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SOME PROPERTIES OF THE SOLUTION OF A SECOND ORDER ELLIPTIC ABSTRACT DIFFERENTIAL EQUATION

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ABSTRACT. In this paper we study a class of non regular boundary value problems for elliptic differential-operator equation of second order with an operator in boundary conditions. We give conditions which guarantee the coerciveness of the solution of the considered problem, the completeness of system of root vectors in Banach-valued functions spaces and we establish the Abel basis property of this system in Hilbert spaces. Finally, we apply this abstract results to a partial differential equation in cylindrical domain.

Key words and phrases: Abstract differential equation, coerciveness, completeness, Abel basis.

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1. INTRODUCTION

The regular boundary value problems for elliptic differential-operator equation with a spectral parameter have been studied by many authors [2, 4, 19, 20]. Such parameter may appear in both the equation and the boundary conditions.

However, non-local problems, not satisfying Sapiro-Lopatinski condition or complementing condition, are less studied. In the papers and monographs [1, 5, 6, 7, 19, 20] sufficient conditions for coerciveness estimate and the completeness of system of root vectors to hold are given in Hilbert-valued function spaces. We quote in particular [20], where a number of such problems is considered.

The Abel basis property of a system of root vectors of an unbounded operators, introduced in B. Lidskii [13] (see also V. Matsaev and S. Agranovich), is used and developed in the book by S. Yakubov and Ya. Yakubov [20] and some papers. But in this book, the results mostly obtained for case when the spectral parameter may appear in both the equation and the boundary conditions.

A. Aibeche in [7, 8] considered a non-local boundary value problems for elliptic differential-operator equation of second order whith an operator in boundary conditions, and established the coerciveness estimate, the completeness of root vectors, when the principal part of the corresponding spectral problem is selfadjoint and the Fredholm property for nonselfadjoint operators.

The main objective of the present paper is to discuss similar problem as those in [7, 8], we study the coerciveness, the completeness and the Abel basis property in the corresponding Hilbert spaces. Moreover, we used the results obtained by V. Shakhmurov [1, 15, 16], to give conditions which guarantee the coerciveness estimate and the completeness of root vectors in Banach-valued L_p spaces.

These results are applied to non-local boundary value problems for elliptic partial differential equation with parameter in cylindrical domains.

More precisely, in Section 2, we give some background preliminaries, more precisely, we recall Dore-Yakubov Theorem and the multiplier Theorem in $L_p\left(R^n, E\right)$. The principal boundary value problem for abstract differential equations is studied in Section 3, where the isomorphism and the coerciveness are proved. In Section 4, the completeness and the Abel basis property of the root vectors of differential operator generated by our problem are shown. Finally, in Section 5, we apply the obtained abstracts results to some boundary value problems for elliptic partial differential equation in a cylinder.

2. PRELIMINARIES

Let E be a Banach space, A a linear closed operator in E and $D\left(A\right)$ its domain.

We denote by $L_p\left(\Omega, E\right)$, the space of strongly measurable E-valued functions that are defined on a domain $\Omega \subset \mathbb{R}^n$ with the norm

$$\|u\|_{L_p(\Omega,E)} = \left(\int_\Omega \|u\left(x\right)\|_E^p \, dx\right)^{\frac{1}{p}}, \ 1 \leq p \leq \infty.$$

By $L_p\left(\Omega\right), \ W_p^l\left(\Omega\right)$, we will denote a p-summable function space and Sobolev space.

The Banach space E is said to be $\xi-$convex if there exists on $E \times E$ a symmetric valued function ξ wich is convex with respect to each of the variables and satisfies the conditions

$$\begin{aligned} \xi\left(0,0\right) &> 0, \\ \xi\left(u,v\right) &\leq \|u+v\|, \text{ for } \|u\|_E = \|v\|_E = 1. \end{aligned}$$

The $\xi-$convex Banach space E is often called a UMD space. $L_p, \ l_p$ spaces and Lorentz spaces L_{pq} with $p, q \in (1, \infty)$ are UMD spaces.

Let $\mathbb{C}$ be the set of complex numbers and

$$S_{\varphi}=\{\lambda\in\mathbb{C},\ |\arg\lambda|\leq\varphi\}\cup\{0\},\ 0<\varphi\leq\pi.$$

Definition 2.1. [1] A linear operator A is said to be $\varphi-$positive in Banach space E with bound $M>0$, if $D(A)$ is dense in E and

$$\left\|(A-\lambda I)^{-1}\right\|_{L(E)}\leq\frac{M}{1+|\lambda|}$$

with $\lambda\in S_{\varphi},\ \varphi\in(0,\pi]$, where I is the identity operator in E and $L(E)$ is the space of bounded linear operators acting on E.

Let $E(A)$ denote the space $D(A)$ with graphical norm defined as

$$\|u\|_{E(A)}=\left(\|u\|^{p}+\|Au\|^{p}\right)^{\frac{1}{p}},\ 1\leq p<\infty.$$

Let E_0 and E be two Banach spaces and let E_0 be continuously and densely embedded into E. By $(E_0,E)_{\theta,p},\ 0<\theta<1$, we will denote interpolation spaces for $\{E_0,E\}$ by the K-method ([17, section 1.3.1]). We denote by $B_{p,q}^{s}(\Omega)$, where Ω is a regular domain of $\mathbb{R}^n$ the space

$$B_{p,q}^{s}(\Omega)=\left(W_p^{s_0}(\Omega),W_p^{s_1}(\Omega)\right)_{\theta,q};$$

where $0\leq s_0,\ s_1$ are integers, $0<\theta<1,\ 1<p<\infty,\ 1\leq q\leq\infty$ and $s=(1-\theta)s_0+\theta s_1$.

Consider the Banach space

$$W_p^{l}(0,1;E(A),E)=\left\{u,\ Au\in L_p(0,1,E),\ u^{(l)}\in L_p(0,1,E)\right\}$$

with the norm

$$\|u\|_{W_p^l(0,1;E(A),E)}=\|Au\|_{L_p(0,1,E)}+\left\|u^{(l)}\right\|_{L_p(0,1,E)}<\infty.$$

Let E_0 and E_1 be two Banach spaces.

Definition 2.2. [2] A function $\psi\in C(\mathbb{R}^n;L(E_1,E_2))$, is called a multiplier from $L_p(\mathbb{R}^n,E_1)$ to $L_q(\mathbb{R}^n,E_2)$, if there exists a constant $C>0$ with

$$\left\|F^{-1}\psi(\xi)Fu\right\|_{L_q(\mathbb{R}^n,E_2)}\leq M\|u\|_{L_p(\mathbb{R}^n,E_1)}$$

for all $u\in L_p(\mathbb{R}^n,E_1)$, where F is the Fourier transform.

The set of all multipliers from $L_p(\mathbb{R}^n,E_1)$ to $L_q(\mathbb{R}^n,E_2)$ will be denoted by $M_p^q(E_1,E_2)$. For $E_1=E_2=E$, it will be denoted by $M_p^q(E)$.

