The Australian Journal of Mathematical Analysis and Applications

http://ajmaa.org

Volume 5, Issue 2, Article 14, pp. 1-15, 2009

A SUBCLASS OF MEROMORPHICALLY MULTIVALENT FUNCTIONS WITH APPLICATIONS TO GENERALIZED HYPERGEOMETRIC FUNCTIONS

M. K. AOUF

Received 20 November, 2006; accepted 21 November, 2007; published 8 June, 2009.

MATHEMATICS DEPARTMENT, FACULTY OF SCIENCE, MANSOURA UNIVERSITY 35516, EGYPT.
mkaouf127@yahoo.com

ABSTRACT. In this paper a new subclass of meromorphically multivalent functions, which is defined by means of a Hadamard product (or convolution) involving some suitably normalized meromorphically p-valent functions. The main object of the present paper is to investigate the various important properties and characteristics of this subclass of meromorphically multivalent functions. We also derive many interesting results for the Hadamard products of functions belonging to this subclass. Also we consider several applications of our main results to generalized hypergeomtric functions.

Key words and phrases: Meromorphic functions, Hadamard product, Starlike functions, Convex functions, Generalized hypergeometric functions.

2000 *Mathematics Subject Classification.* 30C45.

ISSN (electronic): 1449-5910

1. INTRODUCTION

Let Σ_p denote the class of meromorphic functions $f(z)$ normalized by

$$f(z) = z^{-p} + \sum_{k=p}^{\infty} a_k z^k \quad (a_k \geq 0; p \in N = \{1, 2, \dots\}), \tag{1.1}$$

which are analytic and p-valent in the punctured disc $U^* = U^*(1)$, where

$$U^*(r) = \{z : z \in C \text{ and } 0 < |z| < r\, (0 < r \leq 1)\} = U(r) \backslash \{0\}\, (U(1) \equiv U).$$

A function $f(z) \in \Sigma_p$ is said to be meromorphically p-valent starlike of order $\delta\, (0 \leq \delta < p)$ in $U(r)$ if

$$\operatorname{Re}\left\{\frac{zf^{'}(z)}{f(z)}\right\} < -\delta \quad (z \in U(r);\; 0 < r \leq 1; 0 \leq \delta < p). \tag{1.2}$$

On the other hand, a function $f(z) \in \Sigma_p$ is said to be meromorphically p-valent convex of order $\delta\, (0 \leq \delta < p)$ in $U(r)$ if

$$\operatorname{Re}\left\{1 + \frac{zf^{''}(z)}{f'(z)}\right\} < -\delta \quad (z \in U(r);\; 0 < r \leq 1; 0 \leq \delta < p). \tag{1.3}$$

The Hadamard product (or convolution) of the function $f(z)$ defined by (1.1) with the functions $g(z)$ and $h(z)$ given, respectively, by

$$g(z) = z^{-p} + \sum_{k=1}^{\infty} b_k z^{k-p} \quad (b_k \geq 0; p \in N) \tag{1.4}$$

and

$$h(z) = z^{-p} + \sum_{k=1}^{\infty} c_k z^{k-p} \quad (c_k \geq 0; p \in N) \tag{1.5}$$

can be expressed as follows:

$$(f * g)(z) = z^{-p} + \sum_{k=p}^{\infty} a_k b_{k+p} z^k = (g * f)(z) \tag{1.6}$$

and

$$(f * h)(z) = z^{-p} + \sum_{k=p}^{\infty} a_k c_{k+p} z^k = (h * f)(z)\ , \tag{1.7}$$

where we have assumed that

$$b_j = c_j = 0\, (j = 1, 2, \dots, 2p-1; p \in N).$$

Let $f(z)$ and $g(z)$ be analytic in U. Then we say that the function $g(z)$ is subordinate to the function $f(z)$ and we write $g(z) \prec f(z)(z \in U)$, if there exists a Schwarz function $w(z)$ with $w(0) = 0$ and $|w(z)| < 1(z \in U)$ such that $g(z) = f(w(z))(z \in U)$.

In recent years, various subclasses of the class Σ_p defined by using convolution were studied by Raina and Srivastava [18] and Kumar et al. [10].

Making use the above subordination definition, we introduce here a new class $C_m(g, h; A, B, \lambda, \gamma)$ of meromorphically multivalent functions, which is defined as follows.

Definition 1.1. A function $f(z) \in \Sigma_p$ is said to be in the class $C_m(g, h; A, B, \lambda, \gamma)$ if it satisfies the following subordination condition:

$$\gamma \frac{((f*g)(z))^{(m)}}{((f*h)(z))^m} \prec \gamma - \frac{(A-B)(p-\lambda)z}{1+Bz} (z \in U^*) \tag{1.8}$$

$$(\gamma > 0; 0 \leq B < A \leq 1, 0 \leq \lambda < p; b_k \geq c_k \geq 0 (k \geq p; p \in N); p \geq m; \; m \in N_0^* = \{2j-1 : j \in N\} \cup \{0\}) \, , \tag{1.9}$$

provided that $((f*h)(z))^{(m)} \neq 0 \, (z \in U^*)$. We note that $C_m(g, h; A, B, 0, \gamma) = C_m(g, h; A, B, \gamma)$ (Raina and Srivastava [18]).

Meromorphically multivalent functions have been extensively studied by (for example) Mogra ([15] and [16]), Uralegaddi and Ganigi [22], Aouf ([1] and [2]), Aouf and Hossen [3], Chen et al. [5], Srivastava et al. [20], Owa et al. [17], Joshi and Aouf [7], Joshi and Srivastava [8], Aouf et al. [4], Raina and Srivastava [18], Kulkarni et al. [9], Liu [11], Liu and Srivastava ([12], [13] and [14]), Uralegaddi and Somanatha [23] and Yang ([24] and [25]).

In this paper we obtain the coefficient estimates, distortion properties and the radii of starlikeness and convexity for functions in the class $C_m(g, h; A, B, \lambda, \gamma)$. We also derive many interesting results for the Hadamard products of functions belonging to the class $C_m(g, h; A, B, \lambda, \gamma)$. Several applications of the main results involving generalized hypergeometric functions are considered. All the results are sharp.

