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SOME GENERALIZED DIFFERENCE SEQUENCE SPACES DEFINED BY ORLICZ FUNCTIONS

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ABSTRACT. In this paper, we define the sequence spaces: $[V, M, p, u, \Delta]$, $[V, M, p, u, \Delta]_0$ and $[V, M, p, u, \Delta]_\infty$, where for any sequence $x = (x_n)$, the difference sequence Δx is given by $\Delta x = (\Delta x_n)_{n=1}^{\infty} = (x_n - x_{n-1})_{n=1}^{\infty}$. We also study some properties and theorems of these spaces. These are generalizations of those defined and studied by Savas and Savas [10] and some others before.

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1. INTRODUCTION

Let X be a linear space. A function $p : X \to \mathbb{R}$ is called paranorm if the following are satisfied :

(i) $p(0) \geq 0$

(ii) $p(x) \geq 0$ for all $x \in X$

(iii) $p(x) = p(-x)$ for all $x \in X$

(iv) $p(x+y) \leq p(x) + p(y)$ for all $x \in X$ (triangle inequality)

(v) if (λ_n) is a sequence of scalars with $\lambda_n \to \lambda$ $(n \to \infty)$ and (x_n) is a sequence of vectors with $p(x_n - x) \to 0$ $(n \to \infty)$, then $p(\lambda_n x_n - \lambda x) \to 0$ $(n \to \infty)$ (continuity of multiplication by scalars).

A paranorm p for which $p(x) = 0$ implies $x = 0$ is called total. It is well known that the metric of any linear metric space is given by some total paranorm (cf.[11]).

Let $\Lambda = (\lambda_n)$ a nondecreasing sequence of positive reals tending to infinity and $\lambda_1 = 1$ and $\lambda_{n+1} \leq \lambda_n + 1$.

The generalized de la Vallėe-Poussin means is defined by :

$$t_n(x) = \frac{1}{\lambda_n} \sum_{k \in I_n} x_k,$$

where $I_n = [n - \lambda_n + 1, n]$. A sequence $x = (x_k)$ is said to be $(V, \lambda)-$summable to a number l (see [2]) if $t_n(x) \to l$, as $n \to \infty$.

We write

$$[V, \lambda]_0 = \{x = (x_k) : \lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} | x_k | = 0\}$$

$$[V, \lambda] = \{x = (x_k) : \lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} | x_k - le | = 0, \text{ for some } l \in \mathbb{C}\}$$

and

$$[V, \lambda]_\infty = \{x = (x_k) : \sup_n \frac{1}{\lambda_n} \sum_{k \in I_n} | x_k | < \infty\}.$$

For the set of sequences that are strongly summable to zero, strongly summable and strongly bounded by the de la Vallėe-Poussin method. If $\lambda_n = n$ for $n = 1, 2, 3, \cdots$, then these sets reduce to ω_0, ω and ω_∞ introduced and studied by Maddox [4].

Following Lidenstrauss and Tzafriri [3], we recall that an Orlicz function M is continuous, convex, nondecreasing function defined for $x \geq 0$ such that $M(0) = 0$ and $M(x) \geq 0$ for $x > 0$ (see [1]).

If convexity of M is replaced by $M(x+y) \leq M(x) + M(y)$, then it is called a modulus function, defined and studied by Nakano [7], Ruckle [9], Maddox [5] and others.

An Orlicz function M is said to satisfy the Δ_2-condition for all values of u, if there exist a constant $K > 0$ such that

$$M(2u) \leq KM(u) \; (u \geq 0).$$

It is easy to see that always $K > 2$. The Δ_2-condition is equivalent to the satisfaction of the inequality

$$M(lu) \leq KlM(u),$$

for all values of u and for $l > 1$.

Lidenstrauss and Tzafriri used the idea of Orlicz function to construct the Orlicz sequence space :

$$l_M := \{x = (x_k) : \sum_{k=1}^{\infty} M(\frac{| x_k |}{\rho}) < \infty, \text{ for some } \rho > 0\},$$

which is a Banach space with the norm :

$$\| x \|_M = \inf\{\rho > 0 : \sum_{k=1}^{\infty} M(\frac{| x_k |}{\rho}) \leq 1\}.$$

If $M(x) = x^p, 1 \leq p < \infty$, the space l_M coincide with the classical sequence space l_p.