Definition 2.3. [18] A set $K\subset B(E_1,E_2)$ is called $\mathcal{R}-$bounded if there exists a constant $C>0$ such that for all $T_1,T_2,\cdots,T_m\in K$ and $u_1,\cdots,u_m\in E_1,\ m\in\mathbb{N}$

$$\int_0^1\left\|\sum_{j=1}^{m}r_j(y)T_ju_j\right\|_{E_2}dy\leq C\int_0^1\left\|\sum_{j=1}^{m}r_j(y)u_j\right\|_{E_1}dy$$

where $\{r_j\}$ is a sequence of independent symmetric $[-1,1]$-valued random variables on $[0,1]$.

Now, let

$$\begin{aligned}U_n&=\{\beta=(\beta_1,\cdots,\beta_n);\ \beta_i\in(0,1),\ i=1,\cdots,n\}\\V_n&=\{\xi=(\xi_1,\cdots,\xi_n)\in\mathbb{R}^n;\ \xi_i\neq0,\ i=1,\cdots,n\}\end{aligned}$$

Definition 2.4. [1] A Banach space E is said to be a space satisfying a multiplier condition with respect to $p \in (1,\infty)$ if the following condition holds: if $\psi \in C(\mathbb{R}^n; L(E))$ and the set

$$\left\{\xi^\beta D_\xi^\beta \psi(\xi);\ \xi \in V_n,\ \beta \in U_n\right\}$$

is $\mathcal{R}-$bounded, then $\psi \in M_p^p(E)$.

Definition 2.5. [1] The positive operator A is said to be $\mathcal{R}-$positive in the Banach space E if there exists $\varphi \in (0,\pi]$ such that the set

$$\left\{(1+|\xi|)(A-\xi I)^{-1},\ \xi \in S_\varphi\right\}$$

is $\mathcal{R}-$bounded.

For two sequences $\{a_j\}_1^\infty,\ \{b_j\}_1^\infty$ of positive numbers, the expression $a_j \sim b_j$ means that there exist positive numbers $C_1,\ C_2$ such that $C_1 a_j \le b_j \le C_2 a_j$. $\forall j \in \mathbb{N}$.

Let $\sigma_\infty(E_1,E_2)$ denote the space of compact operators acting from E_1 to E_2. Denote by $s_j(I)$ and $d_j(I)$ the approximation numbers and d-numbers of the operator I, respectively (see, e.g, [17, 1.16.1]).

$$\sigma_p(E_1,E_2) = \left\{A \in \sigma_\infty(E_1,E_2),\ \sum_{j=1}^\infty s_j^p(A) < \infty,\ 1 \le p < \infty\right\}.$$

Theorem 2.1. [10] *Let E be a Banach space. A be a linear closed operator in E of type φ with bound L. Moreover, let m be a positive integer, $p \in (1,\infty)$ and $\alpha \in \left(\frac{1}{2p}, m+\frac{1}{2p}\right)$. for $\lambda \in S_\varphi$ the operator $-(A+\lambda I)^{\frac{1}{2}}$ generates a semigroup $\exp\left(-x(A+\lambda I)^{\frac{1}{2}}\right)$ which is holomorphic for $x > 0$ and strongly continuous for $x \ge 0$. Moreover there exists $C \in R^+$ (depending only on L,φ,m,α,p) such that for every $u \in (E, E(A^m))_{\frac{\alpha}{m}-\frac{1}{2mp},p}$ and $\lambda \in S_\varphi$,*

$$\int_0^\infty \left\|(A+\lambda I)^\alpha \exp\left(-x(A+\lambda I)^{\frac{1}{2}}\right)u\right\|^p \le C\left(\|u\|^p_{(E,E(A^m))_{\frac{\alpha}{m}-\frac{1}{2mp},p}} + |\lambda|^{p\alpha-\frac{1}{2}}\|u\|^p\right).$$

3. Solvability of the Principal Problem

Consider in $L_p(0,1,E)$ the boundary value problems for the second order abstract differential equation

$$L_0(\lambda,D)u = -u''(x) + Au(x) = f(x),\ x \in (0,1); \tag{3.1}$$

$$\begin{cases} L_{10}u = \delta u(0) = f_1; \\ L_{20}u = Bu(0) + u'(1) = f_2. \end{cases} \tag{3.2}$$

where A, B, are linear operators and $\delta \in \mathbb{C},\ f_1 \in (E(A),E)_{\frac{1}{2p},p},\ f_2 \in (E(A),E)_{\frac{1}{2}+\frac{1}{2p},p}$.

3.1. Homogeneous problem.

Consider the principal part of the problem (3.1, 3.2) with a parameter

$$L_0(\lambda,D)u = -u''(x) + (A+\lambda I)u(x) = 0,\ x \in (0,1); \tag{3.3}$$

$$\begin{cases} L_{10}u = \delta u(0) = f_1; \\ L_{20}u = Bu(0) + u'(1) = f_2. \end{cases} \tag{3.4}$$

Theorem 3.1. *Assume that the following condition are satisfied*

(1) *A is a closed, positive and densely defined linear operator on E;*
(2) $\delta \neq 0$;

(3) *B is a linear continuous from $E\left(A^{\frac{1}{2}}\right)$ into E and from $E(A)$ into $E\left(A^{\frac{1}{2}}\right)$.*

Then, for λ such that $|\arg\lambda| \le \varphi < \pi$, $|\lambda| \to \infty$, the problem (3.3, 3.4), for $f_1 \in (E(A), E)_{\frac{1}{2p},p}$, $f_2 \in (E(A), E)_{\frac{1}{2}+\frac{1}{2p},p}$, has a unique solution $u \in W_p^2(0,1;E(A), E)$, and for the solution of the problem (3.3, 3.4), the following coercive estimate holds

$$\begin{aligned} &\|u''\|_{L_p(0,1,E)} + \|Au\|_{L_p(0,1,E)} + |\lambda| \, \|u\|_{L_p(0,1,E)} \\ &\le C\Bigg(\|f_1\|_{(E(A),E)_{\frac{1}{2p},p}} + |\lambda|^{1-\frac{1}{2p}} \|f_1\|_E \\ &\qquad + \|f_2\|_{(E(A),E)_{\frac{1}{2}+\frac{1}{2p},p}} + |\lambda|^{\frac{1}{2}-\frac{1}{2p}} \|f_2\|_E \Bigg). \end{aligned}$$

Proof. From the condition (1), by virtue of Theorem 2.1, for $|\arg\lambda| \le \varphi$, there exists the semigroup $\exp(-x(A+\lambda I))$ which is holomorphic for $x > 0$ and strongly continuous for $x \ge 0$. By virtue of [20][Lemma 5.4.2/1], an arbitrary solution of (3.3) belonging to the space $W_p^2(0,1;E(A), E)$ has the form

$$u(x) = \exp\left(-xA_\lambda^{\frac{1}{2}}\right) g_1 + \exp\left(-(1-x)A_\lambda^{\frac{1}{2}}\right) g_2 \tag{3.5}$$

where $g_1,\ g_2 \in (E(A), E)_{\frac{1}{2p},p}$.

