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EQUILIBRIA AND PERIODIC SOLUTIONS OF PROJECTED DYNAMICAL SYSTEMS ON SETS WITH CORNERS

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ABSTRACT. Projected dynamical systems theory represents a bridge between the static worlds of variational inequalities and equilibrium problems, and the dynamic world of ordinary differential equations. A projected dynamical system (PDS) is given by the flow of a projected differential equation, an ordinary differential equation whose trajectories are restricted to a constraint set K. Projected differential equations are defined by discontinuous vector fields and so standard differential equations theory cannot apply. The formal study of PDS began in the 90's, although some results existed in the literature since the 70's. In this paper we present a novel result regarding existence of equilibria and periodic cycles of a finite dimensional PDS on constraint sets K, whose points satisfy a corner condition. The novelty is due to proving existence of boundary equilibria without using a variational inequality approach or monotonicity type conditions.

Key words and phrases: Projected dynamical systems, Projected differential equations, Variational inequalities, Periodic solutions, Differential inclusions.

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1. INTRODUCTION

A projected dynamical system is given by the flow of a projected differential equation [7], an ordinary differential equation whose trajectories are restricted to some constraint set K. Such systems naturally arise when optimization and equilibrium problems are placed in a dynamical context as is often dictated in economics, networking, engineering and other applications (see [4, 16, 17] and the references therein). In particular, the discipline owes much of its development to the study of variational inequality (VI) problems, which has been ongoing since the 1960s [2, 10, 13, 14, 15].

The formal study of projected dynamical systems began in the early 1990s on Euclidean spaces, with papers investigating the existence of solutions to initial value problems of projected differential equations over a variety of constraint sets K [8, 9]. The study of projected differential equations is complicated by the discontinuity of the vector field along the boundary of the constraint set K, which means that much of standard ordinary differential equations theory does not apply. The study has been extended to Hilbert spaces (see [4, 5, 7, 11, 12]). There are existence and uniqueness results for solutions to such equations in the class of absolutely continuous functions over the interval $[0, \infty)$. The study of critical points for projected dynamical systems, including their stability analyses under various monotonicity conditions, has similarly been conducted in both finite ($\mathbb{R}^n$) and infinite dimensional (Hilbert) spaces (see [6, 11, 17]).

In general, the existence of equilibrium points of a PDS is obtained via the well-developed theory of VIs, due to the fact that all critical points of a PDS can be obtained by solving an associated VI. The main result of this paper, however, establishes the existence of boundary equilibrium points for a finite-dimensional PDS without the use of a VI problem, or the use of monotonicity conditions, or of Lyapunov-type constructions. Instead, we use the geometry of the constraint set to obtain the result.

The study of periodic behaviour for PDS began only recently with the publication of [5], which considered the existence of periodic cycles under monotonicity conditions; however, similar results exist in related literature [1]. Here we are interested in finding other approaches to study the existence of such cycles. In this paper we consider conditions on the constraint set K under which periodic solutions may not pass through a corner point. We note that sets with corner points arise frequently in applications. Nonnegativity of investments/resources requires the set $\mathbb{R}^n_+$. Other constraint sets with corners can appear in game theory, when strategies are restricted to some form of hypercube [3], or in linear programming problems where the corners arise at the intersection of linear constraints.

The paper is organized as follows. In Section 2, we present a brief overview of convex cones, projection operators and projected dynamical systems. In Section 3, we outline a few useful results from differential inclusion theory as they relate to our consideration of projected dynamical systems. In Section 4, we state and prove the main results of the paper, and provide an illustrative example. In Section 5, we give a few concluding remarks and possible avenues for future work.

We note that, while many of the definitions and results used in this paper can be formulated in spaces other than $\mathbb{R}^n$, we limit our considerations here to this space. As such, we let $\|\cdot\|$ denote the standard Euclidean norm and $\langle \cdot, \cdot \rangle$ denote the standard Euclidean inner product.

2. CONVEX CONES, PROJECTION OPERATORS AND PROJECTED DYNAMICAL SYSTEMS

In order to define a projected dynamical system, we assume the reader is familiar with the concepts of closed convex subsets and cones in $\mathbb{R}^n$ (see, for example, [1]). Given a nonempty, closed and convex set K, we can associate to each point $x^* \in K$ two closed and convex cones which are crucial to our understanding of projected dynamical systems: the *tangent cone* and the *normal cone*.

Definition 2.1. The **tangent cone** (or **contingent cone**) to a point $x^* \in K$ is defined to be

$$T_K(x^*) = \overline{\bigcup_{\delta>0} \frac{1}{\delta}(K - x^*)}.$$

Definition 2.2. The **normal cone** to a point $x^* \in K$ is defined to be

$$N_K(x^*) = \{y \in \mathbb{R}^n \mid \langle y, x - x^* \rangle \leq 0, \ \forall\, x \in K\}.$$

It is known that if K is closed and convex then these cones are polar to one another, i.e., $\forall\, x \in T_K(x^*), y \in N_K(x^*), \langle x, y \rangle \leq 0$. We now introduce the *projection operator* and *vector projection operator*, which are the tools by which solutions are restricted to our constraint set K.

