The Australian Journal of Mathematical Analysis and Applications

AJMAA

Volume 5, Issue 2, Article 11, pp. 1-44, 2009

INTEGER SUMS OF POWERS OF TRIGONOMETRIC FUNCTIONS (MOD p), FOR PRIME p

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Received 31 October, 2007; accepted 19 March, 2008; published 11 March, 2009.

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ABSTRACT. Many multi–parameter families of congruences (mod p) are found for integer sums of qth powers of the trigonometric functions over various sets of equidistant arguments, where p is any prime factor of q. Those congruences provide sensitive tests for the accuracy of software for evaluating trigonometric functions to high precision.

Key words and phrases: Fermat's little theorem; Chebyshev polynomials; integer sum (mod p); trigonometric functions; harmonic polynomials.

2000 *Mathematics Subject Classification.* Primary 11A07 11C08 33B10, Secondary 11-03 01A50 01A55 01A60 33C52.

ISSN (electronic): 1449-5910

1. INTRODUCTION

The zeros of Chebyshev polynomials (with integer coefficients) can be expressed in terms of cosines.

Consequently, application of Vieta's Rule and Newton's Rule to those polynomials gives integer expressions for various sums of powers (positive and negative) of trigonometric functions. For example,

$$\sec^4\left(\frac{\pi}{9}\right)+\sec^4\left(\frac{3\pi}{9}\right)+\sec^4\left(\frac{5\pi}{9}\right)+\sec^4\left(\frac{7\pi}{9}\right) \;=\; 1120\,, \tag{1.1}$$

and

$$3\left[\mathrm{cosec}^2\left(\frac{\pi}{n}\right)+\mathrm{cosec}^2\left(\frac{2\pi}{n}\right)+\cdots+\mathrm{cosec}^2\left(\left(\frac{n-1}{2}\right)\frac{\pi}{n}\right)\right] \;=\; \frac{n^2-1}{2} \tag{1.2}$$

for odd $n \geq 3$ (Durell & Robson [4], p. 206 & p. 211).

For a monic polynomial equation with integer coefficients

$$x^d + a_1x^{d-1} + a_2x^{d-2} + \cdots + a_d \;= 0, \tag{1.3}$$

and prime p, the sum S_p of the pth powers of the roots of (1.3) satisfies (2.15) the integer congruence $S_p \equiv -a_1 \bmod p$ (cf. Tee [13], Theorem 1). More generally (2.18), if q is any positive integer then $S_{pq} \equiv S_q \mod p$.

Applied with Vieta's Rule and Newton's Rule to the Chebyshev polynomials, these congruences yield many novel congruences for integer sums of powers of cosines and secants. For example (3.22), with[1] prime p and integers $m \geq 0$ and $d \geq 6$,

$$64^{p^m}\sum_{k=1}^{d\div 2}\cos^{6p^m}\left(\frac{[2k-1]\pi}{2d}\right) \;\equiv\; 10d \qquad (\mathrm{mod}\ p)\,. \tag{1.4}$$

Families of polynomials with integer coefficients are constructed (8.12) whose zeros can be expressed in terms of tangents. Applied with Vieta's Rule and Newton's Rule to those polynomials, the congruences (2.15) and (2.18) yield many novel congruences for integer sums of powers of tangents and cotangents.

The significant trigonometric congruences (each in triples) found here are (3.22), (4.13), (5.18), (5.24), (5.29), (6.7), (6.14) and (6.21) for integer sums of powers of cosines; (3.26), (3.28), (4.17), (4.19), (5.20), (5.26), (5.31), (6.9), (6.16), (6.18), (6.23) and (6.25) for integer sums of powers of secants; (9.8), (9.13), (9.20), (9.30), (10.24) and (10.21) for integer sums of powers of tangents; and (10.28) and (10.30) for integer sums of powers of cotangents.

2. SUMS OF POSITIVE POWERS OF ROOTS

A general polynomial equation of degree $d \geq 1$ with complex coefficients

$$\mathcal{Q}(x) \;=\; c_0x^d + c_1x^{d-1} + c_2x^{d-2} + \cdots + c_d \;=\; 0, \tag{2.1}$$

with $c_0 \neq 0$, has the same roots (and multiplicities) as the monic equation

$$\mathcal{P}(x) \;\stackrel{\text{def}}{=}\; \mathcal{Q}(x)/c_0 \;= x^d + a_1x^{d-1} + a_2x^{d-2} + \cdots + a_d \;= 0, \tag{2.2}$$

where

$$a_i \quad\stackrel{\text{def}}{=}\quad c_i/c_0, \qquad (i \;=\; 1, 2, \ldots, d). \tag{2.3}$$

[1] The symbol $\div$ denotes integer division, yielding integer quotient. For integers n and $d > 0$, $\quad q = n \div d$, where $n = qd + r$, with remainder $0 \leq r < d$.

The d roots of equation (2.1) (and of (2.2)) will be denoted by $\alpha, \beta, \gamma, \ldots, \psi, \omega$; and those symmetric functions of the roots which are called *sigma functions* are denoted thus:

$$\begin{aligned}\sum \alpha &\stackrel{\text{def}}{=} \alpha + \beta + \cdots + \omega,\\ \sum \alpha\beta &\stackrel{\text{def}}{=} \alpha\beta + \alpha\gamma + \cdots + \alpha\omega \;+\; \beta\gamma + \cdots + \beta\omega \;+\; \cdots \;+\; \psi\omega,\\ \sum \alpha^3\beta^2 &\stackrel{\text{def}}{=} \alpha^3\beta^2 + \alpha^3\gamma^2 + \cdots + \alpha^3\omega^2 \;+\\ &\qquad \beta^3\gamma^2 + \cdots + \beta^3\omega^2 \;+\; \cdots \;+\; \psi^3\omega^2 + \beta^3\alpha^2 + \gamma^3\alpha^2\\ &\quad + \cdots + \omega^3\alpha^2 \;+\; \gamma^3\beta^2 + \cdots + \omega^3\beta^2 \;+\; \cdots \;+\; \omega^3\psi^2,\\ &\quad \cdots\end{aligned} \tag{2.4}$$

The sigma functions $\sum\alpha,\ \sum\alpha\beta,\ \sum\alpha\beta\gamma, \ldots, \sum\alpha\beta\gamma\ldots\omega$ are called the *elementary symmetric functions* of $\alpha, \beta, \gamma, \ldots, \omega$; and Vieta's Rule expresses them in terms of the coefficients of $\mathcal{P}$:

$$\begin{gathered}\sum\alpha \;=\; -a_1, \quad \sum\alpha\beta \;=\; a_2, \quad \sum\alpha\beta\gamma \;=\; -a_3,\\ \ldots, \quad \sum\alpha\beta\gamma\ldots\omega \;=\; \alpha\beta\gamma\ldots\omega \;=\; (-1)^d a_d.\end{gathered} \tag{2.5}$$

Each symmetric polynomial with integer coefficients can be expressed as a polynomial in the elementary symmetric functions, with integer coefficients (Dickson [3], p. 67).

Therefore, if all coefficients $a_1, \ldots, a_d$ of the monic polynomial $\mathcal{P}$ are integers (positive, negative or zero), each symmetric polynomial (in the roots of (2.2)) with integer coefficients has integer value. In particular, each sigma function then has integer value.

For integer k, denote the sum of the kth powers of the roots as

$$S_k \stackrel{\text{def}}{=} \sum\alpha^k \;=\; \alpha^k + \beta^k + \cdots + \omega^k, \tag{2.6}$$

which is a sigma function if $k > 0$. From equation (2.2), Vieta's Rule gives $S_1 = -a_1$; then $S_2,\ S_3, \ldots$ may be computed successively by Newton's Rule:

$$\begin{aligned}S_k \;&=\; -a_1 S_{k-1} - a_2 S_{k-2} - \cdots - a_{k-1}S_1 - k a_k, \qquad (k \;=\; 1, 2, \ldots, d);\\ S_k \;&=\; -a_1 S_{k-1} - a_2 S_{k-2} - \cdots - a_d S_{k-d}, \qquad (k > d).\end{aligned} \tag{2.7}$$

From Newton's Rule, it follows that each S_k can be expressed as a polynomial in the a_i, with integer coefficients. For example,

$$\begin{gathered}S_1 \;=\; -a_1, \quad S_2 \;=\; a_1^2 - 2a_2, \quad S_3 \;=\; -a_1^3 + 3a_1a_2 - 3a_3,\\ S_4 \;=\; a_1^4 - 4a_1^2a_2 + 4a_1a_3 + 2a_2^2 - 4a_4,\end{gathered} \tag{2.8}$$

and a_i is taken as 0 for $i > d$. Girard's formula[2] of 1629 gives (MacMahon [8] p. 6) the general expression:

$$S_q \quad = \quad \sum \frac{q\cdot(r_1 + r_2 + \cdots + r_d - 1)!}{r_1! r_2! \ldots r_d!} (-a_1)^{r_1}(-a_2)^{r_2} \ldots (-a_d)^{r_d}, \tag{2.9}$$

where the sum is taken over all non-negative exponents such that

$$r_1 + 2r_2 + 3r_3 + \cdots + d r_d \;=\; q. \tag{2.10}$$

For an even polynomial equation of degree $d = 2j$,

$$x^{2j} + a_2 x^{2j-2} + a_4 x^{2j-4} + \cdots + a_{2j-2}x^2 + a_{2j} \;=\; 0, \tag{2.11}$$

[2] Albert Girard, *Invention Nouvelle en l'Algèbre*, Amsterdam, 1629. The formula is often ascribed to Waring, who gave it without proof in 1782.

the roots occur in pairs $(+\alpha, -\alpha)$, including multiplicity; and hence $S_k = 0$ for all odd k. Indeed, α^2 is a root (with multiplicity twice that for $+\alpha$ in (2.11)) of the equation of degree j:

$$z^j + a_2 z^{j-1} + a_4 z^{j-2} + \cdots + a_{2j-2} z + a_{2j} = 0. \tag{2.12}$$

Accordingly, for all positive integers n, S_{2n} (for the roots of (2.11)) equals twice the sum of the nth powers of the roots of (2.12).

Similarly, for an odd polynomial of degree $d = 2j+1$,

$$x^{2j+1} + a_2 x^{2j-1} + a_4 x^{2j-3} + \cdots + a_{2j-2} x^3 + a_{2j} x = 0, \tag{2.13}$$

the roots are 0 and pairs $(+\alpha, -\alpha)$ including multiplicity; and hence $S_k = 0$ for all odd k. As with the even polynomial (2.11), S_{2n} (for the roots of (2.13)) equals twice the sum of the nth powers of the roots of (2.12).