The function u given by the formula (3.5) satisfies the boundary conditions (3.4) if

$$\begin{cases} \delta g_1 + \delta \exp\left(-A_\lambda^{\frac{1}{2}}\right) g_2 = f_1 \\ -A_\lambda^{\frac{1}{2}} \exp\left(-A_\lambda^{\frac{1}{2}}\right) g_1 + A_\lambda^{\frac{1}{2}} g_2 + Bg_1 + B\exp\left(-A_\lambda^{\frac{1}{2}}\right) g_2 = f_2. \end{cases}$$

which we can write in matrix form as

$$\left[\begin{pmatrix} \delta I & 0 \\ B & A_\lambda^{\frac{1}{2}} \end{pmatrix} + \begin{pmatrix} 0 & \delta\exp\left(-A_\lambda^{\frac{1}{2}}\right) \\ -A_\lambda^{\frac{1}{2}}\exp\left(-A_\lambda^{\frac{1}{2}}\right) & B\exp\left(-A_\lambda^{\frac{1}{2}}\right) \end{pmatrix}\right] \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}. \tag{3.6}$$

The first matrix of operators is invertible, its inverse is

$$\begin{pmatrix} \frac{1}{\delta} I & 0 \\ -\frac{1}{\delta} A_\lambda^{-\frac{1}{2}} B & A_\lambda^{-\frac{1}{2}} \end{pmatrix} \tag{3.7}$$

Multiplying the two hand-sides of (3.6) by the inverse matrix (3.7), we get the following system

$$\begin{cases} g_1 + \exp\left(-A_\lambda^{\frac{1}{2}}\right) g_2 = \frac{1}{\delta} f_1 \\ -\exp\left(-A_\lambda^{\frac{1}{2}}\right) g_1 + g_2 = -\frac{1}{\delta} A_\lambda^{-\frac{1}{2}} B f_1 + A_\lambda^{-\frac{1}{2}} f_2 \end{cases}$$

Hence the solution is written as

$$\begin{cases} g_1 = \frac{1}{\delta} f_1 + R_{11}(\lambda) f_1 + R_{12}(\lambda) f_2; \\ g_2 = -\frac{1}{\delta}(I + T(\lambda)) A_\lambda^{-\frac{1}{2}} B f_1 + (I + T(\lambda)) A_\lambda^{-\frac{1}{2}} f_2 + R_{21}(\lambda) f_1 \end{cases}$$

where $R_{ij}(\lambda)$ are given by

$$\begin{cases} R_{11}(\lambda) = -\frac{1}{\delta}(I + T(\lambda))\exp\left(-A_\lambda^{\frac{1}{2}}\right) + \frac{1}{\delta}(I + T(\lambda))\exp\left(-A_\lambda^{\frac{1}{2}}\right) A_\lambda^{-\frac{1}{2}} B \\ R_{12}(\lambda) = -(I + T(\lambda)) A_\lambda^{-\frac{1}{2}} \exp\left(-A_\lambda^{\frac{1}{2}}\right) \\ R_{21}(\lambda) = \frac{1}{\delta}(I + T(\lambda))\exp\left(-A_\lambda^{\frac{1}{2}}\right). \end{cases}$$

and satisfy $\|R_{ij}(\lambda)\| \to 0$ when $|\lambda| \to \infty$. $(I + T(\lambda))$ is the inverse of $I + \exp\left(-2A_\lambda^{\frac{1}{2}}\right)$.

Finally, the solution u is given by

$$u(x) = \exp\left(-xA_\lambda^{\frac{1}{2}}\right)\left(\frac{1}{\delta}f_1 + R_{11}(\lambda) f_1 + R_{12}(\lambda) f_2\right) +$$
$$\exp\left(-(1-x)A_\lambda^{\frac{1}{2}}\right)\left(-\frac{1}{\delta}(I + T(\lambda)) A_\lambda^{-\frac{1}{2}} B f_1 + R_{21}(\lambda) f_1 + (I + T(\lambda)) A_\lambda^{-\frac{1}{2}} f_2\right).$$

From the assumptions of Theorem 3.1 and the properties of interpolation spaces, the following applications are continuous,

$$\begin{aligned} (I + T(\lambda)) A_\lambda^{-\frac{1}{2}} B &: (E(A), E)_{\frac{1}{2p},p} \to (E(A), E)_{\frac{1}{2p},p} \\ (I + T(\lambda)) A_\lambda^{-\frac{1}{2}} &: (E(A), E)_{\frac{1}{2}+\frac{1}{2p},p} \to (E(A), E)_{\frac{1}{2p},p} \end{aligned}$$

Let $u(x) = v_1 + v_2 + v_3$, where

$$\begin{aligned} v_1 &= \exp\left(-xA_\lambda^{\frac{1}{2}}\right)\frac{1}{\delta}f_1; \\ v_2 &= \exp\left(-(1-x)A_\lambda^{\frac{1}{2}}\right)\frac{1}{\delta}(I + T(\lambda)) A_\lambda^{-\frac{1}{2}} B f_1; \\ v_3 &= \exp\left(-(1-x)A_\lambda^{\frac{1}{2}}\right)(I + T(\lambda)) A_\lambda^{-\frac{1}{2}} f_2; \end{aligned}$$

Then,

$$\|u''\|_{L_p(0,1,E)} + \|Au\|_{L_p(0,1,E)} + |\lambda| \|u\|_{L_p(0,1,E)} \leq$$
$$\|A_\lambda v_1\|_{L_p(0,1,E)} + \|A_\lambda v_2\|_{L_p(0,1,E)} + \|A_\lambda v_3\|_{L_p(0,1,E)} + \|Av_1\|_{L_p(0,1,E)} + \|Av_2\|_{L_p(0,1,E)}$$
$$+ \|Av_3\|_{L_p(0,1,E)} + |\lambda| \|v_1\|_{L_p(0,1,E)} + |\lambda| \|v_2\|_{L_p(0,1,E)} + |\lambda| \|v_3\|_{L_p(0,1,E)}.$$

However, from Theorem 2.1, we have

$$\begin{aligned} \|A_\lambda v_1\|_{L_p(0,1,E)} &= \left\|A_\lambda \exp\left(-xA_\lambda^{\frac{1}{2}}\right)\frac{1}{\delta}f_1\right\|_{L_p(0,1,E)} \\ &\leq C\left(\|f_1\|_{(E(A),E)_{\frac{1}{2p},p}} + |\lambda|^{1-\frac{1}{2p}}\|f_1\|_E\right). \end{aligned}$$

Similarly we estimate the other terms. ∎

3.2. **Non homogeneous problem.** Consider, now the Non homogeneous problem equation with a parameter

(3.8) $$L_0(\lambda, D) u = -u''(x) + (A + \lambda I) u(x) = f(x),\ x \in (0,1);$$

(3.9) $$\begin{cases} L_{10}u = \delta u(0) = f_1; \\ L_{20}u = Bu(0) + u'(1) = f_2. \end{cases}$$