Definition 2.3. The **projection operator**, or **closest element mapping**, $P_K : \mathbb{R}^n \to K$ is given by $P_K(x) \in K$ such that

$$\|x - P_K(x)\| \leq \|x - y\|, \ \forall\, y \in K.$$

While the projection operator is not differentiable in the classical sense, it does permit a one-sided directional derivative given by:

Definition 2.4. The **vector projection** of a vector $v \in \mathbb{R}^n$ at a point $x \in K$ onto K is defined to be

$$\Pi_K(x, v) = \lim_{\delta \to 0^+} \frac{P_K(x + \delta v) - x}{\delta}. \tag{2.1}$$

Application of this concept is significantly simplified by the following result, which can be found in [19], Lemma 4.6:

Theorem 2.1. *Let $K \subset \mathbb{R}^n$ be a closed and convex set and $v \in \mathbb{R}^n$. Then for every $x \in K$*

$$\Pi_K(x, v) = P_{T_K(x)}(v).$$

We now define a projected differential equation and a projected dynamical system. We note that the accepted convention in the literature is to use $-F$ to denote the vector field, which is a consequence of how projected dynamical systems have arisen with respect to variational inequality theory. This is the convention that will be used throughout this paper.

Definition 2.5. Let $K \subset \mathbb{R}^n$ be a closed and convex set and $-F : K \to \mathbb{R}^n$ be a vector field. Then the **projected differential equation** associated with K and $-F$ is given by

$$\frac{dx(t)}{dt} = \Pi_K(x(t), -F(x(t))). \tag{2.2}$$

To any such equation we can associate the ***initial value problem*** (IVP)

$$\frac{dx(t)}{dt} = \Pi_K(x(t), -F(x(t))), \ x(0) = x_0 \in K. \tag{2.3}$$

By a solution to (2.3) we mean a solution in the following sense.

Definition 2.6. An absolutely continuous function $x : [0, \infty) \to \mathbb{R}^n$ such that $x(t) \in K$ for all $t \in [0, \infty)$ and $\dfrac{dx(t)}{dt} = \Pi_K(x(t), -F(x(t)))$, for almost all $t \in [0, \infty)$, is called a **solution** to the initial value problem (2.3).

Supposing that solutions to (2.3) exist on $\mathbb{R}_+$ and are uniquely defined for each initial point $x(0) = x_0$, then we define a PDS as follows:

Definition 2.7. A **projected dynamical system** is given by a mapping $\Phi : K \times \mathbb{R}_+ \to K$ that solves the initial value problem

$$\dot{\Phi}_x(t) = \Pi_K(\Phi_x(t), -F(\Phi_x(t))),\ \Phi_x(0) = x \in K$$

where we take $\Phi_x(t) := \Phi(x, t)$.

We now state an important existence and uniqueness result for solutions of PDS (see [7] for a proof):

Theorem 2.2. *Let X be a Hilbert space of arbitrary dimension and $K \subset X$ be a non-empty, closed and convex subset. Let $-F : K \to X$ be a Lipschitz continuous vector field and $x_0 \in K$. Then the initial value problem*

$$\frac{dx(t)}{dt} = \Pi_K(x(t), -F(x(t))),\ x(0) = x_0$$

has a unique absolutely continuous solution on the interval $[0, \infty)$.

As we consider in this paper only finite-dimensional Euclidean spaces, we take $X := \mathbb{R}^n$.

It is known that the critical points of projected differential equations can be classified in the following way:

Proposition 2.3. *The points $x^* \in K$ satisfying*

$$\Pi_K(x^*, -F(x^*)) = 0. \tag{2.4}$$

coincide with the points $x^* \in K$ satisfying

$$-F(x^*) \in N_K(x^*). \tag{2.5}$$

We note that, since $0 \in N_K(x)$ regardless of the choice of $x \in K$, it is always true that if the vector field $-F$ vanishes at a point $x^* \in K$ then x^* is a critical point. In general, in the PDS literature the existence of critical points as in Proposition 2.3 is obtained via the following result (see [5, 12, 17] for proofs):

Theorem 2.4. *The critical points of (2.2) coincide with the solutions of the variational inequality problem:*

$$\textit{find } x^* \in K \textit{ s.t. } \langle F(x^*), y - x^* \rangle \geq 0,\ \forall\, y \in K.$$

VI theory has numerous existence results for solutions of such problems. The existence of critical points of PDS follows as a result of Theorem 2.4. In this paper, however, we ask the question of finding different criteria for existence of boundary critical points of PDS without involving the use of VI theory. We answer this question by considering the geometry of the constraint set K.

3. DIFFERENTIAL INCLUSIONS AND VIABLE TRAJECTORIES

In applications, it is often useful to consider the derivative (or the "rate of change" of a process) to be an element of a set rather than to equal an explicit expression. Such a consideration gives rise mathematically to what is known as a *differential inclusion*, a field with substantial mathematical literature to date (for a comprehensive introduction to differential inclusions and set-valued mappings, see [1] and the references therein). In this section, we show in brief how a PDS is naturally related to a differential inclusion, and how this relation helps in the study of a projected dynamics. We state here only those definitions and results which are directly relevant to the topic at hand.