For both even and odd polynomials, define

$$\sigma_n \stackrel{\text{def}}{=} \tfrac{1}{2} S_{2n}, \tag{2.14}$$

so that σ_n equals the sum of the $2n$th powers of the roots of the even or odd polynomial equation, using each nonzero pair $(+\alpha, -\alpha)$ once only. The root 0 for an odd polynomial is excluded from the sum in the definition of σ_n, to facilitate the later treatment of negative powers of roots.

Throughout this paper, p denotes any prime, and m denotes the non–negative integer exponent in p^m.

Hereafter, we shall consider only polynomials with integer coefficients.

2.1. **Monic Polynomials With Integer Coefficients.** If all coefficients in the monic polynomial $\mathcal{P}$ are integers, then its zeros are called algebraic integers. Newton's Rules (2.7) shew that S_k then has integer value for all positive integers k. It has been proved (Tee 2.12, Theorem 1) that[3]

$$S_p \equiv -a_1 \pmod{p}. \tag{2.15}$$

Therefore

$$p | S_p \iff p | a_1. \tag{2.16}$$

and (Tee [12], Theorem 2), S_p is an integer multiple of p for all primes p if and only if the coefficient $a_1 = 0$ in $\mathcal{P}$.

For both even and odd polynomials, we get that

$$\tfrac{1}{2} S_{2p} = \sigma_p \equiv -a_2 \pmod{p}. \tag{2.17}$$

Therefore, in an even or odd polynomial, σ_p is an integer multiple of p for all primes p, if and only if $a_2 = 0$.

N.B. Note that the congruence and mod notations, devised by Gauß for relations between integers, have been extended to indicate relations between polynomials. Consider integral polynomials f and g of degree n; i.e. $f(x) = \sum_{r=0}^{n} c_r x^{n-r}$, $g(x) = \sum_{r=0}^{n} e_r x^{n-r}$, with integer coefficients c_r and e_r. If $c_r \equiv e_r \pmod{m}$ for each $r \in [0, \ldots, n]$, then "we say that $f(x)$ and $g(x)$ are congruent to modulus m, and write $f(x) \equiv g(x) \pmod{m}$" (Hardy & Wright [5], page 82). Thus, the symbol "$\equiv$" is used in two different senses: Gauß's meaning for a relation between integers, and the sense just defined, in which it expresses a relation between polynomials which does not imply any particular value (or type) for the variable x. Hardy & Wright

[3] In 1908, L. E. Dickson [2] stated the special case with $a_1 = 0$; but that very muddled note considered only the case of distinct roots for the monic polynomial equation.

explained that "there should be no confusion because, except in the phrase 'the congruence $f(x) \equiv 0$', the variable x will occur only when the symbol is used in the second sense. When we assert that $f(x) \equiv g(x)$, or $f(x) \equiv 0$, we are using it in this sense, and there is no reference to any numerical value of x. But when we make an assertion about 'the roots of the congruence $f(x) \equiv 0$', or discuss 'the solution of the congruence', it is naturally the first sense which we have in mind" ([5], p. 83).

For instance, Comtet writes that for prime p, "$(1+x)^p \ \equiv \ 1+x^p \quad (\mathrm{mod}\ p)$; which means that these two polynomials have the same coefficients in **Z**/p**Z**" ([1], p. 14]). With that definition of "$\equiv$" for integral polynomials, it does follow that "$(x_1+x_2+\cdots+x_m)^p \ \equiv \ x_1^p+x_2^p+\cdots+x_m^p$ mod(p)" (Comtet [1], p. 29). But if $x_1, x_2, \ldots, x_m$ are not integer variables then that is not a standard Gauß–type congruence of integers (unless the expressions on left and right of the congruence happen to have integer values).

Some authors do seem to have become confused by the two meanings of "$\equiv$". For instance, B. H. Neumann & L. G. Wilson [10] published D. H. Lehmer's purported proof of (2.15) for the special case $a_1 = 0$ (with $d = 4$) — but Lehmer's proof is valid only if the roots x, y, z, t of his monic polynomial equation are all integers.

The proof (Tee [12], Theorem 1) of (2.15) applies for a general monic polynomial equation with integer coefficients, whose roots (called algebraic integers) will in general be irrational or complex.

Hereafter, in this paper, "$\equiv$" always denotes congruence of integers.

Edouard Lucas in 1878 ([6], p. 230) attributed to Euler the more general result that if q is any positive integer then

$$S_{pq} \ \equiv \ S_q \qquad (\mathrm{mod}\ p), \tag{2.18}$$

which reduces to the congruence (2.15) when $q = 1$. But Lucas did not give any reference, and it has not been possible to locate that generalized congruence (2.18) within Euler's colossal output.

The congruence (2.15) generalizes readily to give (2.18). For the monic polynomial whose zeros are the q-th powers of the zeros of $\mathcal{P}$:

$$\mathcal{R}(\mu) \ \stackrel{\mathrm{def}}{=} \ \mu^d + r_1\mu^{d-1} + r_r\mu^{d-2} + \cdots + r_d \tag{2.19}$$

Vieta's Rule (2.5) becomes:

$$\begin{gathered} \sum \alpha^q \ = \ -r_1, \quad \sum \alpha^q\beta^q \ = \ r_2, \quad \sum \alpha^q\beta^q\gamma^q \ = -r_3, \\ \ldots, \quad \sum \alpha^q\beta^q\gamma^q\ldots\omega^q \ = \ \alpha^q\beta^q\gamma^q\ldots\omega^q \ = \ (-1)^d r_d. \end{gathered} \tag{2.20}$$

Each of those symmetric functions is a sigma function for the monic polynomial $\mathcal{P}$ with integer coefficients, and hence (Dickson [3], p.67) each coefficient $r_1, r_2, \ldots, r_d$ of the monic polynomial $\mathcal{R}$ is an integer. Therefore, the congruence (2.15) for $\mathcal{P}$ can be applied to $\mathcal{R}$, to give:

$$S_{pq} \ \equiv \ -r_1 \qquad (\mathrm{mod}\ p). \tag{2.21}$$

Now (cf. (2.20)) $S_q = -r_1$, and hence the generalized congruence (2.18) does follow from (2.15).

If n is any non-negative multiple of p with $n = p^m q$, where $m > 0$, then it follows from (2.9) and (2.18) by induction on m that

$$S_n = S_{p^m q} \equiv S_q = \sum \frac{q\cdot(r_1 + r_2 + \cdots + r_d - 1)!}{r_1! r_2! \ldots r_d!}(-a_1)^{r_1}(-a_2)^{r_2}\ldots(-a_d)^{r_d} \pmod{p}, \tag{2.22}$$

with $r_1, \ldots, r_d$ as in (2.10). The congruence (2.22) also holds trivially with $m = 0$, so that $n = q$.

It follows from (2.8) and (2.22) that for prime p and positive integer m:

$$\begin{aligned} S_{p^m} &\equiv -a_1 \pmod{p}, \\ S_{2p^m} &\equiv a_1^2 - 2a_2 \pmod{p}, \\ S_{3p^m} &\equiv -a_1^3 + 3a_1a_2 - 3a_3 \pmod{p}, \\ S_{4p^m} &\equiv a_1^4 - 4a_1^2a_2 + 4a_1a_3 + 2a_2^2 - 4a_4 \pmod{p}, \quad \textit{et cetera}, \end{aligned} \tag{2.23}$$

with $a_j = 0$ for $j > d$.

Likewise, for odd or even monic polynomials,

$$\begin{aligned} \sigma_{p^m} &\equiv -a_2 \pmod{p}, \\ \sigma_{2p^m} &\equiv a_2^2 - 2a_4 \pmod{p}, \\ \sigma_{3p^m} &\equiv -a_2^3 + 3a_2a_4 - 3a_6 \pmod{p}, \\ \sigma_{4p^m} &\equiv a_2^4 - 4a_2^2a_4 + 4a_2a_6 + 2a_4^2 - 4a_8 \pmod{p}, \quad \textit{et cetera}. \end{aligned} \tag{2.24}$$

2.1.1. *Fermat's Little Theorem for Algebraic Integers.* For $d = 1$ the polynomial equation (2.2) reduces to

$$x + a_1 = 0, \tag{2.25}$$

and (2.15) becomes $(-a_1)^p \equiv -a_1 \pmod{p}$. For odd prime p, this becomes $-a_1^p \equiv -a_1 \pmod{p}$, and for $p = 2$ this becomes $a_!^2 \equiv -a_1 \equiv a_1$ (mod 2).

Hence, for all primes p, (2.15) reduces to Fermat's Little Theorem for prime p and integer a_1:

$$a_1^p \equiv a_1 \pmod{p}; \tag{2.26}$$

so that the congruence (2.15) is a generalization of Fermat's Little Theorem from arithmetic integers to algebraic integers.

Fermat's Little Theorem is frequently given in the ($p - 1$)–power version:

$$a^{p-1} \equiv 1 \pmod{p} \tag{2.27}$$

for integer a which is not a multiple of p. Can this ($p - 1$)–power version of Fermat's Little Theorem be generalized to algebraic integers, as the p–power version can?

If the d roots $\alpha, \beta, \ldots, \omega$ of (2.2) are all integers which are not multiples of p, then

$$S_{p-1} = \alpha^{p-1} + \beta^{p-1} + \cdots + \omega^{p-1} \equiv 1 + 1 + \cdots + 1 \equiv d \pmod{p}. \tag{2.28}$$

But consider the monic polynomial of degree $d = 2$ with integer coefficients:

$$x^2 - x + a_2 = 0, \tag{2.29}$$

whose roots are not integers if a_2 is any integer not of the form $b(1 - b)$ with integer b. With the prime $p = 2$ we get that

$$S_{p-1} = S_1 = \alpha + \beta = 1 \equiv d - 1 \pmod{p}, \tag{2.30}$$

which is incompatible with the general result (2.28) for roots which are all integers not divisible by p.

Therefore, the $(p-1)$–power version of Fermat's Little Theorem does not generalize in any simple manner from arithmetic integers to algebraic integers, as does the p–power version.

2.2. **Non-monic Polynomials With Integer Coefficients.** The case of a polynomial $\mathcal{Q}$ with general integer coefficients $(c_0 \neq 0)$ can be reduced to the case of a monic polynomial with integer coefficients, as follows.