Then, we have the result

Theorem 3.2. *Suppose the following conditions satisfied*

(1) *A is a closed, $\mathcal{R}-$positive and densely defined linear operator on E which is UMD;*

(2) $\delta \neq 0$;

(3) *B is a linear continuous from $E\left(A^{\frac{1}{2}}\right)$ into E and from $E(A)$ into $E\left(A^{\frac{1}{2}}\right)$.*

Then, the operator

$$\mathcal{L}_0(\lambda) : u \mapsto \mathcal{L}_0(\lambda)\, u = \left(L_0(\lambda, D)\, u, L_{10}u, L_{20}u\right)$$

for $f_1 \in (E(A), E)_{\frac{1}{2p},p}$, $f_2 \in (E(A), E)_{\frac{1}{2}+\frac{1}{2p},p}$ and λ such that $|\arg\lambda| \leq \varphi < \pi$, $|\lambda| \to \infty$, is an isomorphism from $W_p^2(0,1;E(A), E)$ into $L_p(0,1,E) \times (E(A), E)_{\frac{1}{2p},p} \times (E(A), E)_{\frac{1}{2}+\frac{1}{2p},p}$, $p \in (1,\infty)$, and the following coercive estimate holds

$$\begin{aligned} &\|u''\|_{L_p(0,1,E)} + \|Au\|_{L_p(0,1,E)} + |\lambda|\,\|u\|_{L_p(0,1,E)} \\ &\leq C\Bigg(\|f\|_{L_p(0,1,E)} + \|f_1\|_{(E(A),E)_{\frac{1}{2p},p}} + |\lambda|^{1-\frac{1}{2p}}\|f_1\|_E \\ &\qquad + \|f_2\|_{(E(A),E)_{\frac{1}{2}+\frac{1}{2p},p}} + |\lambda|^{\frac{1}{2}-\frac{1}{2p}}\|f_2\|_E\Bigg); \end{aligned}$$

where C does not depend on λ.

Proof. In Theorem 3.1, we proved the uniqueness. The solution of the problem (3.8, 3.9) belonging to $W_p^2(0,1;E(A), E)$ can be written in the form $u(x) = u_1(x) + u_2(x)$, where $u_1(x)$ is the restriction to $[0,1]$ of $\tilde{u}_1(x)$ solution of the equation

$$L_0(\lambda, D)\,\tilde{u}_1(x) = \tilde{f}(x),\ x \in \mathbb{R} \tag{3.10}$$

with $\tilde{f}(x) = f(x)$ if $x \in [0,1]$ and $\tilde{f}(x) = 0$ otherwise. $u_2(x)$ is the solution of the problem

$$L_0(\lambda, D)\, u_2 = 0,\ L_{10}u_2 = f_1 - L_{10}u_1;\ L_{20}u_2 = f_2 - L_{20}u_1. \tag{3.11}$$

The solution of the equation (3.10) is given by the formula

$$\tilde{u}_1(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp(i\mu x)\, L_0(\lambda, i\mu)^{-1} F\tilde{f}(\mu)\, d\mu; \tag{3.12}$$

where $F\tilde{f}$ is the Fourier transform of the function $\tilde{f}(x)$, $L_0(\lambda,\sigma)$ is the characteristic pencil of the equation (3.10) i.e $L_0(\lambda,\sigma) = A + \lambda I - \sigma^2 I$.

From (3.12), it follows that

$$\begin{aligned} &\|\tilde{u}_1''\|_{L_p(\mathbb{R},E)} + \|A\tilde{u}_1\|_{L_p(\mathbb{R},E)} + |\lambda|\,\|\tilde{u}_1\|_{L_p(\mathbb{R},E)} \\ = &\left\|F^{-1}(i\mu)^2 L_0(\lambda,i\mu)^{-1} F\tilde{f}(\mu)\right\|_{L_p(\mathbb{R},E)} + \left\|F^{-1}AL_0(\lambda,i\mu)^{-1}F\tilde{f}(\mu)\right\|_{L_p(\mathbb{R},E)} \\ &+ \left\|F^{-1}\lambda L_0(\lambda,i\mu)^{-1}F\tilde{f}(\mu)\right\|_{L_p(\mathbb{R},E)}. \end{aligned}$$

where F is the Fourier transform.

We show that the operator-valued functions

$$T(\lambda,\mu) = \lambda L_0(\lambda,i\mu)^{-1},\ T_{k+1}(\lambda,\mu) = (i\mu)^{2k} A^{1-k} L_0(\lambda,i\mu)^{-1},\ k = 0,1$$

are Fourier multiplier in $L_p(\mathbb{R},E)$. For $|\arg\lambda| \leq \varphi$ and $\mu \in \mathbb{R}$, we have $(-\lambda-\mu^2) \in S_\varphi$.

Then by virtue of the resolvent properties of the positive operator A, we obtain

$$\left\{\begin{array}{l} \|T(\lambda,\mu)\| = \left\|\lambda L_0(\lambda,i\mu)^{-1}\right\| \leq C; \\ \|T_1(\lambda,\mu)\| = \left\|AL_0(\lambda,i\mu)^{-1}\right\| \leq C; \\ \|T_2(\lambda,\mu)\| = \left\|\mu^2 L_0(\lambda,i\mu)^{-1}\right\| \leq C; \end{array}\right. \tag{3.13}$$

using (3.13), for all $\mu \in \mathbb{R}\backslash \{0\}$, we obtain

$$\left\| \frac{\partial}{\partial \mu} T(\lambda,\mu) \right\| \leq C |\mu|^{-1}, \quad \left\| \frac{\partial}{\partial \mu} T_{k+1}(\lambda,\mu) \right\| \leq C |\mu|^{-1}. \tag{3.14}$$

From the $\mathcal{R}-$positivity of the operator A, the operator-valued functions $T(\lambda,\mu)$, $T_{k+1}(\lambda,\mu)$ are $\mathcal{R}-$bounded with $\mathcal{R}-$bound independent of λ. Moreover, it is easy to see from (3.13) that the operator-valued functions $\mu\left(\frac{\partial}{\partial \mu}\right) T(\lambda,\mu)$ and $\mu\left(\frac{\partial}{\partial \mu}\right) T_{k+1}(\lambda,\mu)$ are $\mathcal{R}-$bounded with $\mathcal{R}-$bound independent of λ. Then, by the Definition 2.2, it follows that the functions $T(\lambda,\mu)$, $T_{k+1}(\lambda,\mu)$ are Fourier multiplier in $L_p(\mathbb{R},E)$. Then, we have

$$\|\tilde{u}_1''\|_{L_p(\mathbb{R},E)} + \|A\tilde{u}_1\|_{L_p(\mathbb{R},E)} + |\lambda| \|\tilde{u}_1\|_{L_p(\mathbb{R},E)} \leq C \left\|\tilde{f}\right\|_{L_p(\mathbb{R},E)}$$

and so,

$$\|u_1''\|_{L_p(0,1,E)} + \|Au_1\|_{L_p(0,1,E)} + |\lambda| \|u_1\|_{L_p(0,1,E)} \leq C \|f\|_{L_p(0,1,E)}.$$

