_1^2) + \frac{1}{4}B_2 c_1^2.$$

Therefore we have

$$a_3 - \mu a_2^2 = \frac{B_1}{2(3-\psi_3)}\left\{c_2 - c_1^2\Big[\frac{1}{2}\Big(1 - \frac{B_2}{B_1} + \frac{(3-\psi_3)\mu - (2-\psi_2)\psi_2)}{(2-\psi_2)^2}B_1\Big)\Big]\right\}$$
$$= \frac{B_1}{2(3-\psi_3)}[c_2 - \upsilon c_1^2]$$

where

$$v = \frac{1}{2}\Big(1 - \frac{B_2}{B_1} + \frac{(3-\psi_3)\mu - (2-\psi_2)\psi_2}{(2-\psi_2)^2}B_1\Big).$$

If $\mu \leq \sigma_1$, then by applying Lemma 1.1, we get

$$|a_3 - \mu a_2^2| \leq \frac{B_2}{(3-\psi_3)} - \frac{\mu B_1^2}{(2-\psi_2)^2} + \frac{\psi_2 B_1^2}{(3-\psi_3)(2-\psi_2)},$$

which is the first part of assertion (2.1).

Similarly, if $\mu \geq \sigma_2$, we get

$$|a_3 - \mu a_2^2| \leq -\frac{B_2}{(3-\psi_3)} + \frac{\mu B_1^2}{(2-\psi_2)^2} - \frac{\psi_2 B_1^2}{(3-\psi_3)(2-\psi_2)},$$

If $\mu = \sigma_1$, then equality holds if and only if

$$p_1(z) = \Big(\frac{1+\lambda}{2}\Big)\frac{1+z}{1-z} + \Big(\frac{1-\lambda}{2}\Big)\frac{1-z}{1+z} \quad (0 \leq \lambda \leq 1; z \in \mathbf{U})$$

or one of its rotations.
Also, if $\mu = \sigma_2$, then

$$\frac{1}{2}\Big(1 - \frac{B_2}{B_1} + \frac{(3-\psi_3)\mu - (2-\psi_2)\psi_2)}{(2-\psi_2)^2}B_1\Big) = 0.$$

Therefore,

$$\frac{1}{p_1(z)} = \Big(\frac{1+\gamma}{2}\Big)\frac{1+z}{1-z} + \Big(\frac{1-\gamma}{2}\Big)\frac{1-z}{1+z} \quad (0 < \gamma < 1; z \in \mathbf{U})$$

Finally, we see that

$$|a_3 - \mu a_2^2| \quad = \quad \frac{B_1}{2(3-\psi_3)}\left|c_2 - c_1^2\Big[\frac{1}{2}\Big(1 - \frac{B_2}{B_1} + \frac{(3-\psi_3)\mu - (2-\psi_2)\psi_2)}{(2-\psi_2)^2}B_1\Big)\Big]\right|$$

and

$$\max\left|\frac{1}{2}\Big(1 - \frac{B_2}{B_1} + \frac{(3-\psi_3)\mu - (2-\psi_2)\psi_2)}{(2-\psi_2)^2}B_1\Big)\right| \quad (\sigma_1 \leq \mu \leq \sigma_2).$$

Therefore using Lemma 1.1, we get

$$|a_3 - \mu a_2^2| \quad = \quad \frac{B_1|c_1|}{2(3-\psi_3)} \leq \frac{B_1}{(3-\psi_3)}, \quad (\sigma_1 \leq \mu \leq \sigma_2).$$

If $\sigma_1 < \mu < \sigma_2$, then we have

$$p_1(z) = \frac{1+\lambda z^2}{1-\lambda z^2}, \qquad (0 \leq \lambda \leq 1).$$