Parashar and Choudhary [8] have introduced and examined some properties of four sequence spaces defined by using an Orlicz function M, which generalized the well-known Orlicz sequence space l_M and strongly summable sequence spaces $[C, 1, p], [C, 1, p]_0$ and $[C, 1, p]_\infty$.

Let M be an Orlicz function, $p = (p_k)$ be any sequence of strictly positive real numbers and $u = (u_k)$ be any sequence such that $u_k \neq 0 (k = 1, 2, \cdots)$. We define the following sequence spaces :

$$\begin{aligned}
[V, M, p, u, \Delta] &= \{x = (x_k) : \lim_n \frac{1}{\lambda_n} \sum_{k \in I_n}^{\infty} [M(\frac{| u_k \Delta x_k - le |}{\rho})] = 0, \text{ for some } l \\
&\quad \text{and } \rho > 0\} \\
[V, M, p, u, \Delta]_0 &= \{x = (x_k) : \lim_n \frac{1}{\lambda_n} \sum_{k \in I_n}^{\infty} [M(\frac{| u_k \Delta x_k |}{\rho})] = 0, \text{ for some } \rho > 0\} \\
[V, M, p, u, \Delta]_\infty &= \{x = (x_k) : \sup_n \frac{1}{\lambda_n} \sum_{k \in I_n}^{\infty} [M(\frac{| u_k \Delta x_k |}{\rho})] < \infty, \text{ for some } \rho > 0\}.
\end{aligned}$$

If $u = e$ and $\Delta x_k = x_k$ for all k, then these gives the spaces $[V, M, p], [V, M, p]_0$ and $[V, M, p]_\infty$ respectively defined and studied by Savas and Savas [10].

2. Main results

We prove the following theorems :

Theorem 2.1. *For any Orlicz function M and any sequence $p = (p_k)$ of strictly positive real numbers, $[V, M, p, u, \Delta], [V, M, p, u, \Delta]_0$ and $[V, M, p, u, \Delta]_\infty$ are linear spaces over the set of complex numbers.*

Proof. We shall prove only for $[V, M, p, u, \Delta]_0$. The others can be treated similarly. Let $x, y \in [V, M, p, u, \Delta]_0$ and $\alpha, \beta \in \mathbb{C}$. In order to prove the result, we need to find some $\rho_3 > 0$ such that :

$$\lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} [M(\frac{| \alpha u_k \Delta x_k + \beta u_k \Delta y_k |}{\rho_3})]^{p_k} = 0.$$

Since $x, y \in [V, M, p, u, \Delta]_0$, there exists some positive ρ_1 and ρ_2 such that :

$$\lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} [M(\frac{| u_k \Delta x_k |}{\rho_1})]^{p_k} = 0 \text{ and } \lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} [M(\frac{| u_k \Delta y_k |}{\rho_2})]^{p_k} = 0.$$

Define $\rho_3 = \max(2 \mid \alpha \mid \rho_1, 2 \mid \beta \mid \rho_2)$. Since M is nondecreasing and convex,

$$\begin{aligned} & \frac{1}{\lambda_n} \sum_{k \in I_n} [M(\frac{\mid \alpha u_k \Delta x_k + \beta u_k \Delta y_k \mid}{\rho_3})]^{p_k} \\ \leq & \frac{1}{\lambda_n} \sum_{k \in I_n} [M(\frac{\mid \alpha u_k \Delta x_k \mid}{\rho_3} + \frac{\mid \beta u_k \Delta y_k \mid}{\rho_3})]^{p_k} \\ \leq & \frac{1}{\lambda_n} \sum_{k \in I_n} \frac{1}{2^{p_k}} [M(\frac{\mid u_k \Delta x_k \mid}{\rho_1}) + M(\frac{\mid u_k \Delta y_k \mid}{\rho_2})]^{p_k} \\ \leq & \frac{1}{\lambda_n} \sum_{k \in I_n} [M(\frac{\mid u_k \Delta x_k \mid}{\rho_1}) + M(\frac{\mid u_k \Delta y_k \mid}{\rho_2})]^{p_k} \\ \leq & K. \frac{1}{\lambda_n} \sum_{k \in I_n} [M(\frac{\mid u_k \Delta x_k \mid}{\rho_1})]^{p_k} + K. \frac{1}{\lambda_n} \sum_{k \in I_n} [M(\frac{\mid u_k \Delta y_k \mid}{\rho_2})]^{p_k} \to 0, \end{aligned}$$

as $n \to \infty$, where $K = \max(1, 2^{H-1})$, $H = \sup p_k$, so that $\alpha x + \beta y \in [V, M, p, u, \Delta]_0$. This completes the proof.