Thus, by Theorem 3.1, the problem (3.11) has a unique solution $u_2(x)$ that belong to the space $W_p^2(0,1;E(A),E)$ for $|\arg\lambda| \leq \varphi$ and for sufficiently large $|\lambda|$. Moreover, for a solution of the problem (3.11), we have

$$\begin{aligned}
&\|u_2''\|_{L_p(0,1,E)} + \|Au_2\|_{L_p(0,1,E)} + |\lambda| \|u_2\|_{L_p(0,1,E)} \\
&\leq C \Bigg(\|f_1 - L_{10}u_1\|_{(E(A),E)_{\frac{1}{2p},p}} + |\lambda|^{1-\frac{1}{2p}} \|f_1 - L_{10}u_1\|_E \\
&\quad + \|f_2 - L_{20}u_1\|_{(E(A),E)_{\frac{1}{2}+\frac{1}{2p},p}} + |\lambda|^{\frac{1}{2}-\frac{1}{2p}} \|f_2 - L_{20}u_1\|_E \Bigg) \\
&\leq C \Bigg(\|f_1\|_{(E(A),E)_{\frac{1}{2p},p}} + \|L_{10}u_1\|_{(E(A),E)_{\frac{1}{2p},p}} + |\lambda|^{1-\frac{1}{2p}} \|f_1\|_E \\
&\quad + |\lambda|^{1-\frac{1}{2p}} \|L_{10}u_1\|_E + \|f_2\|_{(E(A),E)_{\frac{1}{2}+\frac{1}{2p},p}} + \|L_{20}u_1\|_{(E(A),E)_{\frac{1}{2}+\frac{1}{2p},p}} \\
&\quad + |\lambda|^{\frac{1}{2}-\frac{1}{2p}} \|f_2\|_E + |\lambda|^{\frac{1}{2}-\frac{1}{2p}} \|L_{20}u_1\|_E \Bigg).
\end{aligned}$$

We have

$$\begin{aligned}
\|L_{10}u_1\|_{(E(A),E)_{\frac{1}{2p},p}} &= \|\delta u_1(0)\|_{(E(A),E)_{\frac{1}{2p},p}} \\
&\leq C \|u_1(0)\|_{(E(A),E)_{\frac{1}{2p},p}},
\end{aligned}$$

Therefore, by virtue of [17, section 1.8], we obtain

$$\|L_{10}u_1\|_{(E(A),E)_{\frac{1}{2p},p}} \leq C \|u_1\|_{W_p^2(0,1;E(A),E)} \leq C \|f\|_{L_p(0,1,E)}.$$

We have

$$\begin{aligned}
|\lambda|^{1-\frac{1}{2p}} \|L_{10}u_1\|_E &= |\lambda|^{1-\frac{1}{2p}} \|\delta u_1(0)\|_E \\
&\leq C |\lambda|^{1-\frac{1}{2p}} \|u_1(0)\|
\end{aligned}$$

By Theorem [20, 1.7.7/2], for $\mu \in \mathbb{C}$, $u_1 \in W_p^2(0,1;E)$ and ($E_0 = E$, $E_1 = E$, $\gamma_0 = \gamma_1 = \gamma = 0$, $p_0 = p_1 = p$, $s = 0$, $l = 2$),

$$|\mu|^2 \|u_1(0)\| \leq C \left(|\mu|^{\frac{1}{p}} \|u_1\|_{W_p^2(0,1;E)} + |\mu|^{2+\frac{1}{p}} \|u_1\|_{L_p(0,1,E)} \right) \tag{3.15}$$

Dividing by $|\mu|^{\frac{1}{p}}$ and substituting $\lambda=\mu^2$ for $\lambda\in\mathbb{C}$ and $u_1\in W_p^2(0,1;E)$, from (3.15) we get

$$\begin{aligned}|\lambda|^{1-\frac{1}{2p}}\|u_1(0)\| &\le C\left(\|u_1\|_{W_p^2(0,1;E)}+|\lambda|\|u_1\|_{L_p(0,1,E)}\right)\\ &\le C\|f\|_{L_p(0,1,E)}\end{aligned}$$

Similarly, we get the other bounds and by the same way the coerciveness estimate. ∎

3.3. **Perturbed problem.** Consider, now the general problem with a parameter

$$L(\lambda,D)u=-u''(x)+(A+\lambda I)u(x)+A(x)u(x)=f(x),\ x\in(0,1) \tag{3.16}$$

$$\begin{aligned}L_1u &= \delta u(0)+\sum_{s=1}^{N_1}T_{1s}u(x_{1s})=f_1;\\ L_2u &= Bu(0)+u'(1)+\sum_{s=1}^{N_2}T_{2s}u(x_{2s})=f_2.\end{aligned} \tag{3.17}$$

where $A,B,A(x)$ are linear operators and $\delta\in\mathbb{C},\ f_1\in(E(A),E)_{\frac{1}{2p},p}$, $f_2\in(E(A),E)_{\frac{1}{2}+\frac{1}{2p},p}$, and T_{ks} are, generally speaking, unbounded operators in E.

Theorem 3.3. *Suppose the following conditions satisfied*

(1) A *is a closed,* $\mathcal{R}-$*positive and densely defined linear operator on* E *which is* UMD*;*
(2) $\delta\neq 0$;
(3) B *is a linear continuous from* $E\left(A^{\frac{1}{2}}\right)$ *into* E *and from* $E(A)$ *into* $E\left(A^{\frac{1}{2}}\right)$;
(4) *the imbedding* $E(A)\subset E$ *is compact;*
(5) *for any* $\varepsilon>0$ *and for almost all* $x\in[0,1]$,

$$\|A(x)u\|\le\varepsilon\|Au\|+C(\varepsilon)\|u\|,\ u\in D(A);$$

for $u\in D(A)$ *the function* $A(x)u$ *is measurable on* $[0,1]$ *in* E.
(6) *for* $\varepsilon>0$ *and* $u\in(E(A),E)_{\frac{1}{2p},p}$, *where* $p\in(1,\infty),\ \theta_k=\frac{m_k}{2}+\frac{1}{2p},\ m_k=\{0,1\}$ *we have*

$$\begin{aligned}\|T_{ks}u\|_{(E(A),E)_{\theta_k,p}} &\le \varepsilon\|u\|_{(E(A),E)_{\frac{1}{2p},p}}+C(\varepsilon)\|u\|_E,\ k=1,\ 2;\\ \|T_{1s}u\|_E &\le C(\varepsilon)\|u\|_E;\\ \|T_{2s}u\|_E &\le \varepsilon\|u\|_{(E(A),E)_{\frac{1}{2},p}}+C(\varepsilon)\|u\|_E;\end{aligned}$$