We consider throughout the following differential inclusion:

$$\dot{x}(t) \in -F(x(t)) - \tilde{N}_K(x(t)), \text{ a.a. } t \in [0, \infty), \tag{3.1}$$

where

$$\tilde{N}_K(x) = \{n \in N_K(x) \mid \|n\| \leq \|F(x)\|\}. \tag{3.2}$$

It is known that the solution set of this inclusion contains the specific solution to Equation (2.2) [1]. More explicitly the following theorem follows from a theorem in [1], Ch. 5, taking $-F(x)$ to be a singleton.

Theorem 3.1. *Let $K \subset \mathbb{R}^n$ be a nonempty, closed and convex subset. Then the solutions to the initial value problem (2.3) coincide with the viable solutions to the differential inclusion (3.1).*

We notice by Theorem 2.2 that if $-F$ is Lipschitz continuous then the solution to (2.3) is in fact unique. This result therefore implies the uniqueness of any viable solution of (3.1). (The definition of a *viable trajectory* is given below in Definition 3.2.)

Definition 3.1. Let $A, B \subseteq \mathbb{R}^n$. A **set-valued mapping** $G : A \to 2^B$ is a map that associates to any $x \in A$ a subset $G(x) \subseteq B$, where by 2^B we denote the collection of subsets of B.

Remark 3.1. We are often interested in constructing ϵ-neighbourhoods around sets rather than singletons. Given a non-empty subset $D \subset \mathbb{R}^n$, we take

$$B(D, \epsilon) := \bigcup_{x \in D} B(x, \epsilon)$$

where by $B(x, \epsilon)$ we denote the open ball of radius ϵ centered at $x \in \mathbb{R}^n$.

The concepts of upper hemicontinuity and upper semicontinuity are well-known for set-valued mappings and are related by the following theorem (see [1], Ch. 1).

Theorem 3.2. *Let $A, B \subseteq \mathbb{R}^n$ and $G : A \to 2^B$ be a set-valued mapping such that $G(x)$ is convex and compact for each $x \in A$. Then if G is upper hemicontinuous at x it is upper semicontinuous at x.*

The following results give properties of the set-valued mapping $x \mapsto -F(x) - \tilde{N}_K(x)$ of the differential inclusion above (see [7] for a proof):

Theorem 3.3. *Let $K \subset \mathbb{R}^n$ be a non-empty, closed and convex subset and $-F : K \to \mathbb{R}^n$ be a Lipschitz continuous vector field with Lipschitz constant b. Let $L > 0$ and $x_0 \in K$ arbitrarily fixed. Then the set-valued mapping $x \mapsto -F(x) - \tilde{N}_K(x)$*

(1) *is upper hemicontinuous on $K \cap B(x_0, L)$;*
(2) *has non-empty, compact and convex values for each $x \in K$.*

Using Theorems 3.2 and 3.3 we obtain:

Theorem 3.4. *Assume the hypotheses of Theorem 3.3. Then the set-valued mapping $x \mapsto -F(x) - \tilde{N}_K(x)$ is upper semicontinuous on $K \cap B(x_0, L)$.*

Remark 3.2. We notice that if, in addition to the hypotheses of Theorem 3.3, $K \subset \mathbb{R}^n$ is bounded then the set-valued mapping $x \mapsto -F(x) - \tilde{N}_K(x)$ is upper hemicontinuous and upper semicontinuous on K since we can take $x_0 \in K$ fixed and $L > 0$ large enough so that $K \subset B(x_0, L)$.

The central concept we will be interested in with regards to the inclusion (3.1) is that of *viability*, which is a generalization of the notion of a solution (see Definition 2.6) to the case of a set-valued vector field.

Definition 3.2. Let $K \subset \mathbb{R}^n$ be a non-empty subset and $G : K \to 2^{\mathbb{R}^n}$ be a set-valued mapping. An absolutely continuous trajectory $x(t)$ of the differential inclusion $\dot{x}(t) \in G(x(t))$ is **viable** on K if

$$x(t) \in K, \ \forall\, t \in [0, \infty).$$

The book [1] presents a number of results regarding the existence of viable trajectories. The following is the one we use ([1], Ch. 4).

Theorem 3.5. *Let $K \subset \mathbb{R}^n$ be a nonempty and compact subset and $G : K \to 2^{\mathbb{R}^n}$ be a proper upper hemicontinuous map with compact, convex values. Suppose that for every $x \in K$*

$$G(x) \cap T_K(x) \neq \emptyset.$$

Then for every $x_0 \in K$, there exists a viable trajectory of the differential inclusion $\dot{x}(t) \in G(x), x(0) = x_0$, defined on $[0, \infty)$.

We are interested in how this result applies to the set $G(x) := -F(x) - \tilde{N}_K(x)$. The following is essential to our application of Theorem 3.5 during the proof of the main result of this paper.