∎

Theorem 2.2. *For any Orlicz function M and a bounded sequence $p = (p_k)$ of strictly positive real numbers, $[V, M, p, u, \Delta]_0$ is a total paranormed space with :*

$$g(x) = \inf\{\rho^{p_n/H} : (\frac{1}{\lambda_n} \sum_{k \in I_n} [M(\frac{\mid x_k \mid}{\rho})]^{p_k})^{1/H} \leq 1, \ n = 1, 2, 3, \cdots \},$$

where $H = \max(1, \sup p_k)$.

Proof. Clearly $g(x) = g(-x)$. By using Theorem 2.1, for $\alpha = \beta = 1$, we get $g(x+y) \leq g(x) + g(y)$. Since $M(0) = 0$, we get $\inf\{\rho^{p_n/H}\} = 0$ for $x = 0$. Conversely, suppose $g(x) = 0$, then :

$$\inf\{\rho^{p_n/H} : (\frac{1}{\lambda_n} \sum_{k \in I_n} [M(\frac{\mid x_k \mid}{\rho})]^{p_k})^{1/H} \leq 1\} = 0.$$

This implies that for a given $\epsilon > 0$, there exists some ρ_ϵ $(0 < \rho_\epsilon < \epsilon)$ such that :

$$(\frac{1}{\lambda_n} \sum_{k \in I_n} [M(\frac{\mid x_k \mid}{\rho_\epsilon})]^{p_k})^{1/H} \leq 1.$$

Thus,

$$(\frac{1}{\lambda_n} \sum_{k \in I_n} [M(\frac{\mid x_k \mid}{\epsilon})]^{p_k})^{1/H} \leq (\frac{1}{\lambda_n} \sum_{k \in I_n} [M(\frac{\mid x_k \mid}{\rho_\epsilon})]^{p_k})^{1/H} \leq 1,$$

for each n.

Suppose that $x_{n_m} \neq 0$ for some $m \in I_n$, then $(\frac{x_{n_m}}{\epsilon}) \to \infty$. It follows that :

$$(\frac{1}{\lambda_n} \sum_{k \in I_n} [M(\frac{\mid x_{n_m} \mid}{\epsilon})]^{p_k})^{1/H} \to \infty$$

which is a contradiction. Therefor $x_{n_m} = 0$ for all m. Finally we prove that scalar multiplication is continuous. Let μ be any complex number, then by definition,

$$g(\mu x) = \inf\{\rho^{p_n/H} : (\frac{1}{\lambda_n}\sum_{k\in I_n}[M(\frac{|\mu x_k|}{\rho})]^{p_k})^{1/H} \leq 1,\ n = 1,2,3,\cdots\}.$$

Then

$$g(\mu x) = \inf\{(|\mu| s)^{p_n/H} : (\frac{1}{\lambda_n}\sum_{k\in I_n}[M(\frac{|x_k|}{s})]^{p_k})^{1/H} \leq 1,\ n = 1,2,3,\cdots\},$$

where $s = \rho/|\mu|$. Since $|\mu|^{p_n} \leq \max(1, |\mu|^{\sup p_n})$, we have

$$g(\mu x) \leq (\max(1,|\mu|^{\sup p_n}))^{1/H}.\inf\{(s)^{p_n/H} : (\frac{1}{\lambda_n}\sum_{k\in I_n}[M(\frac{|x_k|}{s})]^{p_k})^{1/H} \leq 1,\ n = 1,2,3,\cdots\}$$

which converges to zero as x converges to zero in $[V,M,p,u,\Delta]_0$.