Then, the operator

$$\mathcal{L}(\lambda):u\to\mathcal{L}(\lambda)u=(L(\lambda,D)u,L_1u,L_2u)$$

for $f_1\in(E(A),E)_{\frac{1}{2p},p}$, $f_2\in(E(A),E)_{\frac{1}{2}+\frac{1}{2p},p}$ *and* λ *such that* $|\arg\lambda|\le\varphi<\pi$, $|\lambda|\to\infty$, *is an isomorphism from* $W_p^2(0,1;E(A),E)$ *into* $L_p(0,1,E)\times(E(A),E)_{\frac{1}{2p},p}\times(E(A),E)_{\frac{1}{2}+\frac{1}{2p},p}$, $p\in(1,\infty)$, *and the following coercive estimate holds*

$$\begin{aligned}&\|u''\|_{L_p(0,1,E)}+\|Au\|_{L_p(0,1,E)}+|\lambda|\|u\|_{L_p(0,1,E)}\\ &\le C\left(\|f\|_{L_p(0,1,E)}+\|f_1\|_{(E(A),E)_{\frac{1}{2p},p}}+|\lambda|^{1-\frac{1}{2p}}\|f_1\|_E\right.\\ &\left.\quad+\|f_2\|_{(E(A),E)_{\frac{1}{2}+\frac{1}{2p},p}}+|\lambda|^{\frac{1}{2}-\frac{1}{2p}}\|f_2\|_E\right);\end{aligned}$$

where C *does not depend on* λ.

Proof. Let u be a solution of (3.16, 3.17) belonging to $W_p^2(0,1;E(A),E)$. Then u is a solution of the problem

$$\begin{cases} L_0(\lambda,D)\,u = f(x) - A(x)\,u(x)\,,\ x\in(0,1)\,; \\ L_{10}u = f_1 - \sum_{s=1}^{N_1} T_{1s}u(x_{1s})\,; \\ L_{20}u = f_2 - \sum_{s=1}^{N_2} T_{2s}u(x_{2s})\,. \end{cases}$$

From Theorem 3.2, we get the estimate

$$\begin{aligned} &\|u''\|_{L_p(0,1,E)} + \|Au\|_{L_p(0,1,E)} + |\lambda|\,\|u\|_{L_p(0,1,E)} \\ &\le C\Bigg(\|f - A(x)\,u\|_{L_p(0,1,E)} + \left\|f_1 - \sum_{s=1}^{N_1} T_{1s}u(x_{1s})\right\|_{(E(A),E)_{\frac{1}{2p},p}} \\ &\quad + |\lambda|^{1-\frac{1}{2p}}\left\|f_1 - \sum_{s=1}^{N_1} T_{1s}u(x_{1s})\right\|_E + |\lambda|^{\frac{1}{2}-\frac{1}{2p}}\left\|f_2 - \sum_{s=1}^{N_2} T_{2s}u(x_{2s})\right\|_E \\ &\quad + \left\|f_2 - \sum_{s=1}^{N_2} T_{2s}u(x_{2s})\right\|_{(E(A),E)_{\frac{1}{2}+\frac{1}{2p},p}}\Bigg) \\ &\le C\Bigg(\|f\|_{L_p(0,1,E)} + \|A(x)\,u\|_{L_p(0,1,E)} + \|f_1\|_{(E(A),E)_{\frac{1}{2p},p}} \\ &\quad + \left\|\sum_{s=1}^{N_1} T_{1s}u(x_{1s})\right\|_{(E(A),E)_{\frac{1}{2p},p}} + |\lambda|^{1-\frac{1}{2p}}\|f_1\|_E \\ &\quad + |\lambda|^{1-\frac{1}{2p}}\left\|\sum_{s=1}^{N_1} T_{1s}u(x_{1s})\right\|_E + \|f_2\|_{(E(A),E)_{\frac{1}{2}+\frac{1}{2p},p}} \\ &\quad + \left\|\sum_{s=1}^{N_2} T_{2s}u(x_{2s})\right\|_{(E(A),E)_{\frac{1}{2}+\frac{1}{2p},p}} + |\lambda|^{\frac{1}{2}-\frac{1}{2p}}\|f_2\|_E \\ &\quad + |\lambda|^{\frac{1}{2}-\frac{1}{2p}}\left\|\sum_{s=1}^{N_2} T_{2s}u(x_{2s})\right\|_E\Bigg). \end{aligned}$$

From the Condition (5) and Lemma [20, 5.2.1/2], for $u \in W_p^2(0,1;E(A),E)$,

$$\|A(x)\,u\|_{L_p(0,1,E)} \le \varepsilon\,\|u\|_{W_p^2(0,1;E(A),E)} + C(\varepsilon)\,\|u\|_{L_p(0,1,E)}\,,$$

In view of Theorem [20, 1.7.7/1] and the Condition (6), it follows that for $\varepsilon > 0$ and $u \in W_p^2(0,1;E(A),E)$

$$\|T_{1s}u(x_{1s})\|_{(E(A),E)_{\frac{1}{2p},p}} \le \varepsilon\,\|u\|_{W_p^2(0,1;E(A),E)} + C(\varepsilon)\,\|u\|_{L_p(0,1,E)}$$

Similarly we estimate the other terms.

Then, we have

$$\begin{aligned} &(1-C\varepsilon)\,\|u''\|_{L_p(0,1,E)} + (1-C\varepsilon)\,\|Au\|_{L_p(0,1,E)} + (1-C\varepsilon)\,|\lambda|\,\|u\|_{L_p(0,1,E)} \\ &\le C\Bigg(\|f\|_{L_p(0,1,E)} + \|f_1\|_{(E(A),E)_{\frac{1}{2p},p}} + |\lambda|^{1-\frac{1}{2p}}\|f_1\|_E \\ &\qquad + \|f_2\|_{(E(A),E)_{\frac{1}{2}+\frac{1}{2p},p}} + |\lambda|^{\frac{1}{2}-\frac{1}{2p}}\|f_2\|_E\Bigg). \end{aligned}$$

Choosing ε such that $C\varepsilon < 1$, the coerciveness estimate follows easily. Consequently, for $|\arg\lambda| \le \varphi$ and $|\lambda|$ sufficiently large, a solution of problem (3.16, 3.17) in $W_p^2(0,1;E(A),E)$ is unique. By virtue of Theorem 3.2, the operator $\mathcal{L}_0(\lambda)$ from $W_p^2(0,1;E(A),E)$ into $L_p(0,1,E)\times (E(A),E)_{\frac{1}{2p},p} \times (E(A),E)_{\frac{1}{2}+\frac{1}{2p},p}$ is an isomorphism, then Fredholm and by virtue of Lemma

[20, 1.2.7/2]), it follows that the operator $\mathcal{L}_1(\lambda)$

$$\begin{array}{l}\mathcal{L}_1(\lambda): u \mapsto \left(A(x)u(x),\ \sum_{s=1}^{N_1} T_{1s}u(x_{1s}),\ \sum_{s=1}^{N_2} T_{2s}u(x_{2s})\right) \\ W_p^2(0,1;E(A),E) \mapsto L_p(0,1,E) \times (E(A),E)_{\frac{1}{2p},p} \times (E(A),E)_{\frac{1}{2}+\frac{1}{2p},p}\end{array}$$

from $W_p^2(0,1;E(A),E)$ into $L_p(0,1,E) \times (E(A),E)_{\frac{1}{2p},p} \times (E(A),E)_{\frac{1}{2}+\frac{1}{2p},p}$ is compact. The proof of the theorem is completed by applying Theorem [20, 1.2.8,] to operator $\mathcal{L}(\lambda) = \mathcal{L}_0(\lambda) + \mathcal{L}_1(\lambda)$. ∎