Proposition 3.6. *Let $K \subset \mathbb{R}^n$ be a non-empty, closed and convex subset. Then, $\forall\, x \in K$,*

$$[-F(x) - \tilde{N}_K(x)] \cap T_K(x) \neq \emptyset. \tag{3.3}$$

Proof. From Propositions 2 and 3, Ch. 0, of [1], we have that

$$P_{T_K(x)}(-F(x)) + P_{N_K(x)}(-F(x)) = -F(x)$$

with

$$\|P_{T_K(x)}(-F(x))\|^2 + \|P_{N_K(x)}(-F(x))\|^2 = \|F(x)\|^2.$$

The latter implies that $\|P_{N_K(x)}(-F(x))\| \leq \|F(x)\|$, which by definition implies that $P_{N_K(x)}(-F(x)) \in \tilde{N}_K(x)$. Since $P_{T_K(x)}(-F(x)) \in T_K(x)$ trivially, the result follows. □

4. PERIODIC CYCLES ON SETS WITH CORNERS

In this section we consider $K \subset \mathbb{R}^n$ to be a non-empty, compact and convex set which contains some point(s) $x^* \in K$ satisfying the condition

$$\forall\, v_1, v_2 \in T_K(x^*), \langle v_1, v_2 \rangle \geq 0. \tag{4.1}$$

In the most general terms, this condition is a measure of the sharpness of a corner point. In this section we state and prove several properties of systems on constraint sets with points satisfying this condition (Lemma 4.2 and Theorem 4.3). The section concludes with a result (Corollary 4.5) regarding the nature of periodic orbits as they relate to such points.

We note that sets with corner points satisfying this condition arise frequently in applications. Nonnegativity of investments/resources, for example, often dictates such a condition at the origin of the constraint set $K = \mathbb{R}^n_+$. The condition is also satisfied in game theory when strategies are typically restricted to some form of hypercube (i.e. $K = [0, 1]^n$ where n is the number of players [3]). It is also applicable to linear programming problems where the condition can arise at the intersection of linear constraints.

The proof of the main result of this paper (Theorem 4.3) requires several classical results. We state Theorem 4.1 in a formulation most suitable for our purposes (see [18]).

Theorem 4.1. [Hyperplane Separation Theorem] *Let C_1 and C_2 be closed convex cones in $\mathbb{R}^n$. Then C_1 and C_2 intersect only at the origin and C_1 does not contain a line through the origin if and only if there exists an $\alpha \in \mathbb{R}^n$, $\alpha \neq 0$ such that*

$$\langle \alpha, x \rangle > 0, \ \forall\, x \in C_1, x \neq 0, \text{ and}$$
$$\langle \alpha, y \rangle \leq 0, \ \forall\, y \in C_2.$$

The proof of the main result utilizes the following lemma.

Lemma 4.2. *Let $K \subset \mathbb{R}^n$ be a nonempty, closed and convex subset such that there exists $x^* \in K$ satisfying condition (4.1). Then*

$$-T_K(x^*) \subseteq N_K(x^*). \tag{4.2}$$

Proof. By assumption, $\forall\, v_1, v_2 \in T_K(x^*)$, $\langle v_1, v_2 \rangle \geq 0$. This implies $\langle -v_1, v_2 \rangle \leq 0$. Recalling the definition of the tangent cone (Definition 2.1), we substitute $v_2 := x - x^*$ (where $x \in K$ is chosen arbitrarily) to get $\langle -v_1, x - x^* \rangle \leq 0$. Since this applies for all $x \in K$, by the definition of the normal cone (Definition 2.2), we have $-v_1 \in N_K(x^*)$. Since $-v_1 \in -T_K(x^*)$ was chosen arbitrarily, $-T_K(x^*) \subseteq N_K(x^*)$. □

We now state and prove the main result of this section. This theorem states that if a trajectory reaches a sufficiently sharp corner point, as defined by condition (4.1), then the corner point is an equilibrium for this trajectory. The section concludes with Corollary 4.5 which relates this theorem to periodic cycles of projected dynamical systems.

Theorem 4.3. *Let $K \subset \mathbb{R}^n$ be a nonempty, compact and convex subset and $-F : K \to \mathbb{R}^n$ be a Lipschitz continuous vector field. Suppose there exists $x^* \in K$ satisfying condition (4.1) and a solution to the IVP (2.3), denoted $x(t)$, such that $x(0) \neq x^*$ and $x(t^*) = x^*$ for some $t^* > 0$. Then $x(t) = x^*$, $\forall\, t > t^*$.*

Proof. Since $x(t) = x^*$, $\forall\, t > t^*$ implies $-F(x^*) \in N_K(x^*)$ by Proposition 2.3, this result is equivalent to showing that if $-F(x^*) \notin N_K(x^*)$ then there exists no trajectory originating in $K \setminus \{x^*\}$ that reaches x^* in finite time.

We will assume $-F(x^*) \notin N_K(x^*)$. This implies that

$$N_K(x^*) \cap [-F(x^*) - \tilde{N}_K(x^*)] = \emptyset. \tag{4.3}$$

To see that this is the case, assume the contrary. If $N_K(x^*) \cap [-F(x^*) - \tilde{N}_K(x^*)] \neq \emptyset$ we have that $\exists\, n_1, n_2 \in N_K(x^*)$ (since $\tilde{N}_K(x^*) \subset N_K(x^*)$) such that $n_1 = -F(x^*) - n_2$, which implies $n_1 + n_2 = -F(x^*)$. However, since $N_K(x^*)$ is a cone, this implies $-F(x^*) \in N_K(x^*)$, which contradicts our assumption.