2. PROPERTIES OF THE P-VALENTLY MEROMORPHIC FUNCTION CLASS $C_m(g, h; A, B, \lambda, \gamma)$

We first determine a necessary and sufficient condition for a function $f(z) \in \Sigma_p$ of the form (1.1) to be in the class $C_m(g, h; A, B, \lambda, \gamma)$ of meromorphically p-valent functions with positive coefficients.

Theorem 2.1. *Let $f(z) \in \Sigma_p$ be given by (1.1). Then $f(z) \in C_m(g, h; A, B, \lambda, \gamma)$ if and only if*

$$\sum_{k=p}^{\infty} a_k \left\{\gamma(1+B)b_{k+p} + [(A-B)(p-\lambda) - \gamma(1+B)]c_{k+p}\right\} \binom{k}{m} \leq (A-B)(p-\lambda)\binom{p+m-1}{m} \; (k \geq p; p \in N) \, . \tag{2.1}$$

Proof. Let $f(z) \in C_m(g, h; A, B, \lambda, \gamma)$ be given by (1.1). Then, in view of (1.6) to (1.8), we find for $m \in N_0^*$ that

$$\left| \frac{\gamma \sum_{k=p}^{\infty} a_k (b_{k+p} - c_{k+p}) \binom{k}{m} z^{k+p}}{(A-B)(p-\lambda)\binom{p+m-1}{m} - \sum_{k=p}^{\infty} a_k \left\{\gamma B b_{k+p} + [(A-B)(p-\lambda) - \gamma B] c_{k+p}\right\} \binom{k}{m} z^{k+p}} \right| < 1 (z \in U) \, . \tag{2.2}$$

Putting $z = r (0 \leq r < 1)$, and noting the fact that the denominator in the inequality (2.2) remains positive by virtue of the constraints stated in (1.9) for all $r \in [0, 1)$, we easily arrive at the desired inequality (2.1) by letting $z \to 1^-$ in (2.2).

Conversely, we assume that (2.1) holds true. Then from (1.1) and (2.1), we get

$$\left|\frac{\gamma[((f*g)(z))^{(m)}-((f*h)(z))^{(m)}]}{\gamma B[((f*g)(z))^{(m)}-((f*h)(z))^{(m)}]+(A-B)(p-\lambda)((f*h)(z))^{m}}\right|$$

$$\leq \frac{\gamma\sum_{k=p}^{\infty}a_k(b_{k+p}-c_{k+p})\binom{k}{m}}{(A-B)(p-\lambda)\binom{p+m-1}{m}-\sum_{k=p}^{\infty}a_k\left\{\gamma Bb_{k+p}+[(A-B)(p-\lambda)-\gamma B]c_{k+p}\right\}\binom{k}{m}}$$

$$< 1 (z\in U). \tag{2.3}$$

Hence, by the maximum modulus theorem, we have $f(z)\in C_m(g,h;A,B,\lambda,\gamma)$, we complete the proof of Theorem 2.1. ∎

Corollary 2.2. *Let the function $f(z)$ defined by (1.1) be in the class $C_m(g,h;A,B,\lambda,\gamma)$. Then*

$$a_k \leq \frac{(A-B)(p-\lambda)\binom{p+m-1}{m}}{\{\gamma(1+B)b_{k+p}+[(A-B)(p-\lambda)-\gamma(1+B)]c_{k+p}\}\binom{k}{m}} \quad (k\geq p; p\in N), \tag{2.4}$$

where the equality holds true for the function $f(z)$ given by

$$f(z)=z^{-p}+\frac{(A-B)(p-\lambda)\binom{p+m-1}{m}}{\left\{\gamma(1+B)b_{k+p}+[(A-B)(p-\lambda)-\gamma(1+B)]c_{k+p}\right\}\binom{k}{m}}z^k \quad (k\geq p; p\in N). \tag{2.5}$$

Next, we prove the following growth and distortion properties for the class $C_m(g,h;A,B,\lambda,\gamma)$.

Theorem 2.3. *Let a function $f(z)\in\Sigma_p$ of the form (1.1) belong to the class $C_m(g,h;A,B,\lambda,\gamma)$. If the sequence $\{\eta_k\}$ is nondecreasing, then*

$$r^{-p}-\frac{(A-B)(p-\lambda)\binom{p+m-1}{m}}{\eta_p}r^p \leq |f(z)|\leq r^{-p}+\frac{(A-B)(p-\lambda)\binom{p+m-1}{m}}{\eta_p}r^p$$
$$(0 < |z|=r<1), \tag{2.6}$$

where

$$\eta_k = \eta_k(p,A,B,\lambda,\gamma,m)$$
$$= \{\gamma(1+B)b_{k+p}+[(A-B)(p-\lambda)-\gamma(1+B)]c_{k+p}\}\binom{k}{m} \quad (k\geq p; p\in N). \tag{2.7}$$

If the sequence $\left\{\frac{\eta_k}{k}\right\}$ is nondecreasing, then

$$-pr^{-p-1}-\frac{p(A-B)(p-\lambda)\binom{p+m-1}{m}}{\eta_p}r^{p-1}\leq\left|f^{'}(z)\right|\leq -pr^{-p-1}+$$

$$\frac{p(A-B)(p-\lambda)\binom{p+m-1}{m}}{\eta_p}r^{p-1} \; (0<|z|=r<1). \tag{2.8}$$

The results (2.6) and (2.8) are sharp with the extremal function $f(z)$ given by

$$f(z) = z^{-p} + \frac{(A-B)(p-\lambda)\binom{p+m-1}{m}}{\eta_p} z^p \; (p \in N) \; . \tag{2.9}$$

Proof. Let a function $f(z)$ of the form (1.1) belong to the class $C_m(g,h;A,B,\lambda,\gamma)$. If the sequence $\{\eta_k\}$ is nondecreasing and positive, by Theorem 2.1, we have

$$\sum_{k=p}^{\infty} a_k \leq \frac{(A-B)(p-\lambda)\binom{p+m-1}{m}}{\eta_p} \tag{2.10}$$

and if the sequence $\left\{\frac{\eta_k}{k}\right\}$ is nondecreasing and positive, by Theorem 2.1, we have

$$\sum_{k=p}^{\infty} k a_k \leq \frac{p(A-B)(p-\lambda)\binom{p+m-1}{m}}{\eta_p} . \tag{2.11}$$

Making use of the conditions (2.10) and (2.11), in conjunction with the definition (1.1), we readily obtain the assertions (2.6) and (2.8) of Theorem 2.3. ∎

Remark 2.1. Putting $\lambda = 0$ in Theorem 2.3, we obtain the correct result for the class $C_m(g,h;A,B,\gamma)$ obtained by Raina and Srivastava [[18], Theorem 2].