4. COMPLETENESS AND THE ABEL BASIS PROPERTY

4.1. Completeness of root vectors. Consider a particular case of problem (3.16, 3.17), namely

$$L(\lambda, D)u = -u''(x) + (A + \lambda I)u(x) + A(x)u(x) = 0,\ x \in (0,1); \tag{4.1}$$

$$\begin{aligned} L_1 u &= \delta u(0) + \sum_{s=1}^{N_1} T_{1s}u(x_{1s}) = 0; \\ L_2 u &= Bu(0) + u'(1) + \sum_{s=1}^{N_2} T_{2s}u(x_{2s}) = 0. \end{aligned} \tag{4.2}$$

Let us define, in the space $\mathcal{E} = L_p(0,1,E)$ the operator $\mathcal{A}$ by

$$\begin{aligned} D(\mathcal{A}) &= W_p^2(0,1;\ E(A), E;\ L_k u = 0,\ k = 1,\ 2) \\ \mathcal{A}u &= -u''(x) + Au(x) + A(x)u(x) \end{aligned}$$

A system of roots vectors of problem (3.16, 3.17) is complete in the space $L_p(0,1,E)$ if a system of roots vectors of the operator $\mathcal{A}$ is complete in the space $L_p(0,1,E)$.

Lemma 4.1. *(see, B. Shakhmurov,* [1, Theorem 2.8]*). Suppose that* $s_j(I, E(A), E,) \sim j^{-q}$*, then*

$$s_j\left(J, W_p^2(0,1;E(A),E),\ L_p(0,1,E)\right) \sim j^{-\frac{2q}{2+q}}.$$

I (resp. J) is the imbedding of $E(A)$ in E (resp. of $W_p^2(0,1;E(A),E)$ in $L_p(0,1,E)$) and $s_j(I, E(A), E,)$ are the approximation numbers of the operator I from $E(A)$ to E.

Then, we have the result

Theorem 4.2. *Let all conditions of Theorem 3.3 be satisfied with $s_j(I, E(A), E) \leq Cj^{-q}$, $q > 0$. Then the system of root vectors of the of problem (3.16, 3.17) is complete in the space $L_p(0,1,E)$.*

Proof. From Theorem 3.3, we have

$$\|R(-\lambda, \mathcal{A})\| \leq C|\lambda|^{-1},\ |\arg\lambda| \leq \varphi < \pi,\ |\lambda| \to \infty.$$

Using Lemma 4.1, we get

$$s_j(J, \mathcal{E}(\mathcal{A}), \mathcal{E}) \leq Cs_j\left(J, W_p^2(0,1;\ E(A), E), L_p(0,1,E)\right) \leq Cj^{-\frac{2q}{2+q}},$$

i.e $s_j(J, \mathcal{E}(\mathcal{A}), \mathcal{E}) \leq Cj^{-q'}$, $q' > \frac{2q}{2+q}$.

So, for the operator $\mathcal{A}$, all the condition of [9, Theorem.126, 2.3, p.50], are fulfilled. This achieves the proof of the Theorem. ∎

4.2. **The Abel basis property of root vectors.** Let us define, in the Hilbert space $\mathcal{H} = L_2(0,1,H)$ the operator $\mathcal{A}'$ by

$$\begin{aligned} D(\mathcal{A}') &= W_2^2(0,1;\ E(A), E;\ L_k u = 0,\ k = 1,\ 2) \\ \mathcal{A}'u &= -u''(x) + Au(x) + A(x)u(x) \end{aligned}$$

A system of root vectors of problem (3.16, 3.17) is called an Abel basis of order α in the space $L_2(0,1,H)$ if a system of root vectors of the operator $\mathcal{A}'$ forms an Abel basis of order α in the space $L_2(0,1,H)$.

Theorem 4.3. *Suppose the following*

(1) *The conditions (1)-(5) of Theorem 3.3 be satisfied;*
(2) *for some $q > 0$ it holds that $s_j(I, E(A), E) \le Cj^{-q},\ \frac{2\pi}{q+2} < \varphi < \pi,\ j = 1, \dots, \infty$;*
(3) *the operators T_{1s} from $(H(A), H)_{\frac{1}{4},2}$ into $(H(A), H)_{\frac{1}{4},2}$, from H into H and T_{2s} from $(H(A), H)_{\frac{1}{4},2}$ into $(H(A), H)_{\frac{3}{4},2}$, from $(H(A), H)_{\frac{1}{2},2}$ into H are compact.*

Then, a system of root vectors of the problem (3.16, 3.17) forms an Abel basis of order $\alpha \in \left(\frac{q+2}{2q}, \frac{\pi}{2(\pi-\varphi)}\right)$ in the space $L_2(0,1,H)$.

Proof. Let us apply Theorem [20, 2.2.3] to the operator $\mathcal{A}'$. Using Lemma 3.10, we get

$$s_j(J, \mathcal{H}(\mathcal{A}), \mathcal{H}) \le Cs_j\left(J, W_2^2(0,1;\ E(A), E), L_2(0,1,E)\right) \le Cj^{-\frac{2q}{2+q}},$$

i.e $s_j(J, \mathcal{H}(\mathcal{A}), \mathcal{H}) \le Cj^{-p},\ p > \frac{2q}{2+q}$.

By Condition (3) and Lemma [20, 1.2.7/3,], for operators T_{ks}, the Condition (6) of Theorem 3.3 is fulfilled. Consequently, by virtue of Theorem 3.3, we have

$$\|R(-\lambda, \mathcal{A})\| \le C|\lambda|^{-1},\ |\arg(-\lambda)| \ge \pi - \varphi,\ |\lambda| \to \infty.$$

Since $\varphi > \frac{2\pi}{q+2}$, then $\pi - \varphi < \frac{q\pi}{2+q}$. Consequently, all the conditions of Theorem [20, 2.2.3] are fulfilled. This achieves the proof of the Theorem. ∎