Theorem 2.1. *For a polynomial $\mathcal{Q}$ with general integer coefficients $(c_0 \neq 0)$, let b be such that $c_0 | b^i c_i$ (for $i = 1, 2, \ldots, d$). Then $b^k S_k$ is an integer, for positive integer k.*

Proof. Put $x = z/b$ so that $z = xb$, and hence (2.1) becomes

$$\frac{c_0}{b^d} z^d + \frac{c_1}{b^{d-1}} z^{d-1} + \frac{c_2}{b^{d-2}} z^{d-2} + \cdots + c_d = 0. \tag{2.31}$$

Multiply (2.31) by b^d/c_0 to get the monic equation

$$z^d + a_1 z^{d-1} + a_2 z^{d-2} + \cdots + a_d = 0, \tag{2.32}$$

where each coefficient

$$a_i = \frac{b^d c_i}{c_0 b^{d-i}} = \frac{c_i}{c_0} b^i \qquad (i = 1, 2, \ldots, d) \tag{2.33}$$

is an integer, by hypothesis.

Thus, the general polynomial equation (2.1) in x with integer coefficients is converted to the monic equation (2.32) in z with integer coefficients.

Denote the roots of the monic equation (2.32) by $z = \alpha_1, \ldots, \alpha_d$, so that the roots of the general equation (2.1) are $x = z/b = \gamma_1, \ldots, \gamma_d$, where

$$\alpha_i = b\gamma_i \qquad (i = 1, 2, \ldots, d). \tag{2.34}$$

Hence with S_k for equation (2.1) we get that

$$b^k S_k = \big(b\gamma_1\big)^k + \big(b\gamma_2\big)^k + \cdots + \big(b\gamma_d\big)^k = \alpha_1^k + \alpha_2^k + \cdots + \alpha_d^k; \tag{2.35}$$

and that has integer value, since it is the sum of kth powers of roots of the monic equation (2.32) with integer coefficients. ∎

Corollary 2.2. *If (2.1) is an odd or even polynomial, then (2.33) holds for all odd i and hence it need be tested only for even i. Then, $b^{2k}\sigma_k$ has integer value.*

Corollary 2.3. *For any equation (2.1), c_0 satisfies the condition for b; since $z = c_0 x$ satisfies the monic polynomial equation*

$$z^d + c_1 z^{d-1} + c_0 c_2 z^{d-2} + c_0^2 c_3 z^{d-3} + \cdots + c_0^{d-1} c_d = 0, \tag{2.36}$$

and hence $c_0^k S_k$ has integer value.

If (2.1) is an odd or even polynomial, then from $b^2 = c_0$ we get that $c_0^k \sigma_k$ has integer value.

Corollary 2.4. *If k is a prime p, then it follows from (2.23), (2.32) and (2.33) that*

$$b^{p^m} S_{p^m} \equiv -a_1 \equiv \frac{-bc_1}{c_0} \qquad (\bmod\ p); \tag{2.37}$$

In the case $b = c_0$, which works for every polynomial with integer coefficients, this reduces to

$$c_0^{p^m} S_{p^m} \equiv -c_1 \qquad (\bmod\ p). \tag{2.38}$$

If a value of b less than c_0 (in modulus) can be found which satisfies the conditions for Theorem 2.1, then (2.37) will give results stronger than (2.38) concerning S_p for equation (2.1).

Corollary 2.5. *If b satisfies the conditions for Theorem 2.1, then $b^{p^m} S_{p^m}$ is an integer multiple of p for all primes p and positive integers m, if and only if $c_1 = 0$.*

For an odd or even polynomial with integer coefficients, $c_0^{p^m} \sigma_{p^m}$ is an integer multiple of p for all primes p, if and only if $c_2 = 0$.

Corollary 2.6. *For a general polynomial with integer coefficients, corresponding to (2.23) we get:*

$$\begin{aligned}
b^{p^m} S_{p^m} &\equiv \frac{-bc_1}{c_0} \qquad (\bmod\ p), \\
b^{2p^m} S_{2p^m} &\equiv \left(\left(\frac{c_1}{c_0} \right)^2 - \frac{2c_2}{c_0} \right) b^2 \qquad (\bmod\ p), \\
b^{3p^m} S_{3p^m} &\equiv \left(-\left(\frac{c_1}{c_0} \right)^3 + \frac{3c_1 c_2}{c_0^2} \frac{3c_3}{c_0} \right) b^3 \qquad (\bmod\ p), \\
b^{4p^m} S_{4p^m} &\equiv \left(\left(\frac{c_1}{c_0} \right)^4 - \frac{4c_1^2 c_2}{c_0^3} + \frac{4c_1 c_2 + 2c_2^2}{c_0^2} - \frac{4c_4}{c_0} \right) b^4 \\
&\qquad (\bmod\ p), \qquad et\ cetera,
\end{aligned} \tag{2.39}$$

with $c_j = 0$ for $j > d$.

Taking c_0 for b, these become:

$$\begin{aligned}
c_0^{p^m} S_{p^m} &\equiv -c_1 \qquad (\bmod\ p), \\
c_0^{2p^m} S_{2p^m} &\equiv c_1^2 - 2c_2 c_0 \qquad (\bmod\ p), \\
c_0^{3p^m} S_{3p^m} &\equiv -c_1^3 + 3c_1 c_2 c_0 - 3c_3 c_0^2 \qquad (\bmod\ p), \\
c_0^{4p^m} S_{4p^m} &\equiv c_1^4 - 4c_1^2 c_2 c_0 + (4c_1 c_2 + 2c_2^2) c_0^2 - 4c_4 c_0^3 \\
&\qquad (\bmod\ p), \qquad et\ cetera.
\end{aligned} \tag{2.40}$$

Similarly, for odd or even polynomials with integer coefficients, then $b^2 = c_0$ gives

$$\begin{aligned}
c_0^{p^m} \sigma_{p^m} &\equiv -c_2 \qquad (\bmod\ p), \\
c_0^{2p^m} \sigma_{2p^m} &\equiv c_2^2 - c_4 c_0 \qquad (\bmod\ p), \\
c_0^{3p^m} \sigma_{3p^m} &\equiv -c_2^3 + 3c_2 c_4 c_0 - 3c_6 c_0^2 \qquad (\bmod\ p), \\
c_0^{4p^m} \sigma_{4p^m} &\equiv c_2^4 - 4c_2^2 c_4 c_0 + (4c_2 c_4 + 2c_4^2) c_0^2 - 4c_8 c_0^3 \\
&\qquad (\bmod\ p), \qquad et\ cetera,
\end{aligned} \tag{2.41}$$

with $c_j = 0$ for $j > d$.

2.3. **Sums of Negative Powers of Roots.** If $c_d \neq 0$, the polynomial equation inverse to (2.1):

$$y^d Q\left(\frac{1}{y}\right) = c_0 + c_1 y + c_2 y^2 + \cdots + c_{d-1} y^{d-1} + c_d y^d = 0 \tag{2.42}$$

has roots $\omega_1, \ldots, \omega_d$ which are the inverses of the roots $\gamma_1, \ldots, \gamma_d$ of (2.1), including multiplicity. Hence for integer $k > 0$,

$$S_{-k} = \gamma_1^{-k} + \cdots + \gamma_d^{-k} = \omega_1^k + \cdots + \omega_d^k, \tag{2.43}$$

which is related to the coefficients of the inverse polynomial (2.42) by Newton's Rule. (This holds, even if the coefficients of (2.1) are not integers.)

If $c_d = \pm 1$, then the inverse polynomial (2.42) is a monic polynomial with integer coefficients; and accordingly, for integer $k > 0$, S_{-k} has integer value.

Applying Theorem 2.1 and its Corollaries to the inverse polynomial (2.42), we get the following results for sums of negative powers of roots of (2.1):

Theorem 2.7. *Let b be any integer such that $c_d | b^i c_{d-i}$ (for $i = 1, 2, \ldots, d$). Then $b^k S_{-k}$ is an integer, for positive integer k.*

Corollary 2.8. *If (2.1) is an odd or even polynomial, then $b^{2k}\sigma_{-k}$ has integer value.*

Note that the definition (2.15) of σ_n excludes the root 0 for an odd polynomial, and so σ_{-k} is defined for an odd polynomial.

Corollary 2.9. *For any equation (2.42), c_d satisfies the conditions for b; for $z = yc_d$ satisfies the monic polynomial equation*

$$c_d^{d-1}c_0 + c_d^{d-2}c_1 z + \cdots + c_d c_{d-2} z^{d-2} + c_{d-1} z^{d-1} + z^d \;=\; 0, \tag{2.44}$$

and hence $c_d^k S_{-k}$ has integer value. If (2.1) is an odd or even polynomial, then $c_d^k \sigma_{-k}$ has integer value.

Corollary 2.10. *If k is a prime p and m is a positive integer, then*

$$b^{p^m} S_{-p^m} \;\equiv\; \frac{-bc_{d-1}}{c_d} \qquad (\mathrm{mod}\ p). \tag{2.45}$$

In the case $b = c_d$, which works for every polynomial with integer coefficients, this reduces to

$$c_d^{p^m} S_{-p^m} \;\equiv\; -c_{d-1} \qquad (\mathrm{mod}\ p). \tag{2.46}$$

Corollary 2.11. *If b satisfies the conditions for Theorem 2.7, then $b^{p^m} S_{-p^m}$ is an integer multiple of p for all primes p and positive integers m, if and only if $c_{d-1} = 0$.*

Corollary 2.12. *Applying (2.39) to the inverse (2.42) of a polynomial with integer coefficients, we get that*

$$\begin{aligned}
b^{p^m} S_{-p^m} &\equiv \frac{-bc_{d-1}}{c_d} \qquad (\mathrm{mod}\ p),\\
b^{2p^m} S_{-2p^m} &\equiv \left(\left(\frac{c_{d-1}}{c_d}\right)^2 - \frac{2c_{d-2}}{c_d}\right) b^2 \qquad (\mathrm{mod}\ p) \quad (d \geq 2),\\
b^{3p^m} S_{-3p^m} &\equiv \left(-\left(\frac{c_{d-1}}{c_d}\right)^3 + \frac{3c_{d-1}c_{d-2}}{c_d^2} - \frac{3c_{d-3}}{c_d}\right) b^3\\
&\qquad (\mathrm{mod}\ p)\ (d \geq 3), \qquad et\ cetera,
\end{aligned} \tag{2.47}$$

with $c_j = 0$ for $j < 0$; and likewise in the other formulæ for sums of negative powers.