We next determine the radii of meromorphically p-valent starlikeness of order $\delta (0 \leq \delta < p)$ and meromorphically p-valent convexity of order $\delta (0 \leq \delta < p)$ for functions in the class $C_m(g,h;A,B,\lambda,\gamma)$, which are given by Theorem 2.4 below.

Theorem 2.4. *Let the function $f(z)$ defined by (1.1) be in the class $C_m(g,h;A,B,\lambda,\gamma)$. Then*

(i) $f(z)$ is meromorphically p-valent starlike of order δ $(0 \leq \delta < p)$ in the disc $|z| < r_1$, that is,

$$\text{Re}\left\{-\frac{zf'(z)}{f(z)}\right\} > \delta \; (|z| < r_1; 0 \leq \delta < p; p \in N) \; , \tag{2.12}$$

where

$$r_1 = \inf_{k \geq p} \left\{ \frac{(p-\delta)\eta_k}{(k+\delta)(A-B)(p-\lambda)\binom{p+m-1}{m}} \right\}^{\frac{1}{k+p}} . \tag{2.13}$$

(ii) $f(z)$ is meromorphically p-valent convex of order δ $(0 \leq \delta < p)$ in the disc $|z| < r_2$, that is,

$$\text{Re}\left\{-(1 + \frac{zf''(z)}{f'(z)})\right\} > \delta \;\; (|z| < r_2; 0 \leq \delta < p; \; p \in N) \; , \tag{2.14}$$

where

$$r_2 = \inf_{k \geq p} \left\{ \frac{p(p-\delta)\eta_k}{k(k+\delta)(A-B)(p-\lambda)\binom{p+m-1}{m}} \right\}^{\frac{1}{k+p}} . \tag{2.15}$$

The sequence $\{\eta_k\}$ occurring in (2.13) and (2.15) is given by (2.7). Each of these results is sharp for the function $f(z)$ given by (2.5).

Proof. (i) From the definition (1.1), we easily get

$$\left|\frac{\frac{zf'(z)}{f(z)}+p}{\frac{zf'(z)}{f(z)}-p+2\delta}\right| \leq \frac{\sum_{k=p}^{\infty}(k+p)a_k|z|^{k+p}}{2(p-\delta)-\sum_{k=p}^{\infty}(k-p+2\delta)a_k|z|^{k+p}} . \tag{2.16}$$

Thus we have the desired inequality :

$$\left|\frac{\frac{zf'(z)}{f(z)}+p}{\frac{zf'(z)}{f(z)}-p+2\delta}\right| \leq 1\,(0\leq\delta<p; p\in N) \tag{2.17}$$

if

$$\sum_{k=p}^{\infty}(\frac{k+\delta}{p-\delta})a_k|z|^{k+p} \leq 1 . \tag{2.18}$$

Hence, by Theorem 2.1, (2.18) will be true if

$$(\frac{k+\delta}{p-\delta})|z|^{k+p} \leq \left\{\frac{\eta_k}{(A-B)(p-\lambda)\binom{p+m-1}{m}}\right\}(k\geq p; p\in N) . \tag{2.19}$$

The last inequality (2.19) leads us immediately to the disc $|z|<r_1$, where r_1 is given by (2.13).

(ii) In order to prove the second assertion of Theorem 2.4, we find from the definition (1.1) that

$$\left|\frac{1+\frac{zf''(z)}{f'(z)}-p}{1+\frac{zf''(z)}{f'(z)}-p+2\delta}\right| \leq \frac{\sum_{k=p}^{\infty}k(k+p)a_k|z|^{k+p}}{2p(p-\delta)-\sum_{k=p}^{\infty}k(k-p+2\delta)a_k|z|^{k+p}} . \tag{2.20}$$

Thus we have the desired inequality:

$$\left|\frac{1+\frac{zf''(z)}{f'(z)}+p}{1+\frac{zf''(z)}{f'(z)}-p+2\delta}\right| \leq 1\,(0\leq\delta<p; p\in N) , \tag{2.21}$$

if

$$\sum_{k=p}^{\infty}\frac{k(k+\delta)}{p(p-\delta)}a_k|z|^{k+p} \leq 1 . \tag{2.22}$$

Hence, by Theorem 2.1, (2.22) will be true if

$$\frac{k(k+\delta)}{p(p-\delta)}|z|^{k+p} \leq \left\{\frac{\eta_k}{(A-B)(p-\lambda)\binom{p+m-1}{m}}\right\}(k\geq p; p\in N) . \tag{2.23}$$

This last inequality (2.23) readily yields the disc $|z|<r_2$, with r_2 is defined by (2.15), and the proof of Theorem 2.4 is completed by merely verifying that each assertion is sharp for the function $f(z)$ given by (2.5). ∎

3. CONVOLUTION PROPERTIES FOR THE P-VALENTLY MEROMORPHIC FUNCTION CLASS $C_m(g, h; A, B, \lambda, \gamma)$

For the functions

$$f_j(z) = z^{-p} + \sum_{k=p}^{\infty} a_{k,j} z^k \ (a_{k,j} \geq 0; j = 1, 2; p \in N), \tag{3.1}$$

we denote by $(f_1 \otimes f_2)(z)$ the Hadamard product (or convolution) of the functions $f_1(z)$ and $f_2(z)$, that is,

$$(f_1 \otimes f_2)(z) = z^{-p} + \sum_{k=p}^{\infty} a_{k,1} a_{k,2} z^k . \tag{3.2}$$

Throughout this section, we assume that the sequence $\{\eta_k\}$ is nondecreasing, where η_k is given by (2.7).