From Lemma 4.2 we have that $-T_K(x^*) \subseteq N_K(x^*)$. This implies by Equation (4.3) that

$$-T_K(x^*) \cap [-F(x^*) - \tilde{N}_K(x^*)] = \emptyset. \tag{4.4}$$

Since $-T_K(x^*)$ and $-F(x^*) - \tilde{N}_K(x^*)$ are disjoint, both sets are closed, and $-F(x^*) - \tilde{N}_K(x^*)$ is bounded, we have that $\exists\, \epsilon_1 > 0$ such that

$$-T_K(x^*) \cap \overline{B(-F(x^*) - \tilde{N}_K(x^*), \epsilon_1)} = \emptyset. \tag{4.5}$$

Now consider the sets

$$-T_K(x^*) \text{ and } C := \bigcup_{h \geq 0} h\overline{B(-F(x^*) - \tilde{N}_K(x^*), \epsilon_1)}.$$

The mapping $-F(x^*) - \tilde{N}_K(x^*)$ is non-empty, closed and convex by Theorem 3.3, which implies that $\overline{B(-F(x^*) - \tilde{N}_K(x^*), \epsilon_1)}$ is non-empty, closed and convex. We therefore have that C is a closed convex cone. We also see that for any fixed $h > 0$, $-T_K(x^*)$ and $h\overline{B(-F(x^*) - \tilde{N}_K(x^*), \epsilon_1)}$ are disjoint since $-T_K(x^*)$ is a convex cone, and $-T_K(x^*)$ does not contain a line through the origin since for all $v_1, v_2 \in T_K(x^*)$ we have $\langle v_1, v_2 \rangle \geq 0$. We can therefore apply the Hyperplane Separation Theorem (Theorem 4.1) to show that there exists an $\alpha \in \mathbb{R}^n$, $\alpha \neq 0$ such that

$$\langle \alpha, -v \rangle > 0, \ \forall \ -v \in -T_K(x^*), -v \neq 0, \text{ and} \tag{4.6}$$

$$\langle \alpha, hB(-F(x^*) - n, \epsilon_1) \rangle \leq 0, \ \forall \ n \in \tilde{N}_K(x^*), \ \forall \ h \geq 0, \tag{4.7}$$

where we have omitted the points on the boundary of $B(-F(x^*) - \tilde{N}_K(x^*), \epsilon_1)$ for simplicity.

Since $K \subset \mathbb{R}^n$ is non-empty, compact and convex, by Theorem 3.4 $-F(x) - \tilde{N}_K(x)$ is upper semicontinuous on K. This means that $\forall \ \epsilon > 0, \exists \ \delta > 0$ such that $\forall \ y \in B(x, \delta)$, $-F(y) - \tilde{N}_K(y) \subset B(-F(x) - \tilde{N}_K(x), \epsilon)$. Taking $\epsilon := \epsilon_1$ to be as in Equation (4.5) we have that $\exists \ \delta_1 > 0$ such that $\forall \ y \in K \cap B(x^*, \delta_1)$, $-F(y) - \tilde{N}_K(y) \subset B(-F(x^*) - \tilde{N}_K(x^*), \epsilon_1)$. By Equation (4.7), after multiplying through by h, we have that, $\forall \ y \in K \cap B(x^*, \delta_1)$,

$$\langle \alpha, -F(y) - n \rangle \leq 0, \ \forall \ n \in \tilde{N}_K(y). \tag{4.8}$$

We can rewrite Equation (4.6) as $\langle \alpha, v \rangle < 0, \ \forall \ v \in T_K(x^*), v \neq 0$. Taking $v := x - x^* \in T_K(x^*)$ we have that, for every $x \in K \setminus \{x^*\}$,

$$\langle \alpha, x - x^* \rangle < 0. \tag{4.9}$$

We now take $\tilde{x} \in K \setminus \{x^*\}$ and consider the set $\tilde{K} \subset K$ of the form

$$\tilde{K} = \{x \in K \mid \langle \alpha, x - \tilde{x} \rangle \leq 0\}. \tag{4.10}$$

We may view $\tilde{K}$ as K with the corner containing x^* cut out, since $\tilde{K} = K \setminus \{x \in K \mid \langle \alpha, x - \tilde{x} \rangle > 0\}$. We would like Equation (4.8) to hold along $\{x \in K \mid \langle \alpha, x - \tilde{x} \rangle = 0\}$, so we have to guarantee that this slice can occur within the neighbourhood $B(x^*, \delta_1)$. For the application of the viability theorems used later in this proof, we also need this slice to be able to occur arbitrarily close to x^*. We achieve both of these objectives through the following Proposition, which uses the same definitions and assumptions as outlined thus far in the proof.

Proposition 4.4. *For every $\delta > 0$, we can choose a $\tilde{x} \in K \setminus \{x^*\}$ such that*

$$\{x \in K \mid \langle \alpha, x - \tilde{x} \rangle = 0\} \subset B(x^*, \delta).$$

Proof. Suppose otherwise. This implies that $\exists \ \delta > 0$ such that $\forall \ \tilde{x} \in K \setminus \{x^*\}$, $\exists \ x \in K$ such that $\langle \alpha, x - \tilde{x} \rangle = 0$ but $x \notin B(x^*, \delta)$.