With c_d for b, (2.47) becomes:

$$\begin{aligned}
c_d^{p^m} S_{-p^m} &\equiv -c_{d-1} \qquad (\mathrm{mod}\ p),\\
c_d^{2p^m} S_{-2p^m} &\equiv c_{d-1}^2 - 2c_{d-2}c_d \qquad (\mathrm{mod}\ p) \quad (d \geq 2),\\
c_d^{3p^m} S_{-3p^m} &\equiv -c_{d-1}^3 + 3c_{d-2}c_{d-1}c_d - 3c_{d-3}c_d^2\\
&\qquad (\mathrm{mod}\ p) \quad (d \geq 3), \qquad et\ cetera.
\end{aligned} \tag{2.48}$$

Applying (2.41) to the inverse of an even or odd polynomial of degree $2j$ or $2j+1$, with $b^2 = c_{2j}$ we get:

$$
\begin{aligned}
c_{2j}^{p^m}\,\sigma_{-p^m} &\equiv -c_{2j-2} \qquad (\bmod\ p),\\
c_{2j}^{2p^m}\,\sigma_{-2p^m} &\equiv c_{2j-2}^2 \;-\; 2c_{2j-4}c_{2j} \qquad (\bmod\ p) \quad (j \geq 2),\\
c_{2j}^{3p^m}\,\sigma_{-3p^m} &\equiv -c_{2j-2}^3 \;+\; 3c_{2j-4}c_{2j-2}c_{2j} \;-\; 3c_{2j-6}c_{2j}^2\\
&\qquad (\bmod\ p) \quad (j \geq 3), \qquad \textit{et cetera}.
\end{aligned}
\tag{2.49}
$$

More generally (cf. (2.22)), if p is any prime factor of n and the polynomial $\mathcal{Q}$ has integer coefficients, then both $c_0^n S_n$ and $c_d^n S_{-n}$ have integer values which are congruent (mod p) to polynomials in the coefficients $c_0, c_1, \ldots, c_{d-1}, c_d$ (cf. (2.40) and (2.48)). Similarly, if $\mathcal{Q}$ is also odd or even, then both $c_0^n\sigma_n$ and $c_{2j}^n\sigma_{-n}$ have integer values which are congruent (mod p) to polynomials in the coefficients $c_0, c_2, \ldots, c_{2j-2}, c_{2j}$ (cf. (2.41) and (2.49)).

3. Chebyshev Polynomials of First Type

The Chebyshev polynomial of the first type T_d is defined by the initial values

$$T_0(x) \;=\; 1, \qquad T_1(x) \;=\; x, \tag{3.1}$$

with the 3–term recurrence relation for $n > 1$:

$$T_n(x) \;=\; 2xT_{n-1}(x) - T_{n-2}(x). \tag{3.2}$$

By induction on n in (3.2), it follows from (3.1) that, for all integers $d \geq 0,\ T_d(x)$ is a polynomial in x of degree d with integer coefficients. Moreover, T_d is an even polynomial for even d and an odd polynomial for odd d.

The integer coefficients of T_d are given explicitly for $d > 0$ by the formula[4] (Mason & Handscomb [9], p. 24):

$$
\begin{aligned}
T_d(x) \;&=\; \sum_{k=0}^{d\div 2}(-1)^k\frac{2^{d-2k-1}d(d-k-1)!}{k!(d-2k)!}\,x^{d-2k} \;=\\
&2^{d-1}x^d - 2^{d-3}dx^{d-2} + 2^{d-6}d(d-3)x^{d-4} - \frac{2^{d-8}}{3}d(d-4)(d-5)x^{d-6} + \cdots\\
&\cdots + \begin{cases} (-1)^{d\div 2} & \text{(for even } d\text{)},\\ (-1)^{(d-1)\div 2}\,dx & \text{(for odd } d\text{)}.\end{cases}
\end{aligned}
\tag{3.3}
$$

If $d = 2j+1$ is odd, then $T_d(x)$ is an odd polynomial:

$$T_{2j+1}(x) \;=\; (-1)^j\Big[\ldots \;-\; \frac{2}{3}j(j+1)(2j+1)x^3 \;+\; (2j+1)x\Big]; \tag{3.4}$$

and if $d = 2j$ is even, then $T_d(x)$ is an even polynomial:

$$T_{2j}(x) \;= (-1)^j\left[\ldots \;+\; \frac{2}{3}j^2(j^2-1)x^4 \;-2j^2x^2+1\right]. \tag{3.5}$$

For example (Lyusternik *et alia* [7], pp. 168–169), in addition to (3.1),

$$
\begin{aligned}
T_2(x) \;&=\; 2x^2-1, & T_3(x) \;&=\; 4x^3-3x,\\
T_4(x) \;&=\; 8x^4-8x^2+1, & T_5(x) \;&=\; 16x^5-20x^3+5x\ .
\end{aligned}
\tag{3.6}
$$

---

[4]The symbol $\div$ denotes integer division, yielding integer quotient. For integers n and $d > 0$, $\quad q = n \div d$, where $n = qd + r$, with remainder $0 \leq r < d$.

By induction on n in (3.2), it follows from (3.1) that for all complex x and integer $n \geq 0$, with $\vartheta \stackrel{\text{def}}{=} \cos^{-1} x$,

$$T_n(x) \;=\; T_n(\cos\vartheta) \;=\; 2\,\cos\,\vartheta\;\cos((n-1)\vartheta) - \cos((n-2)\vartheta) \;=\; \cos(n\vartheta). \tag{3.7}$$

(Any of the infinitely many values of $\cos^{-1} x$ can be used for ϑ.)

The polynomials $T_d(x)$ are orthogonal on the interval (-1, 1), with weight function $1/\sqrt{1-x^2}$.

The zeros of Chebyshev polynomials are known explicitly in terms of cosines, and hence application of the congruences (2.40), (2.41), (2.48) and (2.49) to various types of Chebyshev polynomials gives various trigonometric congruences, of a type which seems to be novel.

The significant trigonometric congruences (each in triples) found here are (3.22), (4.13), (5.18), (5.24), (5.29), (6.7), (6.14) and (6.21) for integer sums of powers of cosines, and (3.26), (3.28), (4.17), (4.19), (5.20), (5.26), (5.31), (6.9), (6.16), (6.18), (6.23) and (6.25) for integer sums of powers of secants.

3.1. **Modified Chebyshev Polynomial of First Type.** The standard Chebyshev polynomials $T_d(x)$ will now be transformed to monic polynomials, to simplify the expressions for sums of powers of zeros.

The modified Chebyshev polynomial C_d of first type is defined by the initial values

$$C_0(x) \;=\; 2, \qquad C_1(x) \;=\; x, \tag{3.8}$$

with the recurrence relation

$$C_{n+1}(x) \;=\; xC_n(x) - C_{n-1}(x). \tag{3.9}$$

Comparison the of the 3–term recurrence relations for T_d and for C_d shews that

$$C_d(x) \;\equiv\; 2T_d(x/2) \tag{3.10}$$

for all $d \geq 1$ (Lyusternik *et alia* [7], p. 163), and $C_d(x)$ is a monic polynomial in x of degree d with integer coefficients. The coefficients (3.3) of $T_d(x)$ convert, by (3.10), to give the integer coefficients of $C_d(x)$:

$$\begin{aligned} C_d(x) \quad &= \quad d\sum_{k=0}^{d\div 2}(-1)^k\frac{(d-k-1)!}{k!(d-2k)!}x^{d-2k} \\ &= \; x^d \;-\; dx^{d-2} \;+\; \frac{1}{2}d(d-3)x^{d-4} \;-\frac{1}{6}d(d-4)(d-5)x^{d-6} + \;\cdots\;. \end{aligned} \tag{3.11}$$

If $d = 2j+1$ is odd, then $C_d(x)$ is an odd polynomial:

$$\begin{aligned} C_{2j+1}(x) \;=\; (-1)^j\Big[\cdots \;-\; &\frac{1}{5040}j(j^2-1)(j^2-4)(j+3)(2j+1)x^7 \\ + \;\frac{1}{120}j(j^2-1)(j+2)(2j+1)x^5 -\; &\frac{1}{6}j(j+1)(2j+1)x^3 \\ + \;(2j+1)x&\Big]; \end{aligned} \tag{3.12}$$

and if $d = 2j$ is even, then $C_d(x)$ is an even polynomial:

$$\begin{aligned} C_{2j}(x) \;=\; (-1)^j\Big[\cdots \;-\; \frac{1}{360}j^2(j^2-1)(j^2-4)x^6 \;+\; &\frac{1}{12}j^2(j^2-1)x^4 \\ -\; j^2x^2 + 2&\Big]. \end{aligned} \tag{3.13}$$

For example (Lyusternik *et alia* [7], p. 171), in addition to (3.8),

$$C_2(x) \;=\; x^2 - 2, \qquad C_3(x) \;=\; x^3 - 3x, \qquad C_4(x) \;=\; x^4 - 4x^2 + 2,$$
$$C_5(x) = x^5 - 5x^3 + 5x, \qquad C_6(x) \;=\; x^6 - 6x^4 + 9x^2 - 2. \tag{3.14}$$

The polynomials $C_d(x)$ are orthogonal on the interval (-2,2), with weight function $1/\sqrt{4 - x^2}$. Let

$$x \;=\; 2\cos\vartheta \tag{3.15}$$

so that if $-2 \leq x \leq 2$ then ϑ is real. Then it follows from (3.8) and (3.9), by induction on n, that

$$C_d(2\cos\vartheta) \;=\; 2\cos(d\vartheta); \tag{3.16}$$

and accordingly

$$-2 \;\leq\; C_d(x) \;\leq\; 2. \tag{3.17}$$

The zeros of C_d are given by

$$\cos(d\vartheta) = 0, \tag{3.18}$$

so that

$$d\vartheta \;=\; \left(k - \frac{1}{2}\right)\pi \tag{3.19}$$

with integer k, and hence each zero of C_d is of the form

$$\alpha_k \;=\; 2\;\cos\left(\frac{[2k-1]\pi}{2d}\right). \tag{3.20}$$

For $k = 1, 2, \ldots, d$, this formula gives a strictly decreasing sequence of real zeros

$$2 \;>\; \alpha_1 \;>\; \alpha_2 \;>\ldots\;>\; \alpha_d \;>\; -2, \tag{3.21}$$

and hence all d zeros of C_d are given by $\alpha_1, \alpha_2, \ldots, \alpha_d$.