Theorem 3.1. *Let the functions $f_j(z)(j = 1, 2)$ defined by (3.1) be in the class $C_m(g, h; A, B, \lambda, \gamma)$. Then $(f_1 \otimes f_2)(z) \in C_m(g, h; A, B, \zeta, \gamma)$, where*

$$\begin{aligned} \zeta &= p - \\ &\frac{\binom{p+m-1}{m}\binom{p}{m}\gamma(1+B)(b_{2p}-c_{2p})(A-B)(p-\lambda)^2}{[\eta_p(p,A,B,\lambda,\gamma,m)]^2 - \binom{p+m-1}{m}\binom{p}{m}c_{2p}(A-B)^2(p-\lambda)^2}, \end{aligned} \tag{3.3}$$

where $\eta_k(p, A, B, \lambda, \gamma, m)$ is defined by (2.7). The result is sharp for the functions $f_j(z)(j = 1, 2)$ given by

$$f_j(z) = z^{-p} + \frac{(A-B)(p-\lambda)\binom{p+m-1}{m}}{\eta_p(p,A,B,\lambda,\gamma,m)} z^p \ (j = 1, 2; p \in N) . \tag{3.4}$$

Proof. Employing the technique used earlier by Schild and Silverman [19], we need to find the largest ζ such that

$$\sum_{k=p}^{\infty} \frac{\eta_k(p,A,B,\zeta,\gamma,m)}{(A-B)(p-\zeta)\binom{p+m-1}{m}} a_{k,1} a_{k,2} \leq 1 \tag{3.5}$$

for $f_j(z) \in C_m(g, h; A, B, \zeta, \gamma)(j = 1, 2)$. Since $f_j(z) \in C_m(g, h; A, B, \lambda, \gamma)(j = 1, 2)$, we readily see that

$$\sum_{k=p}^{\infty} \frac{\eta_k(p,A,B,\lambda,\gamma,m)}{(A-B)(p-\lambda)\binom{p+m-1}{m}} a_{k,j} \leq 1 \ (j = 1, 2) . \tag{3.6}$$

Therefore, by the Cauchy-Schwarz inequality, we obtain

$$\sum_{k=p}^{\infty} \frac{\eta_k(p,A,B,\lambda,\gamma,m)}{(A-B)(p-\lambda)\binom{p+m-1}{m}} \sqrt{a_{k,1} a_{k,2}} \leq 1 . \tag{3.7}$$

This implies that we only need to show that

$$\frac{\eta_k(p,A,B,\zeta,\gamma,m)}{(p-\zeta)} a_{k,1} a_{k,2} \leq \frac{\eta_k(p,A,B,\lambda,\gamma,m)}{(p-\lambda)} \sqrt{a_{k,1} a_{k,2}} \ (k \geq p; p \in N) . \tag{3.8}$$

or, equivalently, that

$$\sqrt{a_{k,1}a_{k,2}} \leq \frac{(p-\zeta)\eta_k(p,A,B,\lambda,\gamma,m)}{(p-\lambda)\eta_k(p,A,B,\zeta,\gamma,m)} \; (k \geq p; p \in N)\,. \tag{3.9}$$

Hence, by the inequality (3.7), it is sufficient to prove that

$$\frac{(A-B)(p-\lambda)\binom{p+m-1}{m}}{\eta_k(p,A,B,\lambda,\gamma,m)} \leq \frac{(p-\zeta)\eta_k(p,A,B,\lambda,\gamma,m)}{(p-\lambda)\eta_k(p,A,B,\zeta,\gamma,m)} \quad (k \geq p; p \in N)\,. \tag{3.10}$$

It follows from (3.10) that

$$\zeta \leq p - \frac{\binom{p+m-1}{m}\binom{k}{m}\gamma(1+B)(b_{k+p}-c_{k+p})(A-B)(p-\lambda)^2}{[\eta_k(p,A,B,\lambda,\gamma,m)]^2 - \binom{p+m-1}{m}\binom{k}{m}c_{k+p}(A-B)^2(p-\lambda)^2} \quad (k \geq p; p \in N). \tag{3.11}$$

Now, defining the function $\varphi(k)$ by

$$\varphi(k) = p - \frac{\binom{p+m-1}{m}\binom{k}{m}\gamma(1+B)(b_{k+p}-c_{k+p})(A-B)(p-\lambda)^2}{[\eta_k(p,A,B,\lambda,\gamma,m)]^2 - \binom{p+m-1}{m}\binom{k}{m}c_{k+p}(A-B)^2(p-\lambda)^2} \quad (k \geq p; p \in N)\,, \tag{3.12}$$

we see that $\varphi(k)$ is an increasing function of k. Therefore, we conclude that

$$\zeta \leq \varphi(p) = p - \frac{\binom{p+m-1}{m}\binom{p}{m}\gamma(1+B)(b_{2p}-c_{2p})(A-B)(p-\lambda)^2}{[\eta_p(p,A,B,\lambda,\gamma,m)]^2 - \binom{p+m-1}{m}\binom{p}{m}c_{2p}(A-B)^2(p-\lambda)^2}\,, \tag{3.13}$$

which evidently completes the proof of Theorem 3.1.