Since this must hold for all $\tilde{x} \in K \setminus \{x^*\}$, we consider a sequence $\{\tilde{x}_n\}$ such that $\tilde{x}_n \in K$ for all $n > 0$ and $\tilde{x}_n \to x^*$ as $n \to \infty$. For each such $\tilde{x}_n$, there is a corresponding $x_n \in K$ such that $\langle \alpha, x_n - \tilde{x}_n \rangle = 0$ and $\|x_n - x^*\| \geq \delta$ by assumption.

As $n \to \infty$ we have $\langle \alpha, \tilde{x}_n \rangle \to \langle \alpha, x^* \rangle$ which implies $\langle \alpha, x_n \rangle \to \langle \alpha, x^* \rangle$. However, since K is a bounded set, the Bolzano-Weierstrass Theorem implies that the sequence $\{x_n\}$ has a subsequence (which we will also denote by $\{x_n\}$) that converges. Since K is closed, this subsequence converges to an element $x \in K$. As $n \to \infty$ we have that $\|x_n - x^*\| \to \|x - x^*\| \geq \delta > 0$ and $\langle \alpha, x_n \rangle \to \langle \alpha, x \rangle = \langle \alpha, x^* \rangle$ where $x \in K$, i.e. there is a point $x \in K$, $x \neq x^*$ such that $\langle \alpha, x - x^* \rangle = 0$. This contradicts Equation (4.9). Therefore, for every $\delta > 0$ we can choose $\tilde{x} \in K$ such that $\{x \in K \mid \langle \alpha, x - \tilde{x} \rangle = 0\} \subset B(x^*, \delta)$.

□

We now want to investigate the relationship between $-F(x) - \tilde{N}_K(x)$ and $T_{\tilde{K}}(x)$ for $x \in \tilde{K}$ based on our knowledge of $T_K(x)$ for $x \in K$. Since $\tilde{K}$ is the intersection of two convex sets, it is convex, and therefore the cone $T_{\tilde{K}}(x)$ is well defined. We know that $[-F(x) - \tilde{N}_K(x)] \cap T_K(x) \neq \emptyset$ for all $x \in K$ from Equation (3.3). We notice that for every $x \in \tilde{K}$ such that $\langle \alpha, x - \tilde{x} \rangle < 0$ we have that $T_{\tilde{K}}(x) = T_K(x)$ and therefore

$$[-F(x) - N_K(x)] \cap T_{\tilde{K}}(x) \neq \emptyset, \ \forall \ x \in \tilde{K} \text{ s.t. } \langle \alpha, x - \tilde{x} \rangle < 0.$$

We now consider $x \in \tilde{K}$ such that $\langle \alpha, x - \tilde{x} \rangle = 0$. We see that for any such x, $T_{\tilde{K}}(x) = \{v \in T_K(x) \mid \langle \alpha, v \rangle \leq 0\}$. From Equation (4.8) we have that $\exists\ \delta_1 > 0$ such that for all $y \in K \cap B(x^*, \delta_1)$, $\langle \alpha, -F(y) - n \rangle \leq 0$ for all $n \in \tilde{N}_K(y)$, and from Proposition 4.4, choosing $\delta < \delta_1$, we have that we can choose $\tilde{x} \in K$ such that $\{x \in K \mid \langle \alpha, x - \tilde{x} \rangle = 0\} \subset B(x^*, \delta_1)$. From Equation (3.3) we therefore have that

$$[-F(x) - \tilde{N}_K(x)] \cap T_{\tilde{K}}(x) \neq \emptyset,\ \forall\ x \in \tilde{K} \text{ s.t. } \langle \alpha, x - \tilde{x} \rangle = 0.$$

Together these results imply that, for an appropriate choice of $\tilde{x}$,

$$[-F(x) - \tilde{N}_K(x)] \cap T_{\tilde{K}}(x) \neq \emptyset,\ \forall\ x \in \tilde{K}. \tag{4.11}$$

We notice that $\tilde{K}$ is compact since it is the intersection of a compact set, K, and a closed set, $\{x \in \mathbb{R}^n \mid \langle \alpha, x - \tilde{x} \rangle \leq 0\}$. Since $-F(x) - \tilde{N}_K(x)$ is upper hemicontinuous on K by Theorem 3.3, it is upper hemicontinuous on $\tilde{K} \subset K$, and since Equation (4.11) holds for all $x \in \tilde{K}$, the Viability Theorem (Theorem 3.5) implies that there exists a viable trajectory on $\tilde{K}$ for all $x(0) \in \tilde{K}$. This trajectory is clearly also viable on K since $\tilde{K} \subset K$. By Theorem 3.1 we have that viable solutions correspond to solutions to the IVP (2.3), and by Theorem 2.2 we have that the solution is unique through every $x(0) \in \tilde{K}$.