Our result now follows by an application of Lemma 1.1. To show that these bounds are sharp, we define the functions $K_{\phi\delta}(\delta = 2, 3, ...)$ by

$$\frac{zK'_{\phi\delta}(z)}{K_{k(\phi\delta)}(z)} = \phi(z^{\delta-1}), \quad K_{\phi\delta}(0) = 0 = (K_{\phi\delta})'(0) - 1$$

and the function F_λ and G_λ $(0 \leq \lambda \leq 1)$ by

$$\frac{zF'_\lambda(z)}{F_{k(\lambda)}(z)} = \phi(z^{\delta-1}), \quad F_\lambda(0) = 0 = (F_\lambda)'(0) - 1$$

and

$$\frac{zG'_\lambda(z)}{G_{k(\lambda)}(z)} = \phi(z^{\delta-1}), \quad G_\lambda(0) = 0 = (G_\lambda)'(0) - 1$$

Clearly the functions $K_{\phi\delta}, F_\lambda, G_\lambda \in S_s^{(k)}(\phi)$. Also we write $K_\phi := K_{\phi 2}$. If $\mu < \sigma_1$ or $\mu > \sigma_2$, then the equality holds if and only if f is K_ϕ or one of its rotations. When $\sigma_1 < \mu < \sigma_2$, the equality holds if and only if f is $K_{\phi 3}$ or one of its rotations. If $\mu = \sigma_1$ then the equality holds if and only if f is F_λ or one of its rotations. If $\mu = \sigma_2$ then the equality holds if and only if f is G_λ or one of its rotations. ∎

Remark 2.1. If $\sigma_1 \le \mu \le \sigma_2$, then in view of Lemma 1.1, Theorem 2.1 can be improved. Let σ_3 be given by

$$\sigma_3 := \frac{(2-\psi_2)\{(2-\psi_2)B_2 + \psi_2 B_1\}}{(3-\psi_3)B_1^2}$$

If $\sigma_1 \le \mu \le \sigma_3$, then

$$|a_3 - \mu a_2^2| + \frac{(2-\psi_2)^2}{(3-\psi_3)B_1^2}\left[B_1 - B_2 + \frac{(3-\psi_3)\mu - (2-\psi_2)\psi_2}{(2-\psi_2)^2}B_1^2\right]|a_2|^2 \le \frac{B_1}{(3-\psi_3)}.$$

If $\sigma_3 \le \mu \le \sigma_2$, then

$$|a_3 - \mu a_2^2| + \frac{(2-\psi_2)^2}{(3-\psi_3)B_1^2}\left[B_1 + B_2 - \frac{(3-\psi_3)\mu - (2-\psi_2)\psi_2}{(2-\psi_2)^2}B_1^2\right]|a_2|^2 \le \frac{B_1}{(3-\psi_3)}$$

Proof. For the values of $\sigma_1 \le \mu \le \sigma_3$, we have

$$\begin{aligned}
& |a_3 - \mu a_2^2| + (\mu - \sigma_1)|a_2|^2 \\
= \; & \frac{B_1}{2(3-\psi_3)}|c_2 - \upsilon c_1^2| + (\mu - \sigma_1)\frac{B_1^2}{4(2-\psi_2)^2}|c_1|^2 \\
= \; & \frac{B_1}{2(3-\psi_3)}|c_2 - \upsilon c_1^2| + \left(\mu - \frac{(2-\psi_2)\{(B_2 - B_1)(2-\psi_2) + \psi_2 B_1^2\}}{(3-\psi_3)B_1^2}\right)\frac{B_1^2}{4(2-\psi_2)^2}|c_1|^2 \\
= \; & \frac{B_1}{(3-\psi_3)}\left\{\frac{1}{2}\left[|c_2 - \upsilon c_1^2| + \upsilon|c_1|^2\right]\right\} \\
\le \; & \frac{B_1}{(3-\psi_3)}.
\end{aligned}$$