Using arguments similar to those in the proof of Theorem 3.1, we obtain the following result. ∎

Theorem 3.2. *Let the function $f_1(z)$ defined by (3.1) be in the class $C_m(g,h;A,B,\lambda_1,\gamma)$. Suppose also that the function $f_2(z)$ defined by (3.1) be in the class $C_m(g,h;A,B,\lambda_2,\gamma)$. Then $(f_1 \otimes f_2)(z) \in C_m(g,h;A,B,\xi,\gamma)$, where*

$$\xi = p - \frac{\binom{p+m-1}{m}\binom{p}{m}\gamma(1+B)(b_{2p}-c_{2p})(A-B)(p-\lambda_1)(p-\lambda_2)}{[\eta_p(p,A,B,\lambda_1,\gamma,m)][\eta_p(p,A,B,\lambda_2,\gamma,m)] - \Lambda}\,, \tag{3.14}$$

$$\left(\Lambda = \binom{p+m-1}{m}\binom{p}{m}c_{2p}(A-B)^2(p-\lambda_1)(p-\lambda_2)\right)\,.$$

The result is sharp for the functions $f_j(z)(j=1,2)$ *given by*

$$f_1(z) = z^{-p} + \frac{(A-B)(p-\lambda_1)\begin{pmatrix} p+m-1 \\ m \end{pmatrix}}{\eta_p(p,A,B,\lambda_1,\gamma,m)} z^p \ \ (p \in N)\,, \tag{3.15}$$

and

$$f_2(z) = z^{-p} + \frac{(A-B)(p-\lambda_2)\begin{pmatrix} p+m-1 \\ m \end{pmatrix}}{\eta_p(p,A,B,\lambda_2,\gamma,m)} z^p \ \ (p \in N)\,. \tag{3.16}$$

Theorem 3.3. *Let the functions* $f_j(z)(j=1,2)$ *defined by (3.1) be in the class* $C_m(g,h;A,B,\lambda,\gamma)$. *Then the function* $h(z)$ *defined by*

$$h(z) = z^{-p} + \sum_{k=p}^{\infty}(a_{k,1}^2 + a_{k,2}^2)z^k \tag{3.17}$$

belongs to the class $C_m(g,h;A,B,\Im,\gamma)$, *where*

$$\Im = p - \frac{2\begin{pmatrix} p+m-1 \\ m \end{pmatrix}\begin{pmatrix} p \\ m \end{pmatrix}\gamma(1+B)(b_{2p}-c_{2p})(A-B)(p-\lambda)^2}{[\eta_p(p,A,B,\lambda,\gamma,m)]^2 - 2\begin{pmatrix} p+m-1 \\ m \end{pmatrix}\begin{pmatrix} p \\ m \end{pmatrix}c_{2p}(A-B)^2(p-\lambda)^2}\,. \tag{3.18}$$

The result is sharp for the functions $f_j(z)(j=1,2)$ *given already by (3.4).*

Proof. Noting that

$$\sum_{k=p}^{\infty}\frac{[\eta_k(p,A,B,\lambda,\gamma,m)]^2}{[(A-B)(p-\lambda)\begin{pmatrix} p+m-1 \\ m \end{pmatrix}]^2}a_{k,j}^2 \leq$$

$$\left\{\sum_{k=p}^{\infty}\frac{\eta_k(p,A,B,\lambda,\gamma,m)}{(A-B)(p-\lambda)\begin{pmatrix} p+m-1 \\ m \end{pmatrix}}a_{k,j}\right\}^2 \leq 1 \ (j=1,2), \tag{3.19}$$

for $f_j(z) \in C_m(g,h;A,B,\lambda,\gamma)(j=1,2)$, we have

$$\sum_{k=p}^{\infty}\frac{1}{2}\frac{[\eta_k(p,A,B,\lambda,\gamma,m)]^2}{[(A-B)(p-\lambda)\begin{pmatrix} p+m-1 \\ m \end{pmatrix}]^2}(a_{k,1}^2 + a_{k,2}^2) \leq 1\,. \tag{3.20}$$

Therefore, we have to find the largest $\Im$ such that

$$\frac{\eta_k(p,A,B,\Im,\gamma,m)}{(p-\Im)} \leq \frac{[\eta_k(p,A,B,\lambda,\gamma,m)]^2}{2(A-B)(p-\lambda)^2\begin{pmatrix} p+m-1 \\ m \end{pmatrix}} \ (k \geq p; p \in N) \tag{3.21}$$

that is, that

$$\Im \leq p - \frac{2\binom{p+m-1}{m}\binom{k}{m}\gamma(1+B)(b_{k+p}-c_{k+p})(A-B)(p-\lambda)^2}{[\eta_k(p,A,B,\lambda,\gamma,m)]^2 - 2\binom{p+m-1}{m}\binom{k}{m}c_{k+p}(A-B)^2(p-\lambda)^2} \quad (k \geq p; p \in N). \tag{3.22}$$

Now, defining a function $\Psi(k)$ by

$$\Psi(k) = p - \frac{2\binom{p+m-1}{m}\binom{k}{m}\gamma(1+B)(b_{k+p}-c_{k+p})(A-B)(p-\lambda)^2}{[\eta_k(p,A,B,\lambda,\gamma,m)]^2 - 2\binom{p+m-1}{m}\binom{k}{m}c_{k+p}(A-B)^2(p-\lambda)^2} \quad (k \geq p; p \in N), \tag{3.23}$$

we observe that $\Psi(k)$ is an increasing function of k. We thus conclude that

$$\Im \leq \Psi(p) = p - \frac{2\binom{p+m-1}{m}\binom{p}{m}\gamma(1+B)(b_{2p}-c_{2p})(A-B)(p-\lambda)^2}{[\eta_p(p,A,B,\lambda,\gamma,m)]^2 - 2\binom{p+m-1}{m}\binom{p}{m}c_{2p}(A-B)^2(p-\lambda)^2}, \tag{3.24}$$

which completes the proof of Theorem 3.3. ∎

4. Applications Involving Generalized Hypergeometric Functions

In order to obtain some applications of Theorems 2.1-3.3 to the generalized hypergeometric functions, we first put the sequences $\{b_k\}$ and $\{c_k\}$, which are involved in (1.4) and (1.5), as follows:

$$b_k = \left(\frac{\alpha_1+k}{\alpha_1}\right)c_k = \frac{(\alpha_1+1)_k(\alpha_2)_k\ldots(\alpha_q)_k}{(\beta_1)_k\ldots(\beta_s)_k}\,\frac{1}{k!}\,, \tag{4.1}$$

where

$$c_k = \frac{(\alpha_1)_k\ldots(\alpha_q)_k}{(\beta_1)_k\ldots(\beta_s)_k}\,\frac{1}{k!}\,. \tag{4.2}$$