3.2. **Sums of Even Powers of Cosines.** Applying (2.24) to the expansion (3.12) of the monic Chebyshev polynomial C_d (which is even or odd with d), we get trigonometric congruences in prime p and integers $d > 0$ and $m \geq 0$:

$$\begin{aligned}
\sigma_{p^m} \;&=\; 4^{p^m}\sum_{k=1}^{d\div 2}\cos^{2p^m}\left(\frac{[2k-1]\pi}{2d}\right)\\
&\equiv\; d \qquad (\mathrm{mod}\ p) \qquad (d \geq 2),\\
\sigma_{2p^m} \;&=\; 16^{p^m}\sum_{k=1}^{d\div 2}\cos^{4p^m}\left(\frac{[2k-1]\pi}{2d}\right)\\
&\equiv\; d^2 - d(d-3) = 3d \qquad (\mathrm{mod}\ p)\ (d \geq 4),\\
\sigma_{3p^m} \;&=\; 64^{p^m}\sum_{k=1}^{d\div 2}\cos^{6p^m}\left(\frac{[2k-1]\pi}{2d}\right)\\
&\equiv\; d^3 \;-\; \frac{3}{2}d^2(d-3) \;-\; \frac{1}{2}d(d-4)(d-5)\\
&=\; 10d \qquad (\mathrm{mod}\ p) \quad (d \geq 6), \qquad et\ cetera.
\end{aligned} \tag{3.22}$$

3.2.1. *Computed examples.* Each significant trigonometric congruence has been checked, by evaluating each of those trigonometric sums numerically for several sets of parameters. The computations were performed on a Macintosh computer in THINK Pascal, using **extended** arithmetic with about 19 significant decimal figures. One numerical example is printed after each triple of congruences. In each case (including the many not printed here), the congruence was verified within **extended** accuracy.

For example, with $p = 5$, $m = 2$ and $d = 3$,

$$\frac{1}{p}\left[4^{p^m}\sum_{k=1}^{d\div 2}\cos^{2p^m}\left(\frac{[2k-1]\pi}{2d}\right) - d\right] = 169457721888{\cdot}000\,. \tag{3.23}$$

Hence (Tee [12], Theorem 1), for prime p and $m \geq 1$, the positive zeros of C_d give the result that $p|\sigma_{p^m}$ if and only if $p|d$; $p|\sigma_{2p^m}$ if and only if $p|3d$, and $p|\sigma_{3p^m}$ if and only if $p|10d$.

3.3. **Sums of Even Powers of Secants.** For the Chebyshev polynomial C_d, if $d = 2j+1$ is odd then application of (2.49) to (3.12) shews that

$$\begin{aligned}
(2j+1)^{p^m}\sigma_{-p^m} &\equiv \frac{1}{6}j(j+1)(2j+1) \qquad (\mathrm{mod}\ p) \quad (j \geq 1),\\
(2j+1)^{2p^m}\sigma_{-2p^m} &\equiv \left(\frac{1}{6}j(j+1)(2j+1)\right)^2 - \frac{1}{60}j(j^2-1)(j+2)(2j+1)^2\\
&= \frac{1}{90}j(j+1)(2j+1)^2(j^2+j+3) \qquad (\mathrm{mod}\ p) \quad (j \geq 2),\\
(2j+1)^{3p^m}\sigma_{-3p^m} &\equiv \left(\frac{1}{6}j(j+1)(2j+1)\right)^3\\
&\quad -3\times\frac{1}{120}j(j^2-1)(j+2)(2j+1)\cdot\frac{1}{6}j(j+1)(2j+1)^2\\
&\quad +3\times\frac{1}{5040}j(j^2-1)(j^2-4)(j+3)(2j+1)\cdot(2j+1)^2\\
&= \frac{1}{7560}j(j+1)(2j+1)^3\left(8j^4+16j^3+35j^2+27j+54\right)\\
&\qquad (\mathrm{mod}\ p) \quad (j \geq 3), \qquad \textit{et cetera}.
\end{aligned} \tag{3.24}$$

If $d = 2j$ is even then application of (2.49) to (3.13) shews that

$$\begin{aligned}
2^{p^m}\sigma_{-p^m} &\equiv j^2 \qquad (\mathrm{mod}\ p) \quad (j \geq 1),\\
4^{p^m}\sigma_{-2p^m} &\equiv j^4 - \frac{4}{12}j^2(j^2-1) = \frac{1}{3}j^2(2j^2+1) \qquad (\mathrm{mod}\ p) \quad (j \geq 2),\\
8^{p^m}\sigma_{-3p^m} &\equiv j^6 - \frac{1}{2}j^2(j^2-1) + \frac{1}{30}j^2(j^2-1)(j^2-4)\\
&= \frac{1}{15}j^2\left(8j^4+5j^2+2\right) \qquad (\mathrm{mod}\ p) \quad (j \geq 3), \qquad \textit{et cetera}.
\end{aligned} \tag{3.25}$$

The zeros of C_d are given explicitly by (3.20), and hence (3.24) yields trigonometric congruences for prime p and integers $j > 0$ and $m \geq 0$:

$$\begin{aligned}
&\left(\frac{2j+1}{4}\right)^{p^m} \sum_{k=1}^{j} \sec^{2p^m}\left(\frac{[2k-1]\pi}{4j+2}\right) \equiv \\
&\qquad \frac{1}{6} j(j+1)(2j+1) \qquad (\bmod\ p)\ (j \geq 1), \\
&\left(\frac{2j+1}{4}\right)^{2p^m} \sum_{k=1}^{j} \sec^{4p^m}\left(\frac{[2k-1]\pi}{4j+2}\right) \equiv \\
&\frac{1}{90} j(j+1)(2j+1)^2(j^2+j+3) \qquad (\bmod\ p) \quad (j \geq 2), \\
&\left(\frac{2j+1}{4}\right)^{3p^m} \sum_{k=1}^{j} \sec^{6p^m}\left(\frac{[2k-1]\pi}{4j+2}\right) \equiv \\
&\frac{1}{7560} j(j+1)(2j+1)^3\left(8j^4+16j^3+35j^2+27j+54\right) \\
&\qquad (\bmod\ p)\ (j \geq 3), \qquad \textit{et cetera.}
\end{aligned} \tag{3.26}$$

For example, with $j = 5$, $p = 5$ and $m = 1$,

$$\frac{1}{p}\left[\left(\frac{2j+1}{4}\right)^{2p^m} \sum_{k=1}^{j} \sec^{4p^m}\left(\frac{[2k-1]\pi}{4j+2}\right) - \frac{1}{90} j(j+1)(2j+1)^2(j^2+j+3)\right]$$

$$= 498444890400952{\cdot}000\ . \tag{3.27}$$

Likewise, (3.25) yields trigonometric congruences for prime p and integers $j > 0$ and $m \geq 0$:

$$\begin{aligned}
\left(\frac{1}{2}\right)^{p^m} \sum_{k=1}^{j} \sec^{2p^m}\left(\frac{[2k-1]\pi}{4j}\right) &\equiv j^2 \qquad (\bmod\ p) \quad (j \geq 1), \\
\left(\frac{1}{4}\right)^{p^m} \sum_{k=1}^{j} \sec^{4p^m}\left(\frac{[2k-1]\pi}{4j}\right) &\equiv \frac{1}{3} j^2(2j^2+1) \qquad (\bmod\ p) \quad (j \geq 2), \\
\left(\frac{1}{8}\right)^{p^m} \sum_{k=1}^{j} \sec^{6p^m}\left(\frac{[2k-1]\pi}{4j}\right) &\equiv \frac{1}{15} j^2\left(8j^4+5j^2+2\right) \\
&\qquad (\bmod\ p)\ (j \geq 3), \quad \textit{et cetera.}
\end{aligned} \tag{3.28}$$

For example, with $j = 5$, $p = 11$ and $m = 1$,

$$\frac{1}{p}\left[\left(\frac{1}{2}\right)^{p^m} \sum_{k=1}^{j} \sec^{2p^m}\left(\frac{[2k-1]\pi}{4j}\right) - j^2\right] = 23548521916600{\cdot}000\ . \tag{3.29}$$

4. CHEBYSHEV POLYNOMIALS OF SECOND TYPE

The modified Chebyshev polynomial $S_d(x)$ of second type is defined by the initial values

$$S_0(x) = 1, \qquad S_1(x) = x, \tag{4.1}$$

with the same recurrence relation as for C_d:

$$S_{n+1}(x) = xS_n(x) - S_{n-1}(x). \tag{4.2}$$

By induction on n, it follows that for all $d \geq 0,\ S_d(x)$ is a monic polynomial in x of degree d with integer coefficients:

$$\begin{aligned} S_d(x) &= \sum_{k=0}^{d \div 2} (-1)^k \binom{d-k}{k} x^{d-2k} \\ &= x^d - (d-1)x^{d-2} + \binom{d-2}{2} x^{d-4} - \binom{d-3}{3} x^{d-6} + \cdots . \end{aligned} \tag{4.3}$$

For odd $d = 2j+1,\ S_d(x)$ is an odd polynomial:

$$S_{2j+1}(x) \quad =$$

$$(-1)^{j-1} \left[\cdots - \binom{j+4}{7} x^7 + \binom{j+3}{5} x^5 - \binom{j+2}{3} x^3 + (j+1)x \right]; \tag{4.4}$$

and for even $d = 2j,\ S_d(x)$ is an even polynomial:

$$S_{2j}(x) = (-1)^j \left[\cdots - \binom{j+3}{6} x^6 + \binom{j+2}{4} x^4 - \binom{j+1}{2} x^2 + 1 \right]. \tag{4.5}$$

For example (Lyusternik *et alia* [7], pp. 172–173), in addition to (4.1),

$$\begin{aligned} S_2(x) = x^2 - 1, \qquad S_3(x) = x^3 - 2x, \qquad S_4(x) = x^4 - 3x^2 + 1, \\ S_5(x) = x^5 - 4x^3 + 3x, \qquad S_6(x) = x^6 - 5x^4 + 6x^2 - 1. \end{aligned} \tag{4.6}$$

The polynomials $S_d(x)$ are orthogonal on the interval (-2,2), with weight function $\sqrt{4-x^2}$. Let

$$x = 2\cos\vartheta \tag{4.7}$$

so that if $-2 \leq x \leq 2$ then ϑ is real. Then it follows from (4.1) and (4.2), by induction on n, that (Lyusternik *et alia* [7], p. 163)

$$S_d(2\cos\vartheta)\sin\vartheta = \sin((d+1)\vartheta). \tag{4.8}$$

Accordingly, the zeros of $S_d(x)$ are given by

$$\sin((d+1)\vartheta) = 0, \tag{4.9}$$

so that

$$(d+1)\vartheta = k\pi \tag{4.10}$$

with integer k, and hence each zero of $S_d(x)$ is of the form

$$\alpha_k = 2\cos\left(\frac{k\pi}{d+1}\right). \tag{4.11}$$

For $k = 1, 2, \ldots, d$, this formula gives a strictly decreasing sequence of real zeros

$$2 > \alpha_1 > \alpha_2 > \ldots > \alpha_d > -2, \tag{4.12}$$

and hence all d zeros of S_d are given by $\alpha_1, \alpha_2, \ldots, \alpha_d$.