Similarly, for the values of $\sigma_3 \le \mu \le \sigma_2$, we write

$$\begin{aligned}
& |a_3 - \mu a_2^2| + (\sigma_2 - \mu)|a_2|^2 \\
= \; & \frac{B_1}{2(3-\psi_3)}|c_2 - \upsilon c_1^2| + (\sigma_2 - \mu)\frac{B_1^2}{4(2-\psi_2)^2}|c_1|^2 \\
= \; & \frac{B_1}{2(3-\psi_3)}|c_2 - \upsilon c_1^2| + \left(\frac{(2-\psi_2)\{(B_2 + B_1)(2-\psi_2) + \psi_2 B_1^2\}}{(3-\psi_3)B_1^2} - \mu\right)\frac{B_1^2}{4(2-\psi_2)^2}|c_1|^2 \\
= \; & \frac{B_1}{(3-\psi_3)}\left\{\frac{1}{2}\left[|c_2 - \upsilon c_1^2| + (1-\upsilon)|c_1|^2\right]\right\} \\
\le \; & \frac{B_1}{(3-\psi_3)}.
\end{aligned}$$

Thus, the proof of Remark 2.1 is evidently completed. ∎

Example 2.1. *Let $-1 \leq B < A \leq 1$. If $f(z)$ given by (1.1) belongs to $S_s^{(k)}[A;B]$, then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{B(B-A)}{(3-\psi_3)} - \frac{\mu(A-B)^2}{(2-\psi_2)^2} + \frac{\psi_2(A-B)^2}{(3-\psi_3)(2-\psi_2)} & if \ \ \mu \leq \sigma_1 \ ; \\ \frac{A-B}{3-\psi_3} & if \ \ \sigma_1 \leq \mu \leq \sigma_2 \ ; \\ -\frac{B(B-A)}{(3-\psi_3)} + \frac{\mu(A-B)^2}{(2-\psi_2)^2} - \frac{\psi_2(A-B)^2}{(3-\psi_3)(2-\psi_2)} & if \ \ \mu \geq \sigma_2, \end{cases}$$

where

$$\begin{aligned} \sigma_1 &:= \frac{-(2-\psi_2)\{(B+1)(2-\psi_2) - \psi_2(A-B)\}}{(3-\psi_3)(A-B)}, \\ \sigma_2 &:= \frac{(2-\psi_2)\{(1-B)(2-\psi_2) + \psi_2(A-B)\}}{(3-\psi_3)(A-B)}. \end{aligned}$$

The result is sharp.
In particular, let $0 \leq \alpha < 1$, if $f \in S_s^{(k)}(\alpha)$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{2(1-\alpha)}{(3-\psi_3)} - \frac{4\mu(1-\alpha)^2}{(2-\psi_2)^2} + \frac{4\psi_2(1-\alpha)^2}{(3-\psi_3)(2-\psi_2)} & if \ \ \mu \leq \sigma_1 \ ; \\ \frac{2(1-\alpha)}{3-\psi_3} & if \ \ \sigma_1 \leq \mu \leq \sigma_2 \ ; \\ -\frac{2(1-\alpha)}{(3-\psi_3)} + \frac{4\mu(1-\alpha)^2}{(2-\psi_2)^2} - \frac{4\psi_2(1-\alpha)^2}{(3-\psi_3)(2-\psi_2)} & if \ \ \mu \geq \sigma_2, \end{cases}$$

where

$$\begin{aligned} \sigma_1 &:= \frac{(2-\psi_2)\psi_2}{(3-\psi_3)}, \\ \sigma_2 &:= \frac{(2-\psi_2)\{(2-\psi_2) + \psi_2(1-\alpha)\}}{(3-\psi_3)(1-\alpha)}. \end{aligned}$$

The result is sharp.

Since $f \in C_s^{(k)}(\phi)$ if and only if $zf' \in S_s^{(k)}(\phi)$, Theorem 2.1, with an obvious change of the parameter μ, leads to the following Corollary.