We assume further that

$$\alpha_j > 0\,(j = 1,\ldots,q) \text{ and } \beta_j > 0 \ \ (j = 1,\ldots,s)\,.$$

The corresponding functions $g(z)$ and $h(z)$ defined by (1.4) and (1.5) then become

$$g(z) = z^{-p}\,{}_qF_s(\alpha_1+1,\alpha_2,\ldots,\alpha_q;\beta_1,\ldots,\beta_s;z) \tag{4.3}$$

and

$$h(z) = z^{-p}\,{}_qF_s(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s;z)\,, \tag{4.4}$$

where ${}_qF_s(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s;z)$ is the familiar generalized hypergeometric function defined by (see, for example, [21], p. 19)

$$ {}_qF_s(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s;z) = \sum_{k=0}^{\infty}\frac{(\alpha_1)_k\ldots(\alpha_q)_k}{(\beta_1)_k\ldots(\beta_s)_k}\cdot\frac{z^k}{k!}\ (q \leq s+1; q,s\in N_0 = N\cup\{0\}; z\in U)\,. \tag{4.5}$$

where $(\theta)_\nu$ is the Pochhammer symbol defined, in terms of the Gamma function, by

$$(\theta)_\nu = \frac{\Gamma(\theta+\nu)}{\Gamma(\theta)} = \begin{cases} 1, & (\nu = 0; \theta\in C\backslash\{0\}),\\ (\theta+1)\ldots(\theta+k-1), & (\nu\in N; \theta\in C)\,. \end{cases} \tag{4.6}$$

Making use of (4.4), the Hadamard product defined by (1.7) can be used to represent the Dziok-Srivastava linear operator (cf. [6], p. 3)

$$H_p(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s):\Sigma_p\to\Sigma_p$$

by means of the following relation :

$$\begin{aligned}(H_p(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s)f)(z) &= z^{-p}\,{}_qF_s(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s;z) * f(z)\\ &= z^{-p} + \sum_{k=p}^{\infty} a_k c_{k+p} z^k = (f*h)(z)\,,\end{aligned} \tag{4.7}$$

where c_k is given by (4.2). Also, in view of (4.1), the Hadamard product defined by (1.6) represents a relation similar to (4.7), involving the Dziok-Srivastava operator $H_p(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s)$, which is given by

$$\begin{aligned}(H_p(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s)f)(z) &= z^{-p}\,{}_qF_s(\alpha_1+1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s;z) * f(z)\\ &= z^{-p} + \sum_{k=p}^{\infty} a_k(\frac{\alpha_1+k+p}{\alpha_1}) c_{k+p} z^k = (f*g)(z)\,,\end{aligned} \tag{4.8}$$

where c_k is given by (4.2).

The subordination relation (1.8) in conjunction with (4.7) and (4.8) takes the following form:

$$\gamma\frac{((H_p(\alpha_1+1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s)f)(z))^{(m)}}{((H_p(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s)f)(z))^{(m)}} \prec \gamma - \frac{(A-B)(p-\lambda)z}{1+Bz}$$

$$(\gamma>0; 0\leq B<A\leq 1; 0\leq\lambda<p; p\geq m; p,q,s\in N; m\in N_0^*). \tag{4.9}$$

Definition 4.1. A function $f(z)\in\Sigma_p$ of the form (1.1) is said to in the class $C_{p,q,s,m}(\alpha_1; A, B, \lambda, \gamma)$ if it satisfies the subordination relation (4.9) above.

We note that:
(i) $C_{p,q,s,m}(\alpha_1; A, B, 0, \gamma) = C_{p,q,s,m}(\alpha_1; A, B, \gamma)$ (Raina and Srivastava [18]);
(ii) $C_{p,q,s,1}(\alpha_1; A, B, 0, \alpha_1) = \Omega^+_{p,q,s}(\alpha_1; A, B)$ (Liu and Srivastava [13]).

The following consequences of Theorem 2.1 to Theorem 3.3 can be deduced by applying (4.1) and (4.2) along with Definition 4.1.

Corollary 4.1. *A function $f(z)\in\Sigma_p$ of the form (1.1) belongs to the class $C_{p,q,s,m}(\alpha_1; A, B, \lambda, \gamma)$ if and only if*

$$\sum_{k=p}^{\infty} a_k \left[\frac{\gamma(k+p)(1+B)}{\alpha_1} + (A-B)(p-\lambda)\right] \binom{k}{m} c_{k+p} \tag{4.10}$$
$$\leq (A-B)(p-\lambda) \binom{p+m-1}{m},$$

where c_k is given by (4.2).

Corollary 4.2. *Let the function $f(z)$ defined by (1.1) be in the class $C_{p,q,s,m}(\alpha_1; A, B, \lambda, \gamma)$. Then*

$$a_k \leq \frac{(A-B)(p-\lambda)\binom{p+m-1}{m}}{\left[\frac{\gamma(k+p)(1+B)}{\alpha_1} + (A+B)(p-\lambda)\right]\binom{k}{m} c_{k+p}} \; (k \geq p; p \in N)\,. \tag{4.11}$$

The result is sharp for the function $f(z)$ given by

$$f(z) = z^{-p} + \frac{(A-B)(p-\lambda)\binom{p+m-1}{m}}{\left[\frac{\gamma(k+p)(1+B)}{\alpha_1} + (A+B)(p-\lambda)\right]\binom{k}{m} c_{k+p}} z^k \; (k \geq p; p \in N)\,. \tag{4.12}$$

Corollary 4.3. *Let a function $f(z) \in \Sigma_p$ of the form (1.1) belong to the class $C_{p,q,s,m}(\alpha_1; A, B, \lambda, \gamma)$. If the sequence $\{\eta_k^*\}$ is nondecreasing, then*

$$r^{-p} - \frac{(A-B)(p-\lambda)\binom{p+m-1}{m}}{\eta_p^*} r^p \leq |f(z)| \leq r^{-p} +$$
$$\frac{(A-B)(p-\lambda)\binom{p+m-1}{m}}{\eta_p^*} r^p (0 < |z| = r < 1)\,, \tag{4.13}$$

where

$$\eta_k^* = \eta_k^*(p, A, B, \lambda, \gamma, m, \alpha_1)$$
$$= \left[\frac{\gamma(k+p)(1+B)}{\alpha_1} + (A-B)(p-\lambda)\right]\binom{k}{m} c_{k+p} (k \geq p; p \in N)\,. \tag{4.14}$$