4.1. **Sums of Even Powers of Cosines.** Applying (2.24) to the expansion (4.3) of the monic polynomial $S_d(x)$, we get trigonometric congruences for prime p and integers $d > 0$ and $m \geq 0$:

$$\begin{aligned}
\sigma_{p^m} &= 4^{p^m} \sum_{k=1}^{d \div 2} \cos^{2p^m}\left(\frac{k\pi}{d+1}\right) \equiv d-1 \qquad (\mathrm{mod}\ p) \quad (d \geq 2), \\
\sigma_{2p^m} &= 16^{p^m} \sum_{k=1}^{d \div 2} \cos^{4p^m}\left(\frac{k\pi}{d+1}\right) \equiv (d-1)^2 - 2\binom{d-2}{2} \\
&\quad = 3d-5 \qquad (\mathrm{mod}\ p) \quad (d \geq 4), \\
\sigma_{3p^m} &= 64^{p^m} \sum_{k=1}^{d \div 2} \cos^{6p^m}\left(\frac{k\pi}{d+1}\right) \\
&\equiv (d-1)^3 - 3(d-1)\binom{d-2}{2} + 3\binom{d-3}{3} \\
&\quad = 10d - 22 \qquad (\mathrm{mod}\ p) \quad (d \geq 6), \quad et\ cetera,
\end{aligned} \tag{4.13}$$

in view of (4.3) and (4.11).

For example, with $d = 5$, $p = 3$ and $m = 3$,

$$\frac{1}{p}\left[4^{p^m} \sum_{k=1}^{d \div 2} \cos^{2p^m}\left(\frac{k\pi}{d+1}\right) - (d-1)\right] = 2541865828328{\cdot}000\ . \tag{4.14}$$

Hence (Tee [12], Theorem 1), the positive zeros of S_d give the result that $p|\sigma_{p^m}$ if and only if $d \equiv 1$ (mod p); $p|\sigma_{2p^m}$ if and only if $p|(3d-5)$, and $p|\sigma_{3p^m}$ if and only if $p|(10d-22)$.

4.2. **Sums of Even Powers of Secants.** For the Chebyshev polynomial $S_d(x)$, if $d = 2j+1$ is odd then application of (2.49) to (4.4) shews that

$$\begin{aligned}
(j+1)^{p^m}\sigma_{-p^m} &\equiv \binom{j+2}{3} = \frac{1}{6}j(j+1)(j+2) \qquad (\mathrm{mod}\ p) \quad (j \geq 1), \\
(j+1)^{2p^m}\sigma_{-2p^m} &\equiv \binom{j+2}{3}^2 - 2\binom{j+3}{5}(j+1) \\
&= \frac{1}{180}j(j+1)^2(j+2)\left(2j^2+4j+9\right) \qquad (\mathrm{mod}\ p)\ (j \geq 2), \\
(j+1)^{3p^m}\sigma_{-3p^m} &\equiv \binom{j+2}{3}^3 - 3\binom{j+3}{5}\binom{j+2}{3}(j+1) \\
&\qquad + 3\binom{j+4}{7}(j+1)^2 \\
&= \frac{1}{7560}j(j+1)^3(j+2)\left(8j^4+32j^3+77j^2+90j+108\right) \\
&\qquad (\mathrm{mod}\ p) \quad (j \geq 3), \quad et\ cetera.
\end{aligned} \tag{4.15}$$

If $d = 2j$ is even then the inverse polynomial (scaled by $(-1)^j$) is monic, and application of (2.49) to (4.5) shews that

$$\begin{aligned}
\sigma_{-p^m} &\equiv \frac{1}{2}j(j+1) \qquad \pmod p \quad (j \geq 1),\\
\sigma_{-2p^m} &\equiv \left(\frac{1}{2}j(j+1)\right)^2 - 2\binom{j+2}{4}\\
&= \frac{1}{288}j(j+1)\left(71j^2 + 71j + 1\right) \qquad \pmod p \quad (j \geq 2),\\
\sigma_{-3p^m} &\equiv \left(\frac{1}{2}j(j+1)\right)^3 - \frac{3}{2}j(j+1)\binom{j+2}{4} + 3\binom{j+3}{6}\\
&= \frac{1}{120}j(j+1)\left(8j^4 + 16j^3 + 19j^2 + 11j + 6\right)\\
&\qquad \pmod p \ (j \geq 3), \quad \textit{et cetera}.
\end{aligned} \tag{4.16}$$

The zeros of $S_d(x)$ are given explicitly by (4.11), and hence (4.15) yields trigonometric congruences, for prime p and integers $j > 0$ and $m \geq 0$:

$$\begin{aligned}
&\left(\frac{j+1}{4}\right)^{p^m} \sum_{k=1}^{j} \sec^{2p^m}\left(\frac{k\pi}{2(j+1)}\right) \equiv\\
&\qquad \frac{1}{6}j(j+1)(j+2) \qquad \pmod p \quad (j \geq 1),\\
&\left(\frac{j+1}{4}\right)^{2p^m} \sum_{k=1}^{j} \sec^{4p^m}\left(\frac{k\pi}{2(j+1)}\right) \equiv\\
&\frac{1}{180}j(j+1)^2(j+2)\left(2j^2 + 4j + 9\right) \qquad \pmod p \ (j \geq 2),\\
&\left(\frac{j+1}{4}\right)^{3p^m} \sum_{k=1}^{j} \sec^{6p^m}\left(\frac{k\pi}{2(j+1)}\right) \equiv\\
&\frac{1}{7560}j(j+1)^3(j+2)\left(8j^4 + 32j^3 + 77j^2 + 90j + 108\right)\\
&\qquad \pmod p \quad (j \geq 3), \quad \textit{et cetera}.
\end{aligned} \tag{4.17}$$

For example, with $j = 3$, $p = 3$ and $m = 2$,

$$\begin{aligned}
&\frac{1}{p}\left[\left(\frac{j+1}{4}\right)^{2p^m} \sum_{k=1}^{j} \sec^{4p^m}\left(\frac{k\pi}{2(j+1)}\right) - \frac{1}{180}j(j+1)^2(j+2)\left(2j^2 + 4j + 9\right)\right]\\
&= 347256964339012{\cdot}002\ .
\end{aligned} \tag{4.18}$$

Likewise, (4.16) yields trigonometric congruences, for prime p and positive integers $j > 0$ and $m \geq 0$:

$$\left(\frac{1}{4}\right)^{p^m} \sum_{k=1}^{j} \sec^{2p^m}\left(\frac{k\pi}{2j+1}\right) \equiv \frac{1}{2}j(j+1) \quad (\mathrm{mod}\ p) \quad (j \geq 1),$$

$$\left(\frac{1}{16}\right)^{p^m} \sum_{k=1}^{j} \sec^{4p^m}\left(\frac{k\pi}{2j+1}\right) \equiv \frac{1}{288}j(j+1)\left(71j^2+71j+1\right) \quad (\mathrm{mod}\ p)\ (j \geq 2),$$

$$\left(\frac{1}{64}\right)^{p^m} \sum_{k=1}^{j} \sec^{6p^m}\left(\frac{k\pi}{2j+1}\right) \equiv \frac{1}{120}j(j+1)\left(8j^4+16j^3+19j^2+11j+6\right)$$

(4.19) $$(\mathrm{mod}\ p) \quad (j \geq 3), \quad et\ cetera.$$

For example, with $j = 5$, $p = 13$ and $m = 1$,

(4.20) $$\frac{1}{p}\left[\left(\frac{1}{4}\right)^{p^m} \sum_{k=1}^{j} \sec^{2p^m}\left(\frac{k\pi}{2j+1}\right) - \frac{1}{2}j(j+1)\right] = 11878784041151{\cdot}000\ .$$

Standard Chebyshev Polynomial of Second Type

The standard Chebyshev polynomial of the second type is

(4.21) $$U_d(x) = S_d(2x) = (2x)^d - \cdots ,$$

(Lyusternik [7], p. 163, Mason & Handscomb [9], pp. 3–4) and the transformation of $U_d(x)$ by scaling with $b = 2$ in (2.32) just reproduces the congruences (4.13) for $S_d(x)$.

The negative powers of zeros of U_d yield trigonometric congruences which can be obtained from (4.17) and (4.19) by multiplying both sides by 4^{p^m} *et cetera*, using the fact that $4^{p^m} \equiv 4$ (mod p), by Fermat's Little Theorem. The inverse inferences are not so straightforward.

5. ROOTS OF $C_{2j+1}(x) + c = 0$

If $-2 \leq c \leq 2$, then the equation

(5.1) $$C_d(x) + c = 0.$$

has d real roots in $[-2, 2]$. Indeed, writing $c = 2\cos\gamma$ with real γ = arccos(c/2) ($\gamma \in [0, \pi]$), and with $x = 2\cos\vartheta$, equation (5.1) becomes:

(5.2) $$0 = 2\cos(d\vartheta) + 2\cos\gamma = 4\cos\left(\frac{1}{2}(d\vartheta+\gamma)\right)\cos\left(\frac{1}{2}(d\vartheta-\gamma)\right).$$

Hence, the d roots of (5.1) are of the form $\alpha = 2\cos\vartheta$, where

(5.3) $$\cos\left(\frac{1}{2}(d\vartheta \pm \gamma)\right) = 0;$$

so that

(5.4) $$\frac{1}{2}(d\vartheta \pm \gamma) = \left(k - \frac{1}{2}\right)\pi$$

for integer k, and hence

(5.5) $$\vartheta = \frac{(2k-1)\pi \mp \gamma}{d}.$$

Now, write

$$\gamma = \beta\pi, \tag{5.6}$$

so that

$$\beta = \frac{\arccos(c/2)}{\pi} \tag{5.7}$$

with $0 \le \beta \le 1$, and the d roots of (5.1) are each of the form:

$$\alpha = 2\cos\left(\frac{(2k-1\mp\beta)\pi}{d}\right). \tag{5.8}$$

Hence, for $-2 < c < 2$ $(1 > \beta > 0)$, the equation (5.1) has d distinct roots of the form

$$\alpha = 2\cos\left(\frac{\iota\pi}{d}\right), \tag{5.9}$$

given in decreasing order by

$$\iota = 1-\beta, 1+\beta,\ 3-\beta, 3+\beta,\ ,\ldots,\ 2j-1-\beta, 2j-1+\beta,\ 2j+1-\beta \tag{5.10}$$

for odd $d = 2j+1$, and by

$$\iota = 1-\beta, 1+\beta,\ 3-\beta, 3+\beta,\ ,\ldots,\ 2j-1-\beta, 2j-1+\beta, \tag{5.11}$$

for even $d = 2j$.

For $c = 2$ $(\beta = 0)$ there are j pairs of double roots, given in decreasing order by $\iota = 1,1,\ 3,3,\ \ldots,\ 2j-1, 2j-1$, and then (for odd $d = 2j+1$) by $\iota = 2j+1$ $(\alpha = -2)$. For $c = -2$ $(\beta = 1)$ the j pairs of double roots are given in decreasing order by $\iota = 2,2,\ 4,4,\ ,\ldots,\ 2j, 2j$, preceded (for odd $d = 2j+1$) by $\iota = 0$ $(\alpha = 2)$.