Corollary 2.2. *If $f(z)$ given by (1.1) belongs to $C_s^{(k)}(\phi)$, then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{B_2}{3(3-\psi_3)} - \frac{\mu B_1^2}{4(2-\psi_2)^2} + \frac{\psi_2 B_1^2}{3(3-\psi_3)(2-\psi_2)} & if \ \ \mu \leq \sigma_1 \ ; \\ \frac{B_1}{3(3-\psi_3)} & if \ \ \sigma_1 \leq \mu \leq \sigma_2 \ ; \\ -\frac{B_2}{3(3-\psi_3)} + \frac{\mu B_1^2}{4(2-\psi_2)^2} - \frac{\psi_2 B_1^2}{3(3-\psi_3)(2-\psi_2)} & if \ \ \mu \geq \sigma_2, \end{cases}$$

where

$$\begin{aligned} \sigma_1 &:= \frac{4(2-\psi_2)\{(B_2 - B_1)(2-\psi_2) + \psi_2 B_1^2\}}{3(3-\psi_3)B_1^2}, \\ \sigma_2 &:= \frac{4(2-\psi_2)\{(B_2 + B_1)(2-\psi_2) + \psi_2 B_1^2\}}{3(3-\psi_3)B_1^2}. \end{aligned}$$

The result is sharp.

Example 2.2. *Let $-1 \leq B < A \leq 1$. If f(z) given by (1.1) belongs to $C_s^{(k)}[A;B]$, then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{B(B-A)}{3(3-\psi_3)} - \frac{\mu(A-B)^2}{4(2-\psi_2)^2} + \frac{\psi_2(A-B)^2}{3(3-\psi_3)(2-\psi_2)} & if \ \ \mu \leq \sigma_1 \ ; \\ \frac{A-B}{3(3-\psi_3)} & if \ \ \sigma_1 \leq \mu \leq \sigma_2 \ ; \\ -\frac{B(B-A)}{3(3-\psi_3)} + \frac{\mu(A-B)^2}{4(2-\psi_2)^2} - \frac{\psi_2(A-B)^2}{3(3-\psi_3)(2-\psi_2)} & if \ \ \mu \geq \sigma_2, \end{cases}$$

where

$$\begin{aligned} \sigma_1 &:= \frac{-4(2-\psi_2)\big\{(B+1)(2-\psi_2) - \psi_2(A-B)\big\}}{3(3-\psi_3)(A-B)}, \\ \sigma_2 &:= \frac{4(2-\psi_2)\big\{(1-B)(2-\psi_2) + \psi_2(A-B)\big\}}{3(3-\psi_3)(A-B)}. \end{aligned}$$

The result is sharp.
In particular, let $0 \leq \alpha < 1$, if $f \in C_s^{(k)}(\alpha)$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{2(1-\alpha)}{3(3-\psi_3)} - \frac{\mu(1-\alpha)^2}{(2-\psi_2)^2} + \frac{4\psi_2(1-\alpha)^2}{3(3-\psi_3)(2-\psi_2)} & if \ \ \mu \leq \sigma_1 \ ; \\ \frac{2(1-\alpha)}{3-\psi_3} & if \ \ \sigma_1 \leq \mu \leq \sigma_2 \ ; \\ -\frac{2(1-\alpha)}{3(3-\psi_3)} + \frac{\mu(1-\alpha)^2}{(2-\psi_2)^2} - \frac{4\psi_2(1-\alpha)^2}{3(3-\psi_3)(2-\psi_2)} & if \ \ \mu \geq \sigma_2, \end{cases}$$

where

$$\begin{aligned} \sigma_1 &:= \frac{4(2-\psi_2)\psi_2}{3(3-\psi_3)}, \\ \sigma_2 &:= \frac{4(2-\psi_2)\big\{(2-\psi_2) + \psi_2(1-\alpha)\big\}}{3(3-\psi_3)(1-\alpha)}. \end{aligned}$$

The result is sharp.

If $\sigma_1 \leq \mu \leq \sigma_2$, then, in view of Lemma 1.1, Corollary 2.2 can be improved.