If the sequence $\left\{\frac{\eta_k^*}{k}\right\}$ is nondecreasing , then

$$-pr^{-p-1} - \frac{p(A-B)(p-\lambda)\binom{p+m-1}{m}}{\eta_p^*} r^{p-1} \leq |f(z)| \leq -pr^{-p-1} +$$

$$\frac{p(A-B)(p-\lambda)\binom{p+m-1}{m}}{\eta_p^*} r^{p-1} (0 < |z| = r < 1). \tag{4.15}$$

The results (4.13) and (4.15) are sharp with the extremal function $f(z)$ given by

$$f(z) = z^{-p} + \frac{(A-B)(p-\lambda)\binom{p+m-1}{m}}{\eta_p^*} z^k (p \in N)\,. \tag{4.16}$$

Remark 4.1. Putting $\lambda = 0$ in Corollary **??**, we obtain the correct result for the class $C_{p,q,s,m}(\alpha_1; A, B, \gamma)$ obtained by Raina and Srivastava [[18], Corollary 3].

Corollary 4.4. *Let a function $f(z) \in \Sigma_p$ of the form (1.1) belong to the class $C_{p,q,s,m}(\alpha_1; A, B, \lambda, \gamma)$. Then*

(i) $f(z)$ is meromorphically p-valent starlike of order $\delta\,(0 \le \delta < p)$ in the disc $|z| < r_1^*$, where

$$r_1^* = \inf_{k \ge p} \left\{ \frac{(p-\delta)\eta_k^*}{(k+\delta)(A-B)(p-\lambda)\binom{p+m-1}{m}} \right\}^{\frac{1}{k+p}}. \tag{4.17}$$

(ii) $f(z)$ is meromorphically p-valent convex of order $\delta\,(0 \le \delta < p)$ in the disc $|z| < r_2^*$, that is,

$$r_2^* = \inf_{k \ge p} \left\{ \frac{p(p-\delta)\eta_k^*}{k(k+\delta)(A-B)(p-\lambda)\binom{p+m-1}{m}} \right\}^{\frac{1}{k+p}}. \tag{4.18}$$

The sequence $\{\eta_k^*\}$ occuring in (4.17) and (4.18) is given by (4.14). Each of these results is sharp for the function $f(z)$ given by (4.12).

Corollary 4.5. *Let the functions $f_j(z)(j = 1, 2)$ defined by (3.1) be in the class $C_{p,q,s,m}(\alpha_1; A, B, \lambda, \gamma)$. If the sequence $\{\eta_k^*\}$ is nondecreasing, then $(f_1 \otimes f_2)(z) \in C_{p,q,s,m}(\alpha_1; A, B, \zeta, \gamma)$, where*

$$\zeta = p - \frac{\frac{2p}{\alpha_1}\binom{p+m-1}{m}\binom{p}{m}\gamma(1+B)(A-B)(p-\lambda)^2}{[\eta_p^*(p, A, B, \lambda, \gamma, m, \alpha_1)]^2 - \binom{p+m-1}{m}\binom{p}{m} c_{2p}(A-B)^2(p-\lambda)^2}, \tag{4.19}$$

where $\eta_p^*(p, A, B, \lambda, \gamma, m, \alpha_1)$ is given by (4.12). The result is sharp for the functions $f_j(z)(j = 1, 2)$ given by

$$f_j(z) = z^{-p} + \frac{(A-B)(p-\lambda)\binom{p+m-1}{m}}{\eta_p^*(p, A, B, \lambda, \gamma, m, \alpha_1)} z^p \;\; (j = 1, 2; p \in N)\,. \tag{4.20}$$

Corollary 4.6. *Let the function $f_1(z)$ defined by (3.1) be in the class $C_{p,q,s,m}\,(\alpha_1; A, B, \lambda_1, \gamma)$. Suppose also that the function $f_2(z)$ defined by (3.1) be in the class $C_{p,q,s,m}\,(\alpha_1; A, B, \lambda_2, \gamma)$. If* *the sequence* $\{\eta_k^*\}$ *is nondecreasing, then* $(f_1 \otimes f_2)(z) \in C_{p,q,s,m}(\alpha_1; A, B, \xi, \gamma)$*, where*

$$\xi = p - \frac{\frac{2p}{\alpha_1}\begin{pmatrix} p+m-1 \\ m \end{pmatrix}\begin{pmatrix} p \\ m \end{pmatrix}\gamma(1+B)(A-B)(p-\lambda_1)(p-\lambda_2)}{[\eta_p^*(p,A,B,\lambda_1,\gamma,m,\alpha_1)][\eta_p^*(p,A,B,\lambda_2,\gamma,m,\alpha_1)] - \Omega} \tag{4.21}$$

$$\left(\Omega = \begin{pmatrix} p+m-1 \\ m \end{pmatrix}\begin{pmatrix} p \\ m \end{pmatrix} c_{2p}(A-B)^2(p-\lambda_1)(p-\lambda_2)\right).$$

The result is sharp for the functions $f_j(z)(j = 1, 2)$ *given by*

$$f_1(z) = z^{-p} + \frac{(A-B)(p-\lambda_1)\begin{pmatrix} p+m-1 \\ m \end{pmatrix}}{\eta_p^*(p,A,B,\lambda_1,\gamma,m,\alpha_1)} z^p \ \ (p \in N), \tag{4.22}$$

and

$$f_2(z) = z^{-p} + \frac{(A-B)(p-\lambda_2)\begin{pmatrix} p+m-1 \\ m \end{pmatrix}}{\eta_p^*(p,A,B,\lambda_2,\gamma,m,\alpha_1)} z^p \ \ (p \in N). \tag{4.23}$$