Hereafter, we shall consider only rational $c = n/r$, with $r > 0$, $-2r \le n \le 2r$, and $\gcd(n, r) = 1$.

Multiply (5.1) by r, to get a polynomial equation in x of degree d with integer coefficients:

$$rC_d(x) + n = 0. \tag{5.12}$$

If we also restrict β to be rational with rational $c = 2\cos(\beta\pi)$, then the only acceptable values of c in (5.1) are 2, 1, 0, -1 and -2, given by $\beta = 0, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, 1$. The case of $c = 0$ $(\beta = \frac{1}{2})$ has been dealt with in (3.8) to (4.20).

Since C_d is an odd polynomial for odd $d = 2j+1$, then for all complex x and c and non–negative integer j, $C_{2j+1}(x) + c = -(C_{2j+1}(-x) - c)$. Therefore, the set of $2j+1$ zeros of $C_{2j+1}(x) - c$, times -1, is the set of zeros (including multiplicity) of $C_{2j+1}(x) + c$. Accordingly, each sum of powers of zeros of $C_{2j+1}(x) - c$ equals ± 1 times the sum of the same powers of zeros of $C_{2j+1}(x) + c$. Therefore, for the purposes of this paper we need to consider only $c > 0$, and hence $0 < n \le 2r$ with $\gcd(n, r) = 1$.

In view of (3.12), (5.12) becomes the equation with integer coefficients:

$$\begin{aligned} rx^{2j+1} + 0x^{2j} - r(2j+1)x^{2j-1} + 0x^{2j-2} + \cdots \\ \cdots + (-1)^{j-1}\frac{1}{6}rj(j+1)(2j+1)x^3 - 0x^2 \\ +(-1)^j r(2j+1)x + n = 0, \qquad (j > 1). \end{aligned} \tag{5.13}$$

For $j = 0$ and $j = 1$, the polynomial $rC_d(x) + n$ reduces to:

$$rC_1(x) + n = rx + n, \qquad rC_3(x) + n = rx^3 - 3rx + n. \tag{5.14}$$

. For those polynomials (5.13) and (5.14), the congruences (2.40) become:

$$\begin{aligned} r^{p^m} S_{p^m} &\equiv 0 \pmod p \quad (j>0),\\ r^{2p^m} S_{2p^m} &\equiv 2r^2(2j+1) \pmod p \quad (j>0),\\ r^{3p^m} S_{3p^m} &\equiv 0 \pmod p \quad (j>1), \quad \textit{et cetera} \end{aligned} \tag{5.15}$$

for positive powers; and for $j \geq 0$ the congruences (2.48) become:

$$\begin{aligned} n^{p^m} S_{-p^m} &\equiv (-1)^{j+1} r(2j+1) \pmod p,\\ n^{2p^m} S_{-2p^m} &\equiv \big(r(2j+1)\big)^2 \pmod p,\\ n^{3p^m} S_{-3p^m} &\equiv (-1)^{j+1}\left(\big(r(2j+1)\big)^3 - \frac{1}{2} rj(j+1)(2j+1)n^2\right)\\ &\qquad \pmod p, \quad \textit{et cetera} \end{aligned} \tag{5.16}$$

for negative powers.

From (5.7),

$$\beta = \frac{\arccos(n/2r)}{\pi}, \tag{5.17}$$

and we need consider only $0 \leq \beta < \frac{1}{2}$.

Sums of Powers of Cosines

In view of (5.8), the congruences (5.15) yield identities in prime p and integers $j,\ m \geq 0,\ r > 0$ and $n \in [1, \ldots, 2r]$:

$$\begin{aligned} &\sum_{k=1}^{j+1}\left[2r\,\cos\left(\frac{(2k-1-\beta)\pi}{2j+1}\right)\right]^{p^m} + \sum_{k=1}^{j}\left[2r\,\cos\left(\frac{(2k-1+\beta)\pi}{2j+1}\right)\right]^{p^m}\\ &\qquad \equiv 0 \pmod p \quad (j>0),\\ &\sum_{k=1}^{j+1}\left[2r\,\cos\left(\frac{(2k-1-\beta)\pi}{2j+1}\right)\right]^{2p^m} + \sum_{k=1}^{j}\left[2r\,\cos\left(\frac{(2k-1+\beta)\pi}{2j+1}\right)\right]^{2p^m}\\ &\qquad \equiv 2r^2(2j+1) \pmod p \quad (j>0),\\ &\sum_{k=1}^{j+1}\left[2r\,\cos\left(\frac{(2k-1-\beta)\pi}{2j+1}\right)\right]^{3p^m} + \sum_{k=1}^{j}\left[2r\,\cos\left(\frac{(2k-1+\beta)\pi}{2j+1}\right)\right]^{3p^m}\\ &\qquad \equiv 0 \pmod p \quad (j>1), \quad \textit{et cetera.} \end{aligned} \tag{5.18}$$

For example, with $j=1,\ p=3,\ m=2,\ n=5,\ r=4$ and $\beta = \arccos(n/(2r))/\pi$,

$$\begin{aligned} &\frac{1}{p}\left[\sum_{k=1}^{j+1}\left(2r\,\cos\left(\frac{(2k-1-\beta)\pi}{2j+1}\right)\right)^{p^m} + \sum_{k=1}^{j}\left(2r\,\cos\left(\frac{(2k-1+\beta)\pi}{2j+1}\right)\right)^{p^m}\right]\\ &\qquad = -27054080{\cdot}000\,. \end{aligned} \tag{5.19}$$

Sums of Powers of Secants

Likewise, the congruences (5.16) yield identities in prime p and integers $j \geq 0,\ m \geq 0,\ r > 0$ and $n \in [1, \ldots, 2r]$:

$$\left(\frac{n}{2}\right)^{p^m}\left[\sum_{k=1}^{j+1}\sec^{p^m}\left(\frac{(2k-1-\beta)\pi}{2j+1}\right) + \sum_{k=1}^{j}\sec^{p^m}\left(\frac{(2k-1+\beta)\pi}{2j+1}\right)\right]$$

$$\equiv \ (-1)^{j+1} r(2j+1) \qquad (\mathrm{mod}\ p),$$

$$\left(\frac{n}{2}\right)^{2p^m} \left[\sum_{k=1}^{j+1} \sec^{2p^m}\left(\frac{(2k-1-\beta)\pi}{2j+1}\right) + \sum_{k=1}^{j} \sec^{2p^m}\left(\frac{(2k-1+\beta)\pi}{2j+1}\right)\right]$$

$$\equiv \ (r(2j+1))^2 \qquad (\mathrm{mod}\ p),$$

$$\left(\frac{n}{2}\right)^{3p^m} \left[\sum_{k=1}^{j+1} \sec^{3p^m}\left(\frac{(2k-1-\beta)\pi}{2j+1}\right) + \sum_{k=1}^{j} \sec^{3p^m}\left(\frac{(2k-1+\beta)\pi}{2j+1}\right)\right]$$

$$\equiv \ (-1)^{j+1} r(2j+1) \left((r(2j+1))^2 - \frac{1}{2} j(j+1) n^2\right) \qquad (\mathrm{mod}\ p),\ etc. \tag{5.20}$$

For example, with $j=4,\ p=3,\ m=1,\ n=4,\ r=3$ and $\beta = \arccos\left(\frac{n}{2r}\right)/\pi$,

$$\frac{1}{p}\left[\left(\frac{n}{2}\right)^{3p^m} \left(\sum_{k=1}^{j+1} \sec^{3p^m}\left(\frac{(2k-1-\beta)\pi}{2j+1}\right) + \sum_{k=1}^{j} \sec^{3p^m}\left(\frac{(2k-1+\beta)\pi}{2j+1}\right)\right)\right.$$

$$\left. - (-1)^{j+1} r(2j+1)\left((r(2j+1))^2 - \frac{1}{2} j(j+1) n^2\right)\right]$$

$$= \ -1138092652296{\cdot}000\,. \tag{5.21}$$

5.1. **Roots of $C_{2j+1}(x) + 2 \ = \ 0$.** As was noted above, if we want rational c with rational β then the only values are $c \ = \ 2, 1, 0, -1, -2$, given by $\beta \ = \ 0, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, 1$; and we need only consider $c = 2$ ($\beta = 0$) and $c = 1$ ($\beta = \frac{1}{3}$).

With $c = 2,\ r = 1,\ n = 2$ and $\beta = 0$, the identities (5.18) and (5.20) yield identities in prime p and non–negative integers j and m, for sums of powers of cosines and of secants of *rational* multiples of π.

5.1.1. *Sums of Powers of Cosines.* From (5.17), (5.18) and (5.20), there are j double roots of (5.12) given by $k = 1, \ldots, j$, and the simple root -2 given by $k = j+1$. Hence the congruences (5.18) become:

$$2 \times \sum_{k=1}^{j}\left[2\cos\left(\frac{(2k-1)\pi}{2j+1}\right)\right]^{p^m} + (-2)^{p^m} \ \equiv \ 0 \qquad (\mathrm{mod}\ p) \quad (j>0),$$

$$2 \times \sum_{k=1}^{j}\left[2\cos\left(\frac{(2k-1)\pi}{2j+1}\right)\right]^{2p^m} + (-2)^{2p^m} \ \equiv \ 4j+2 \ (\mathrm{mod}\ p) \quad (j>0),$$

$$2 \times \sum_{k=1}^{j}\left[2\cos\left(\frac{(2k-1)\pi}{2j+1}\right)\right]^{3p^m} + (-2)^{3p^m} \ \equiv \ 0 \qquad (\mathrm{mod}\ p) \quad (j>1), \tag{5.22}$$

et cetera.

But, by Fermat's Little Theorem,

$$(-2)^{p^m} \ \equiv \ -2, \quad (-2)^{2p^m} \ = \ 4^{p^m} \ \equiv \ 4, \quad (-2)^{3p^m} \ = \ (-8)^{p^m} \ \equiv \ -8,$$

$$(\mathrm{mod}\ p), \tag{5.23}$$

and hence (5.22) simplifies to the identities in prime p and integers $m \geq 0$ and j:

$$
\begin{aligned}
2^{p^m+1}\sum_{k=1}^{j}\cos^{p^m}\left(\frac{(2k-1)\pi}{2j+1}\right) &\equiv 2 \pmod p \quad (j>0),\\
2^{2p^m+1}\sum_{k=1}^{j}\cos^{2p^m}\left(\frac{(2k-1)\pi}{2j+1}\right) &\equiv 4j-2 \pmod p \quad (j>0),\\
2^{3p^m+1}\sum_{k=1}^{j}\cos^{3p^m}\left(\frac{(2k-1)\pi}{2j+1}\right) &\equiv 8 \pmod p \quad (j>1),
\end{aligned} \tag{5.24}
$$

et cetera.