Corollary 2.3. *Let $f(z)$ given by (1.1) belongs to $C_s^{(k)}(\phi)$. Let σ_3 be given by*

$$\sigma_3 := \frac{4(2-\psi_2)\big\{(2-\psi_2)B_2 + \psi_2 B_1\big\}}{3(3-\psi_3)B_1^2}$$

If $\sigma_1 \leq \mu \leq \sigma_3$, then

$$|a_3 - \mu a_2^2| + \frac{4(2-\psi_2)^2}{3(3-\psi_3)B_1^2}\left[B_1 - B_2 + \frac{3(3-\psi_3)\mu - (2-\psi_2)\psi_2}{(2-\psi_2)^2}B_1^2\right]|a_2|^2 \leq \frac{B_1}{3(3-\psi_3)}.$$

If $\sigma_3 \leq \mu \leq \sigma_2$, then

$$|a_3 - \mu a_2^2| + \frac{4(3-\psi_3)}{3B_1^2}\left[B_1 + B_2 - \frac{3(3-\psi_3)\mu - (2-\psi_2)\psi_2}{(2-\psi_2)^2}B_1^2\right]|a_2|^2 \leq \frac{B_1}{3(3-\psi_3)}$$

3. APPLICATIONS TO FUNCTIONS DEFINED BY FRACTIONAL DERIVATIVES

For two analytic functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, their convolution (or Hadamard product) is defined to be the function $(f * g)(z)$ given by $(f * g)(z) = f(z) * g(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n$. For fixed $g \in \mathcal{A}$, let $S_s^{(k),g}(\phi)$ be the class of functions $f \in \mathcal{A}$ for which $(f * g) \in S_s^{(k)}(\phi)$ and $C_s^{(k),g}(\phi)$ be the class of functions $f \in \mathcal{A}$ for which $(f * g) \in C_s^{(k)}(\phi)$.

Definition 3.1. (see [11], [6]). Let $f(z)$ be analytic in a simply connected region of the z-plane containing the origin. The fractional derivative of f of order λ is defined by

$$D_z^{\lambda} f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\lambda}} d\zeta \qquad (0 \le \lambda < 1).$$

where the multiplicity of $(z-\zeta)^{\lambda}$ is removed by requiring that $\log(z-\zeta)$ is real for $z - \zeta > 0$. Using the above Definition 3.1 and its known extensions involving fractional derivatives and fractional integrals, Owa and Srivastava [5] introduced the operator $\Omega^{\lambda} : \mathcal{A} \to \mathcal{A}$ defined by

$$\Omega^{\lambda} f(z) = \Gamma(2-\lambda) z^{\lambda} D_z^{\lambda} f(z) \qquad (\lambda \neq 2, 3, 4, ...)$$

The class $S_s^{(k),\lambda}(\phi)$ and $C_s^{(k),\lambda}(\phi)$ consists of functions $f \in \mathcal{A}$ for which $\Omega^{\lambda} f \in S_s^{(k)}(\phi)$ and $\Omega^{\lambda} f \in C_s^{(k)}(\phi)$, respectively. Note that $S_s^{(k),\lambda}(\phi)$ is the special case of the class $S_s^{(k),g}(\phi)$ and $C_s^{(k),\lambda}(\phi)$ is the special case of the class $C_s^{(k),g}(\phi)$ when

$$g(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma((2-\lambda)}{\Gamma((n+1-\lambda)} z^n.$$

Let

$$g(z) = z + \sum_{n=2}^{\infty} g_n z^n \qquad (g_n > 0).$$

Since $f \in S_s^{(k),g}(\phi)$ if and only if $f(z) * g(z) \in S_s^{(k)}(\phi)$, and $f \in C_s^{(k),g}(\phi)$ if and only if $f(z) * g(z) \in C_s^{(k)}(\phi)$ we obtain the coefficient estimate for functions in the classes $S_s^{(k),g}(\phi)$ and $C_s^{(k),g}(\phi)$, from the corresponding estimate for functions in the classes $S_s^{(k)}(\phi)$ and $C_s^{(k)}(\phi)$. Applying Theorem 2.1 for the function $f(z) * g(z) = z + a_2 g_2 z^2 + a_3 g_3 z^3 + ...$ we get the following Theorem 3.1 after an obvious change of the parameter μ:

Theorem 3.1. *Let $g(z) = z + \sum_{n=2}^{\infty} g_n z^n, (g_n > 0)$ and let the function $\phi(z)$ be given by $\phi(z) = 1 + \sum_{k=1}^{\infty} B_k z^k$. If $f(z)$ given by (1.1) belongs to $S_s^{(k),g}(\phi)$, then*

$$|a_3 - \mu a_2^2| \le \begin{cases} \frac{B_2}{(3-\psi_3)g_3} - \frac{\mu B_1^2}{(2-\psi_2)^2 g_2^2} + \frac{\psi_2 B_1^2}{(3-\psi_3)(2-\psi_2)g_3} & if \ \ \mu \le \sigma_1 \ ; \\ \frac{B_1}{(3-\psi_3)g_3} & if \ \ \sigma_1 \le \mu \le \sigma_2 \ ; \\ -\frac{B_2}{(3-\psi_3)g_3} + \frac{\mu B_1^2}{(2-\psi_2)^2 g_2^2} - \frac{\psi_2 B_1^2}{(3-\psi_3)(2-\psi_2)g_3} & if \ \ \mu \ge \sigma_2, \end{cases}$$

where

$$\sigma_1 := \frac{g_2^2(2-\psi_2)\{(B_2-B_1)(2-\psi_2)+\psi_2 B_1^2\}}{g_3(3-\psi_3)B_1^2},$$
$$\sigma_2 := \frac{g_2^2(2-\psi_2)\{(B_2+B_1)(2-\psi_2)+\psi_2 B_1^2\}}{g_3(3-\psi_3)B_1^2}.$$

The result is sharp.

Corollary 3.2. *Let* $g(z) = z + \sum_{n=2}^{\infty} g_n z^n, (g_n > 0)$ *and let the function* $\phi(z)$ *be given by* $\phi(z) = 1 + \sum_{k=1}^{\infty} B_k z^k$. *If* $f(z)$ *given by (1.1) belongs to* $C_s^{(k)}(\phi)$, *then*

$$|a_3-\mu a_2^2| \leq \begin{cases} \frac{B_2}{3(3-\psi_3)g_3} - \frac{\mu B_1^2}{4(2-\psi_2)^2 g_2^2} + \frac{\psi_2 B_1^2}{3(3-\psi_3)(2-\psi_2)g_3} & if \;\; \mu \leq \sigma_1 \;; \\ \frac{B_1}{3(3-\psi_3)} & if \;\; \sigma_1 \leq \mu \leq \sigma_2 \;; \\ -\frac{B_2}{3(3-\psi_3)g_3} + \frac{\mu B_1^2}{4(2-\psi_2)^2 g_2^2} - \frac{\psi_2 B_1^2}{3(3-\psi_3)(2-\psi_2)g_3} & if \;\; \mu \geq \sigma_2, \end{cases}$$

where

$$\sigma_1 := \frac{4(2-\psi_2)g_2^2\{(B_2-B_1)(2-\psi_2)+\psi_2 B_1^2\}}{3g_3(3-\psi_3)B_1^2},$$
$$\sigma_2 := \frac{4(2-\psi_2)g_2^2\{(B_2+B_1)(2-\psi_2)+\psi_2 B_1^2\}}{3g_3(3-\psi_3)B_1^2}.$$

The result is sharp.