Since by Proposition 4.4 we are free to choose $\tilde{x}$ in the definition of $\tilde{K}$ arbitrarily close to x^* while maintaining the property (4.11), we can construct sets $\tilde{K}$ which include any arbitrary point $x \in K$ but exclude x^*. This implies that every solution to the IVP (2.3) such that $x(0) \neq x^*$ must exclude x^* on the interval $[0, \infty)$. This is a contradiction, which implies that our assumption $-F(x^*) \notin N_K(x^*)$ was incorrect. Therefore $-F(x^*) \in N_K(x^*)$ and the result follows. □

An immediate consequence of Theorem 4.3 is the following Corollary related to periodic cycles.

Corollary 4.5. *Let $K \subset \mathbb{R}^n$ be a nonempty, compact and convex subset and $-F : K \to \mathbb{R}^n$ be a Lipschitz continuous vector field. Suppose there exists $x^* \in K$ satisfying condition (4.1). Then no periodic cycle can contain the point x^*.*

Proof. By Theorem 4.3 we have that any trajectory originating away from x^* that reaches x^*, at time $t^* > 0$, must remain at x^* for all time $t > t^*$. However, in order to have a periodic cycle through x^* there must be $t_1, t_2 > 0$ satisfying $t_1 < t^* < t_2$, such that $x(t_1) = x(t_2) \neq x^*$, which is a contradiction. The result follows. □

Application of Theorem 4.3/Corollary 4.5

Consider the system

$$\begin{cases} \dfrac{dx}{dt} &= x - ay \\ \dfrac{dy}{dt} &= ax + y,\ a > 0, \end{cases}$$

restricted to the set $K = \{(x, y) \in \mathbb{R}^2 \mid -1 \leq x \leq 1, -1 \leq y \leq 1\}$. The unconstrained system has eigenvalues $\lambda = 1 \pm ai$ which implies that the origin is a spiral source for all $a > 0$. Roughly speaking, the parameter a controls the tightness of spiraling.

Clearly, the four corner points satisfy condition (4.1) which implies by Theorem 4.3 that if a trajectory can reach a corner point, the corner point is a sink, and by Corollary 4.5 that if there is a periodic orbit, it excludes the corner points.

We know from Proposition 2.3 that a point is an equilibrium if and only if $-F(x) \in N_K(x)$. We consider $x^* = (1, 1)$ and see that $-F(x^*) = (1 - a, a + 1)$ and $N_K(x^*) = \{(x, y) \in$

$\mathbb{R}^2 \mid x \geq 0, y \geq 0\}$. This implies that $-F(x^*) \in N_K(x^*)$ for $0 < a \leq 1$. For $a > 1$ we have $-F(x^*) \notin N_K(x^*)$ and therefore from Theorem 4.3 that no trajectory originating away from x^* can reach x^*. The same condition can be easily obtained for each of the other three corner points.

In Figure 1 we can see that the system behaves as predicted. In the range $0 < a \leq 1$, trajectories fall into sinks at the corners; however, as a is scaled into the range $a > 1$, the corner points are no longer sinks, and the system gives rise to periodic orbits which do not enter the corners.

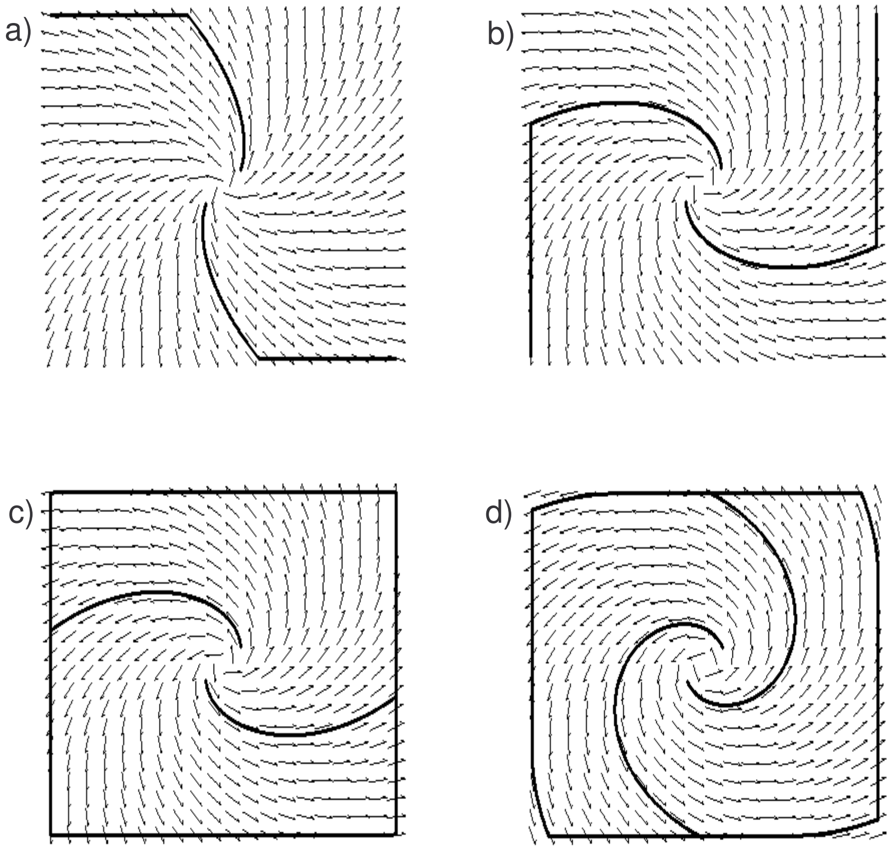

Figure 1: The system $\dot{x} = x - ay$, $\dot{y} = ax + y$ constrained to the set $K = \{(x, y) \in \mathbb{R}^2 | -1 \leq x \leq 1, -1 \leq y \leq 1\}$ with (a) $a = 0.5$, (b) $a = 1$, (c) $a = 1.1$, and (d) $a = 2$.