5. APPLICATION

Let us consider, in the cylindrical domain $\Omega = [0,1] \times G$, where $G \subset \mathbb{R}^r,\ r \ge 2$ is a bounded domain with an $(r-1)$-dimensional boundary ∂G which locally admits rectification, the non local boundary value problem for an elliptic differential equation of the second order

$$(5.1)\quad \begin{cases} L(\lambda, x, y, D_x, D_y)u = \lambda u(x,y) - D_x^2 u(x,y) - \sum_{i,j=1}^r D_i(a_{ij}(y) D_j u(x,y)) \\ \qquad +A(x)u(x,.)\,|_y = f(x,y),\ \ (x,y) \in [0,1] \times G; \\ L_1 u = \delta u(0,y) + \sum_{s=1}^{N_1} T_{1s} u(x_{1s}) = f_1(y),\ \ y \in G; \\ L_2 u = Bu(0,y) + D_x u(1,y) + \sum_{s=1}^{N_2} T_{2s} u(x_{2s}) = f_2(y),\ \ y \in G; \\ Pu = \sum_{|\eta| \le m} b_\eta D_y^\eta u(x,y') = 0;\ (x,y') \in [0,1] \times \Gamma; \end{cases}$$

where $m \le 1,\ \delta \in \mathbb{C}^*,\ D_x = \frac{\partial}{\partial x},\ D_y^\eta = D_1^{\eta_1} \dots D_r^{\eta_r},\ D_j^\eta = \frac{\partial}{\partial y_j},\ \eta = (\eta_1, \dots, \eta_r),\ |\eta| = \eta_1 + \dots + \eta_r,\ T_{ks}$ are generally speaking, unbounded operators in $L_q(G),\ x_{ks} \in [0,1],\ k = 1, 2$.

Theorem 5.1. *Suppose the following conditions satisfied*

(1) $a_{ij} \in C^1\left(\overline{G}\right),\ b_\eta \in C^{2-m}\left(\overline{G}\right),\ \Gamma \in C^2$;
(2) $a_{ij}(y) = \overline{a_{ji}(y)};\ \exists \gamma > 0$ *such that*

$$\sum_{k,j=1}^r a_{kj}(y)\sigma_k\sigma_j \ge \gamma \sum_{k=1}^r \sigma_k^2,\ y \in \overline{G},\ \sigma \in \mathbb{R}^r;$$

(3) *for any* $\varepsilon > 0$ *and for almost all* $x \in [0,1]$,

$$\|A(x)u\|_{L_q(G)} \leq \varepsilon \|u\|_{W_q^2(G)} + C(\varepsilon)\|u\|_{L_q(G)}$$

the function $A(x)u$ *is measurable on* $[0,1]$ *in* $L_q(G)$*;*

(4) *for* $\varepsilon > 0$ *and* $u \in B_{q,p}^{2-\frac{1}{p}}\left(0,1;\ P_s u = 0,\ m < 2 - \frac{1}{p} - \frac{1}{q}\right)$

$$\begin{aligned}
\|T_{1s}u\|_{B_{q,p}^{2-\frac{1}{p}}(G)} &\leq \varepsilon \|u\|_{B_{q,p}^{2-\frac{1}{p}}(G)} + C(\varepsilon)\|u\|_{L_q(G)}, \\
\|T_{2s}u\|_{B_{q,p}^{1-\frac{1}{p}}(G)} &\leq \varepsilon \|u\|_{B_{q,p}^{2-\frac{1}{p}}(G)} + C(\varepsilon)\|u\|_{L_q(G)} \\
\|T_{1s}u\|_{L_q(G)} &\leq C(\varepsilon)\|u\|_{L_q(G)}, \\
\|T_{2s}u\|_{L_q(G)} &\leq \varepsilon \|u\|_{B_{q,p}^{1}(G)} + C(\varepsilon)\|u\|_{L_q(G)}.
\end{aligned}$$

Then,

(1) *The operator*

$$\mathcal{L}' : u \mapsto (L(\lambda, x, y, D_x, D_y)u, L_1 u, L_2 u)$$

for $|\arg \lambda| \leq \varphi < \pi,\ |\lambda| \to \infty$*, is an isomorphism from* $W_p^2\left(0,1; W_q^2(G; Pu = 0), L_q(G)\right)$ *into*

$$L_p(0,1; L_q(G)) \overset{2}{\underset{k=1}{\oplus}} B_{q,p}^{2-m_k-\frac{1}{p}}\left(G;\ Pu = 0,\ m_s < 2 - m_k - \frac{1}{p} - \frac{1}{q}\right)$$

and for this solution we have the coercive estimate

$$\begin{aligned}
&|\lambda| \|u\|_{L_p(0,1;\ L_q(G))} + \|u\|_{W_p^2\left(0,1;\ W_q^2(G;P_s u=0),L_q(G)\right)} \\
\leq\ & C\left(\|f\|_{L_p(0,1;\ L_q(G))} + \sum_{k=1}^{2} \|f_k\|_{B_{q,p}^{2-m_k-\frac{1}{p}}(G)} + |\lambda|^{1-\frac{m_k}{2}-\frac{1}{2p}} \|f_k\|_{L_q(G)}\right);
\end{aligned}$$

(2) *the system of root vectors of the of problem (5.1) is complete in the space* $L_p(0,1,E)$ *;*

(3) *the system of root vectors of the problem (5.1) form an Abel basis of order* $\alpha \in \left(\frac{r+1}{2}, \frac{\pi}{2(\pi-\varphi)}\right)$ *in the space* $L_2(\Omega)$ *.*

Proof. Consider in $E = L_q(G)$ the operators A and $A(x)$ defined by

$$\begin{aligned}
D(A) &= W_q^2(G;\ Pu = 0),\ Au = -\sum_{i,j=1}^{r} D_i(a_{ij}(y) D_j u(y)) \\
D(A(x)) &= L_q(G),\ A(x)u = A(x)u\,|_y\ .
\end{aligned}$$

Then the problem (5.1) can be rewritten in the form

$$\begin{cases}
-u''(x) + (A + \lambda I)u(x) + A(x)u(x) = f(x); \\
\delta u(0) + \sum_{s=1}^{N_1} T_{1s}u(x_{1s}) = f_1; \\
Bu(0) + u'(1) + \sum_{s=1}^{N_2} T_{2s}u(x_{2s}) = f_2;
\end{cases} \tag{5.2}$$

where $u(x) = u(x,.)$, $f(x) = f(x,.)$ and $f_k = f_k(x)$ are functions with values in the Banach space.

We apply Theorem 3.3 to problem (5.2). It is clear that the space $E = L_q(G)$ is an $UMD-$space which satisfies the multiplier condition and the operator A is positive. It is known (see, e.g, [18]) that for $E = L_q(G)$, the definition of $\mathcal{R}-$boundedness is reduced to the formula

$$\left\| \left(\sum_{j=1}^{n} |T_j u_j|^2 \right)^{\frac{1}{2}} \right\|_{L_q} \leq C \left\| \left(\sum_{j=1}^{n} |u_j|^2 \right)^{\frac{1}{2}} \right\|_{L_q};$$

Using positivity of A and by virtue of the above estimate, we obtain that the operator A is $\mathcal{R}-$positive in $L_q(G)$. Moreover, it is known that the embedding $W_q^2(G) \subset L_q(G)$ is compact (see, e.g, [17]). Then using interpolation properties of Sobolev spaces ([17]) we obtain

$$s_j\left(I, W_q^2(G), L_q(G)\right) \leq C j^{-\frac{2}{r}}.$$

Hence all conditions of Theorem 3.3 are fulfilled. This completes the proof. ∎

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