Corollary 4.7. *Let the functions* $f_j(z)(j = 1, 2)$ *defined by (3.1) be in the class* $C_{p,q,s,m}(\alpha_1;\ A,\ B, \lambda,\ \gamma)$ *. If the sequence* $\{\eta_k^*\}$ *is nondecreasing, then the function* $h(z)$ *defined by (3.17) belongs to the class* $C_{p,q,s,m}(\alpha_1;\ A,\ B,\ \Im,\ \gamma)$*, where*

$$\Im = p - \frac{\frac{4p}{\alpha_1}\begin{pmatrix} p+m-1 \\ m \end{pmatrix}\begin{pmatrix} p \\ m \end{pmatrix}\gamma(1+B)(A-B)(p-\lambda)^2}{[\eta_p^*(p,A,B,\lambda,\gamma,m,\alpha_1)]^2 - 2\begin{pmatrix} p+m-1 \\ m \end{pmatrix}\begin{pmatrix} p \\ m \end{pmatrix}c_{2p}(A-B)^2(p-\lambda)^2}. \tag{4.24}$$

The result is sharp for the functions $f_j(z)(j = 1, 2)$ *given already by (4.21).*

Acknowledgement 1. *The author would like to thank the referees of the paper for their helpful suggestions.*

REFERENCES

[1] M. K. AOUF, On a class of meromorphic multivalent functions with positive coefficients, *Math. Japon.*, **35** (1990), pp. 603-608.

[2] M. K. AOUF, A generalization of meromorphic multivalent functions with positive coefficients, *Math. Japon.*, **35** (1990), pp. 609-614.

[3] M. K. AOUF and H. M. HOSSEN, New criteria for meromorphic p-valent starlike functions, *Tsukuba J. Math.*, **17** (1993), pp. 481-486.

[4] M. K. AOUF, H. M. HOSSEN and H. E. ELATTAR, A certain class of meromorphic multivalent functions with positive and fixed second coefficients, *Punjab Univ. J. Math.*, **33** (2000), pp. 115-124.

[5] M. -P. CHEN, H. IRMARK and H. M. SRIVASTAVA, Some families of multivalently analytic functions with negative coefficients, *J. Math. Anal. Appl.*, **214** (1997), pp. 674-690.

[6] J. DZIOK and H. M. SRIVASTAVA, Classes of analytic functions associated with the generalized hypergeometric function, *Appl. Math. Comput.,* **103** (1999), pp. 1-13.

[7] S. B. JOSHI and M. K. AOUF, Meromorphic multivalent functions with positive and fixed second coefficients, *Kyungpook Math. J.*, **35** (1995), pp. 163-169.

[8] S. B. JOSHI and H. M. SRIVASTAVA, A certain family of meromorphically multivalent functions, *Comput. Math. Appl.*, **38** No. (3-4) (1999), pp. 201-211.

[9] S. R. KULKARNI, U. H. NAIK and H. M. SRIVASTAVA, A certain class of meromorphically p-valent quasi-convex functions, *PanAmerican Math. J.*, **8** No. 1 (1998), pp. 57-64.

[10] S. S. KUMAR, V. RAVICHANDRAN and H. C. TANEJEA, Meromorphic functions with positive coefficients defined by using convolution, *J. Ineq. Pure Appl. Math.*, **6** No. 2 Art. 58 (2005), pp. 1-9.

[11] J. -L. LIU, Properties of some families of meromorphic p-valent functions, *Math. Japon.*, **52** No. 3 (2000), pp. 425-434.

[12] J. -L. LIU and H. M. SRIVASTAVA, A linear operator and associated families of meromorphically multivalent functions, *J. Math. Anal. Appl.*, **259** (2000), pp. 566-581.

[13] J. -L. LIU, Classes of meromorphically multivalent functions associated with the generalized hypergeometric function, *Math. Comput. Modelling*, **39** (2004), pp. 21-34.

[14] J. -L. LIU and H. M. SRIVASTAVA, Subclasses of meromorphically multivalent functions associated with a certain linear operator, *Math. Comput. Modelling*, **39** (2004), pp. 35-44.

[15] M. L. MOGRA, Meromorphic multivalent functions with positive coefficients I, *Math. Japon.*, **35** No. 1 (1990), pp. 1-11.

[16] M. L. MOGRA, Meromorphic multivalent functions with positive coefficients II, *Math. Japon.* **35** No. 6 (1990), pp. 1089-1098.

[17] S. OWA, H. E. DARWISH and M. K. AOUF, Meromorphic multivalent functions with positive and fixed second coefficients, *Math. Japon.,* **46** No. 2 (1997), pp. 231-236.

[18] R. K. RAINA and H. M. SRIVASTAVA, A new class of meromorphically multivalent functions with applications to generalized hypergeometric functions, *Math. Comput. Modelling*, **43** (2006), pp. 350-356.

[19] A. SCHILD and H. SILVERMAN, Convolution of univalent functions with negative coefficients, *Ann. Univ. Mariae Curie-Sklodowska Sect. A,* **29** (1975), pp. 99-107.

[20] H. M. SRIVASTAVA, H. M. HOSSEN and M. K. AOUF, A unified presentation of some classes of meromorphically multivalent functions, *Comput. Math. Appl.*, **38** (1999), pp. 63-70.

[21] H. M. SRIVASTAVA and P. W. KARLSON, *Multiple Gaussian Hypergeometric Series*, Ellis Horwood Limited, Chichester, John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1985.

[22] B. A. URALEGADDI and M. D. GANIGI, Meromorphic multivalent functions with positive coefficients, *Napali Math. Sci. Rep.*, **11** No. 2 (1986), pp. 95-102.

[23] B. A. URALEGADDI and C. SOMANATHA, Certain classes of meromorphic multivalent functions, *Tamkang J. Math.,* **23** (1992), pp. 223-231.

[24] D. -G. YANG, On new subclasses of meromorphic p-valent functions, *J. Math. Res. Exposition*, **15** (1995), pp. 7-13.

[25] D. -G. YANG, Subclasses of meromorphically p-valent convex functions, *J. Math. Res. Exposition*, **20** (2000), pp. 215-219.