Since

$$(\Omega^\lambda f)(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} z^n,$$

we have

$$g_2 := \frac{\Gamma(3)\Gamma(2-\lambda)}{\Gamma(3-\lambda)} = \frac{2}{2-\lambda}$$

and

$$g_3 := \frac{\Gamma(4)\Gamma(3-\lambda)}{\Gamma(4-\lambda)} = \frac{6}{(2-\lambda)(3-\lambda)}.$$

For g_2 and g_3 given by above inequalities, Theorem 3.1 reduces to the following:

Theorem 3.3. *Let the function* $\phi(z)$ *be given by* $\phi(z) = 1 + \sum_{n=1}^{\infty} B_n z^n$ *and* $\lambda < 2$. *If* $f(z)$ *given by (1.1) belongs to* $S_s^{(k),\lambda}(\phi)$, *then*

$$|a_3-\mu a_2^2| \leq \begin{cases} \frac{(2-\lambda)(3-\lambda)B_2}{6(3-\psi_3)} - \frac{(2-\lambda)^2 \mu B_1^2}{4(2-\psi_2)^2} + \frac{(2-\lambda)(3-\lambda)\psi_2 B_1^2}{6(3-\psi_3)(2-\psi_2)} & if \;\; \mu \leq \sigma_1 \;; \\ \frac{(2-\lambda)(3-\lambda)B_1}{6(3-\psi_3)} & if \;\; \sigma_1 \leq \mu \leq \sigma_2 \;; \\ -\frac{(2-\lambda)(3-\lambda)B_2}{6(3-\psi_3)} + \frac{(2-\lambda)^2 \mu B_1^2}{4(2-\psi_2)^2} - \frac{(2-\lambda)(3-\lambda)\psi_2 B_1^2}{6(3-\psi_3)(2-\psi_2)} & if \;\; \mu \geq \sigma_2, \end{cases}$$

where

$$\sigma_1 := \frac{4(3-\lambda)(2-\psi_2)\{(B_2-B_1)(2-\psi_2)+\psi_2 B_1^2\}}{6(2-\lambda)(3-\psi_3)B_1^2},$$
$$\sigma_2 := \frac{4(3-\lambda)(2-\psi_2)\{(B_2+B_1)(2-\psi_2)+\psi_2 B_1^2\}}{6(2-\lambda)(3-\psi_3)B_1^2}.$$

The result is sharp.

Corollary 3.4. *Let* $g(z) = z + \sum_{n=2}^{\infty} g_n z^n, (g_n > 0)$ *and let the function* $\phi(z)$ *be given by* $\phi(z) = 1 + \sum_{k=1}^{\infty} B_k z^k$*. If* $f(z)$ *given by (1.1) belongs to* $C_s^{(k)}(\phi)$*, then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{(2-\lambda)(3-\lambda)B_2}{18(3-\psi_3)} - \frac{(2-\lambda)^2\mu B_1^2}{16(2-\psi_2)^2} + \frac{(2-\lambda)3-\lambda)\psi_2 B_1^2}{18(3-\psi_3)(2-\psi_2)} & if \;\; \mu \leq \sigma_1 \; ; \\ \frac{(2-\lambda)(3-\lambda)B_1}{18(3-\psi_3)} & if \;\; \sigma_1 \leq \mu \leq \sigma_2 \; ; \\ -\frac{(2-\lambda)(3-\lambda)B_2}{18(3-\psi_3)} + \frac{(2-\lambda)^2\mu B_1^2}{16(2-\psi_2)^2} - \frac{(2-\lambda)3-\lambda)\psi_2 B_1^2}{18(3-\psi_3)(2-\psi_2)} & if \;\; \mu \geq \sigma_2, \end{cases}$$

where

$$\sigma_1 := \frac{16(3-\lambda)(2-\psi_2)\{(B_2-B_1)(2-\psi_2)+\psi_2 B_1^2\}}{18(2-\lambda)(3-\psi_3)B_1^2},$$
$$\sigma_2 := \frac{16(3-\lambda)(2-\psi_2)\{(B_2+B_1)(2-\psi_2)+\psi_2 B_1^2\}}{18(2-\lambda)(3-\psi_3)B_1^2}.$$

The result is sharp.

Remark 3.1. When $k = 2$ the above theorems reduces to a recent result of Shanmugam et al. [9].

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