Now suppose $\mu_m \to 0$ and x is fixed in $[V,M,p,u,\Delta]_0$. For arbitrary $\epsilon > 0$, let N be a positive integer such that

$$\frac{1}{\lambda_n}\sum_{k\in I_n}[M(\frac{|x_k|}{\rho})]^{p_k} < (\epsilon/2)^H \text{ for some } \rho > 0 \text{ and all } n > N.$$

This implies that

$$\frac{1}{\lambda_n}\sum_{k\in I_n}[M(\frac{|x_k|}{\rho})]^{p_k} < \epsilon/2 \text{ for some } \rho > 0 \text{ and all } n > N.$$

Let $0 < |\mu| < 1$, using convexity of M, for $n > N$, we get

$$\frac{1}{\lambda_n}\sum_{k\in I_n}[M(\frac{|\mu x_k|}{\rho})]^{p_k} < \frac{1}{\lambda_n}\sum_{k\in I_n}[|\mu| M(\frac{|x_k|}{\rho})]^{p_k} < (\epsilon/2)^H .$$

Since M is continuous everywhere in $[0,\infty)$, then for $n \leq N$,

$$f(t) = \frac{1}{\lambda_n}\sum_{k\in I_n}[M(\frac{|tx_k|}{\rho})]^{p_k}$$

is continuous at zero. So there exists $1 > \delta > 0$ such that $|f(t)| < (\epsilon/2)^H$ for $0 < t < \delta$.

Let K be such that $|\mu_m| < \delta$ for $m > K$ and $n \leq N$, then

$$(\frac{1}{\lambda_n}\sum_{k\in I_n}[M(\frac{|\mu_m x_k|}{\rho})]^{p_k})^{1/H} < \epsilon/2.$$

Thus

$$(\frac{1}{\lambda_n}\sum_{k\in I_n}[M(\frac{|\mu_m x_k|}{\rho})]^{p_k})^{1/H} < \epsilon,$$

for $m > K$ and all n, so that $g(\mu x) \to 0$ $(\mu \to 0)$.

∎

Theorem 2.3. *For any Orlicz function M which satisfies the Δ_2-condition, we have $[V,\lambda,u,\Delta] \subseteq [V,M,u,\Delta]$, where*

$$[V,\lambda,u,\Delta] = \{x = (x_k) : \lim_n \frac{1}{\lambda_n}\sum_{k\in I_n} | u_k\Delta x_k - le | = 0,\ \textit{for some } l \in \mathbb{C}\}.$$

Proof. Let $x \in [V, \lambda, u, \Delta]$. Then

$$T_n = \frac{1}{\lambda_n} \sum_{k \in I_n} | u_k \Delta x_k - le | \to 0 \text{ as } n \to \infty, \text{ for some } l.$$

Let $\epsilon > 0$ and choose δ with $0 < \delta < 1$ such that $M(t) < \epsilon$ for $0 \leq t \leq \delta$. Write $y_k = | u_k \Delta x_k - le |$ and consider

$$\frac{1}{\lambda_n} \sum_{k \in I_n} M(| y_k |) = \sum_1 + \sum_2,$$

where the first summation over $y_k \leq \delta$ and the second over $y_k > \delta$. Since M is continuous,

$$\sum_1 < \lambda_n \epsilon$$

and for $y_k > \delta$, we use the fact that $y_k < y_k/\delta < 1 + y_k/\delta$. Since M is nondecreasing and convex, it follows that

$$M(y_k) < M(1 + \delta^{-1} y_k) < \frac{1}{2} M(2) + \frac{1}{2} M(2\delta^{-1} y_k).$$

Since M satisfies the Δ_2-condition, there is a constant $K > 2$ such that $M(2\delta^{-1} y_k) \leq \frac{1}{2} K \delta^{-1} y_k M(2)$, therefor

$$\begin{aligned} M(y_k) &< \frac{1}{2} K \delta^{-1} y_k M(2) + \frac{1}{2} K \delta^{-1} y_k M(2) \\ &= K \delta^{-1} y_k M(2). \end{aligned}$$

Hence

$$\sum_2 M(y_k) \leq K \delta^{-1} M(2) \lambda_n T_n$$

which together with $\sum_1 \leq \epsilon \lambda_n$ yields $[V, \lambda, u, \Delta] \subseteq [V, M, u, \Delta]$. This completes the proof.