5. CONCLUSIONS AND FUTURE WORK

In this paper, we have presented a new result about existence of boundary equilibria and periodic cycles of projected dynamical systems. We have obtained this result without the aid of VI theory or monotonicity results.

The study of projected dynamics is in general made challenging by the discontinuity of the projected vector field on the boundary of the constraint set K. This discipline therefore requires alternative methodology, such as differential inclusions and convex and nonlinear analysis techniques, some of which were used in this work. The study of periodic cycles of projected dynamical systems has only been initiated recently and, consequently, there is significant potential for original research within this field. The key results contained in Section 4 provide many avenues for future work. We highlight two of these. The first one concerns finding an alternative approach which does not require boundedness of the set K. This would allow the result to apply to a wider class of constraint sets, in particular to the cone $\mathbb{R}^n_+$ which arises in many economic and finance related problems. The second one concerns the generalization of such results to spaces other than the finite dimensional space $\mathbb{R}^n$.

It is our feeling that, not only will progress in this field be useful in the theoretical development of projected dynamical systems, but it will also find application in fields such as economics, finance, game theory, epidemiology, etc., where periodic behaviour is commonly observed in the dynamics despite the imposition of a constraint set.

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REFERENCES

[1] J. P. AUBIN and A. CELLINA, *Differential Inclusions*, Springer-Verlag, Berlin, 1984.

[2] H. BREZIS, *Inequations d'Evolution Abstraites*, Comptes Rendus de l'Academie des Sciences, Paris, 1967.

[3] M. G. COJOCARU, C. BAUCH and M. JOHNSTON, Dynamics of vaccination strategies via projected dynamical systems, to appear in *Bull. Math. Biol.*, 2007.

[4] M. G. COJOCARU, P. DANIELE and A. NAGURNEY, Projected dynamical systems and evolutionary variational inequalities via Hilbert spaces and applications, *J. Optim. Theory Appl.*, **127** (2005), no. 3, pp. 549 - 563.

[5] M. G. COJOCARU, Monotonicity and existence of periodic cycles for projected dynamical dystems on hilbert spaces, *Proc. Amer. Math. Soc.*, **134** (2006), pp. 793-804.

[6] M. G. COJOCARU, *Projected Dynamical Systems on Hilbert Spaces*, Ph.D. Thesis, Queen's University, 2002.

[7] M. G. COJOCARU and L. B. JONKER, Projected differential equations in Hilbert spaces, *Proc. Amer. Math. Soc.*, **132** (2004), no. 1, pp. 183-193.

[8] P. DUPUIS and H. ISHII, On Lipschitz continuity of the solution mapping to the Skorokhod problem, with applications, *Stochastics*, **35** (1990), pp. 31-62.

[9] P. DUPUIS and A. NAGURNEY, Dynamical systems and variational inequalities, *Ann. Oper. Res.*, **44** (1993), pp. 9-42.

[10] G. ISAC, *Topological Methods in Complementarity Theory*, Kluwer Academic Publishers, 2000.

[11] G. ISAC and M. G. COJOCARU, The projection operator in a Hilbert space and its directional derivative. Consequences for the theory of projected dynamical systems, *J. Funct. Spaces Appl.*, **2** (2004), no. 1, pp. 71-95.

[12] G. ISAC and M. G. COJOCARU, Variational inequalities, complementarity problems and pseudo-monotonicity: Dynamical aspects, *Seminar on Fixed Point Theory Cluj-Napoca*, **3** (2002), pp. 41-62.

[13] D. KINDERLEHRER and G. STAMPACCHIA, *An Introduction to Variational Inequalities and Their Applications*, Academic Press, 1980.

[14] J. L. LIONS, and G. STAMPACCHIA, Variational Inequalities, *Comm. Pure Appl. Math.*, **22** (1967), pp. 493-519.

[15] G. J. MINTY, Monotone (non-linear) operators in Hilbert space, *Duke Math. J.*, **29** (1962), pp. 341-346.

[16] A. NAGURNEY, *Network Economics. A Variational Inequality Approach*, Kluwer Academic Publishers, 1993.

[17] A. NAGURNEY, and D. ZHANG, *Projected Dynamical Systems and Variational Inequalities with Applications*, Kluwer Academic Publishers, 1996.

[18] R. T. ROCKAFELLAR and R. J-B. WETS, *Variational Analysis*, Springer-Verlag, A Series of Comprehensive Studies in Mathematics, Vol. 317, 1998.

[19] E. ZARANTONELLO, Projections on convex sets in Hilbert space and spectral theory, *Contributions to Nonlinear Functional Analysis*, **27**, Math. Res. Center, Univ. Wisconsin, Academic Press (1971), pp. 237-424.