The method of the proof of Theorem 2.3 shows that for any Orlicz function M which satisfies the Δ_2-condition, we have $[V, \lambda, u, \Delta]_0 \subseteq [V, M, u, \Delta]_0$ and $[V, \lambda, u, \Delta]_\infty \subseteq [V, M, u, \Delta]_\infty$, where

$$[V, \lambda, u, \Delta]_0 = \{x = (x_k) : \lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} | u_k \Delta x_k | = 0\},$$

$$[V, \lambda, u, \Delta]_\infty = \{x = (x_k) : \sup_n \frac{1}{\lambda_n} \sum_{k \in I_n} | u_k \Delta x_k | < \infty\}.$$

∎

Theorem 2.4. *Let* $0 \leq p_k \leq q_k$ *and* (q_k/p_k) *be bounded. Then* $[V, M, q, u, \Delta] \subset [V, M, p, u, \Delta]$

Proof. The proof of Theorem 2.4 used the ideas similar to those used in proving Theorem 7 of Parashar and Choudhary [8].

Mursaleen [6] introduced the concept of statistical convergence as follows :

A sequence $x = (x_k)$ is said to be $\lambda-$statistically convergent or $s_\lambda-$statistically convergent to L if for every $\epsilon > 0$,

$$\lim_n \frac{1}{\lambda_n} | \{k \in I_n : | x_k - L | \geq \epsilon\} | = 0,$$

where the vertical bars indicates the number of elements in the enclosed set. In this case we write $s_\lambda - \lim x = L$ or $x_k \to L\ (s_\lambda)$ and $s_\lambda = \{x : \exists L \in \mathbb{R}: s_\lambda - \lim x = L\}$.

In a similar way, we say that a sequence $x = (x_k)$ is said to be $(\lambda, u, \Delta)-$statistically convergent or $s_\lambda(u, \Delta)-$statistically convergent to L if for every $\epsilon > 0$,

$$\lim_n \frac{1}{\lambda_n} \mid \{k \in I_n : \mid u_k \Delta x_k - Le \mid \geq \epsilon\} \mid = 0,$$

where the vertical bars indicates the number of elements in the enclosed set. In this case we write $s_\lambda(u, \Delta) - \lim x = L$ or $u_k \Delta x_k \to Le\ (s_\lambda)$ and $s_\lambda(u, \Delta) = \{x : \exists L \in \mathbb{R}: s_\lambda - \lim x = L\}$.

∎

Theorem 2.5. *For any Orlicz function* M, $[V, M, u, \Delta] \subset s_\lambda(u, \Delta)$.

Proof. Let $x \in [V, M, u, \Delta]$ and $\epsilon > 0$. Then

$$\begin{aligned} \frac{1}{\lambda_n} \sum_{k \in I_n} M\left(\frac{\mid u_k \Delta x_k - le \mid}{\rho}\right) &\geq \frac{1}{\lambda_n} \sum_{k \in I_n, \mid u_k \Delta x_k - le \mid \geq \epsilon} M\left(\frac{\mid u_k \Delta x_k - le \mid}{\rho}\right) \\ &\geq \frac{1}{\lambda_n} M(\epsilon/\rho). \mid \{k \in I_n : \mid u_k \Delta x_k - le \mid \geq \epsilon\} \mid \end{aligned}$$

from which it follows that $x \in s_\lambda(u, \Delta)$.

To show that $s_\lambda(u, \Delta)$ strictly contain $[V, M, u, \Delta]$, we proceed as in [6]. We define $x = (x_k)$ by $(x_k) = k$ if $n - [\sqrt{\lambda_n}] + 1 \leq k \leq n$ and $(x_k) = 0$ otherwise. Then $x \notin l_\infty(u, \Delta)$ and for every ϵ $(0 < \epsilon \leq 1)$,

$$\frac{1}{\lambda_n} \mid \{k \in I_n : \mid u_k \Delta x_k - 0 \mid \geq \epsilon\} \mid = \frac{[\sqrt{\lambda_n}]}{\lambda_n} \to 0 \text{ as } n \to \infty$$

i.e. $x \to 0\ (s_\lambda(u, \Delta))$, where [ ] denotes the greatest integer function. On the other hand,

$$\frac{1}{\lambda_n} \sum_{k \in I_n} M\left(\frac{\mid u_k \Delta x_k - 0 \mid}{\rho}\right) \to \infty \text{ as } n \to \infty$$

i.e. $x_k \nrightarrow 0\ [V, M, u, \Delta]$. This completes the proof.

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