For example, with $j = 2,\ p = 3$ and $m = 3$,

$$
\frac{1}{p}\left[2\times\sum_{k=1}^{j}\left[2\cos\left(\frac{(2k-1)\pi}{2j+1}\right)\right]^{3p^m} - 8\right] = 56481679380052848{\cdot}000\,. \tag{5.25}
$$

5.1.2. *Sums of Powers of Secants.* The congruences (5.20) become identities in prime p and integers $m \geq 0$ and $j > 0$:

$$
\begin{aligned}
2\times\sum_{k=1}^{j}\sec^{p^m}\left(\frac{(2k-1)\pi}{2j+1}\right) &\equiv 1 + (-1)^{j+1}(2j+1) \pmod p,\\
2\times\sum_{k=1}^{j}\sec^{2p^m}\left(\frac{(2k-1)\pi}{2j+1}\right) &\equiv 4j(j+1) \pmod p,\\
2\times\sum_{k=1}^{j}\sec^{3p^m}\left(\frac{(2k-1)\pi}{2j+1}\right) &\equiv 1 + (-1)^{j+1}(2j+1)(2j^2+2j+1)\\
&\qquad \pmod p, \qquad et\ cetera.
\end{aligned} \tag{5.26}
$$

For example, with $j = 5,\ p = 11$ and $m = 1$,

$$
\frac{1}{p}\left[2\times\sum_{k=1}^{j}\sec^{p^m}\left(\frac{(2k-1)\pi}{2j+1}\right) - (1 - (-1)^j(2j+1))\right] = 374871132{\cdot}000\,. \tag{5.27}
$$

5.2. **Roots of $C_{2j+1}(x) + 1 = 0$.** Here, $c = 1,\ r = 1,\ n = 1$ and $\beta = \frac{1}{3}$, so that the roots are given (cf. (5.9) and (5.10), in decreasing order, as

$$
\begin{aligned}
\alpha &= 2\cos\left(\frac{\left(2k-1\mp\frac{1}{3}\right)\pi}{2j+1}\right)\\
&= 2\cos\left(\frac{(6k-3\mp 1)\pi}{6j+3}\right) = 2\cos\left(\frac{\iota\pi}{6j+3}\right),\\
\iota &= 2,4,\quad 8,10,\quad 14,16,\quad \ldots,\quad 6j-4, 6j-2,\quad 6j+2.
\end{aligned} \tag{5.28}
$$

5.2.1. *Sums of Powers of Cosines.* Hence the congruences (5.18) yield identities in prime p and integers $m \geq 0$ and j:

$$\sum_{k=1}^{j+1}\left[2\cos\left(\frac{(6k-4)\pi}{6j+3}\right)\right]^{p^m} + \sum_{k=1}^{j}\left[2\cos\left(\frac{(6k-2)\pi}{6j+3}\right)\right]^{p^m} \equiv 0 \pmod p \quad (j>0),$$

$$\sum_{k=1}^{j+1}\left[2\cos\left(\frac{(6k-4)\pi}{6j+3}\right)\right]^{2p^m} + \sum_{k=1}^{j}\left[2\cos\left(\frac{(6k-2)\pi}{6j+3}\right)\right]^{2p^m} \equiv 4j+2 \pmod p \quad (j>0),$$

$$\sum_{k=1}^{j+1}\left[2\cos\left(\frac{(6k-4)\pi}{6j+3}\right)\right]^{3p^m} + \sum_{k=1}^{j}\left[2\cos\left(\frac{(6k-2)\pi}{6j+3}\right)\right]^{3p^m} \equiv 0 \pmod p \quad (j>1), \quad et\ cetera. \tag{5.29}$$

For example, with $j=4,\ p=11$ and $m=1$,

$$\frac{1}{p}\left[\sum_{k=1}^{j}\left[2\cos\left(\frac{(6k-2)\pi}{6j+3}\right)\right]^{3p^m} + \sum_{k=1}^{j+1}\left[2\cos\left(\frac{(6k-4)\pi}{6j+3}\right)\right]^{3p^m}\right] = -290296152{\cdot}000\,. \tag{5.30}$$

5.2.2. *Sums of Powers of Secants.* The congruences (5.20) yield identities in prime p and non-negative integers m and j:

$$\left(\frac{1}{2}\right)^{p^m}\left[\sum_{k=1}^{j+1}\sec^{p^m}\left(\frac{(6k-4)\pi}{6j+3}\right) + \sum_{k=1}^{j}\sec^{p^m}\left(\frac{(6k-2)\pi}{6j+3}\right)\right] \equiv (-1)^{j+1}(2j+1) \pmod p,$$

$$\left(\frac{1}{4}\right)^{p^m}\left[\sum_{k=1}^{j+1}\sec^{2p^m}\left(\frac{(6k-4)\pi}{6j+3}\right) + \sum_{k=1}^{j}\sec^{2p^m}\left(\frac{(6k-2)\pi}{6j+3}\right)\right] \equiv (2j+1)^2 \pmod p,$$

$$\left(\frac{1}{8}\right)^{p^m}\left[\sum_{k=1}^{j+1}\sec^{3p^m}\left(\frac{(6k-4)\pi}{6j+3}\right) + \sum_{k=1}^{j}\sec^{3p^m}\left(\frac{(6k-2)\pi}{6j+3}\right)\right] \equiv (-1)^{j+1}\frac{1}{2}(2j+1)(7j^2+7j+2) \pmod p, \quad etc. \tag{5.31}$$

For example, with $j=5,\ p=3$ and $m=2$,

$$\frac{1}{p}\left[\sum_{k=1}^{j}\left[2\cos\left(\frac{(6k-2)\pi}{6j+3}\right)\right]^{-p^m} + \sum_{k=1}^{j+1}\left[2\cos\left(\frac{(6k-4)\pi}{6j+3}\right)\right]^{-p^m} + (-1)^j(2j+1)\right] = 520752892{\cdot}000\,. \tag{5.32}$$

6. ROOTS OF $C_{2j}(x) + c = 0$

For even $d = 2j$, in view of (3.13), (5.12) becomes the equation with integer coefficients:

$$rx^{2j} - 2rjx^{2j-2} + rj(2j-3)x^{2j-4} - \frac{1}{3}rj(2j-4)(2j-5)x^{2j-6} + \cdots$$
$$- \frac{1}{360}(-1)^j rj^2(j^2-1)(j^2-4)x^6 + \frac{1}{12}(-1)^j rj^2(j^2-1)x^4 - (-1)^j rj^2x^2$$
$$(6.1) \qquad + \left[(-1)^j 2r + n\right] = 0, \qquad (j > 2),$$

where $r > 0$, $n \in [-2r, \ldots, 2r]$, and $\gcd(n, r) = 1$. For $j = 1$ and $j = 2$, the polynomial equation (5.12) reduces to:

$$(6.2) \qquad rx^2 + [n - 2r] = 0. \qquad rx^4 - 4rx^2 + [2r + n] = 0.$$

For those even polynomials (6.1) and (6.2), the congruences (2.41) become:

$$\begin{aligned} r^{p^m}\sigma_{p^m} &\equiv 2rj \pmod p \quad (j > 1),\\ r^{2p^m}\sigma_{2p^m} &\equiv 6r^2j \pmod p \quad (j > 2),\\ r^{3p^m}\sigma_{3p^m} &\equiv 20r^3j \pmod p \quad (j > 3), \quad \textit{et cetera}. \end{aligned} \qquad (6.3)$$

For (6.1) and (6.2) we get from (2.49) the congruences

$$\begin{aligned} \left[(-1)^j 2r + n\right]^{p^m}\sigma_{-p^m} &\equiv (-1)^j rj^2 \pmod p,\\ \left[(-1)^j 2r + n\right]^{2p^m}\sigma_{-2p^m} &\equiv r^2j^4 - \frac{1}{6}(-1)^j rj^2(j^2-1)\left[(-1)^j 2r + n\right]\\ &\qquad \pmod p,\\ \left[(-1)^j 2r + n\right]^{3p^m}\sigma_{-3p^m} &\equiv (-1)^j r^3j^6 - \frac{1}{4}r^2j^4(j^2-1)\left[(-1)^j 2r + n\right]\\ &\quad + \frac{1}{120}(-1)^j rj^2(j^2-1)(j^2-4)\left[(-1)^j 2r + n\right]^2\\ &\qquad \pmod p, \quad \textit{et cetera}, \end{aligned} \qquad (6.4)$$

unless $[(-1)^j 2r + n] = 0$, which occurs with $c = 2(-1)^{j+1}$.

With β as in (5.7), in view of (5.9) and (5.11), the j pairs of roots of the even equation (5.12) are given by:

$$\begin{aligned} 2\cos\left(\frac{(2j-1-\beta)\pi}{2j}\right) &= -2\cos\left(\frac{(1+\beta)\pi}{2j}\right),\\ 2\cos\left(\frac{(2j-3-\beta)\pi}{2j}\right) &= -2\cos\left(\frac{(3+\beta)\pi}{2j}\right),\\ &\vdots\\ 2\cos\left(\frac{(1-\beta)\pi}{2j}\right) &= -2\cos\left(\frac{(2j-1+\beta)\pi}{2j}\right); \end{aligned} \qquad (6.5)$$

and hence

$$(6.6) \qquad \sigma_q = \tfrac{1}{2}S_{2q} = \sum_{k=1}^{j}\left[2\cos\left(\frac{(2k-1-\beta)\pi}{2j}\right)\right]^{2q}.$$

Sums of Even Powers of Cosines

Hence, the congruences (6.3) become identities in prime p and integers $j \geq 2,\ r > 0,\ n \in [-2r, \dots, 2r]$:

$$(4r)^{p^m} \sum_{k=1}^{j} \cos^{2p^m}\left(\frac{(2k-1-\beta)\pi}{2j}\right) \equiv 2rj \pmod{p} \quad (j > 1),$$

$$(4r)^{2p^m} \sum_{k=1}^{j} \cos^{4p^m}\left(\frac{(2k-1-\beta)\pi}{2j}\right) \equiv 6r^2 j \pmod{p} \quad (j > 2),$$

$$(4r)^{3p^m} \sum_{k=1}^{j} \cos^{6p^m}\left(\frac{(2k-1-\beta)\pi}{2j}\right) \equiv 20r^3 j \pmod{p} \quad (j > 3), \tag{6.7}$$

et cetera.

For example, with $j = 5,\ p = 7,\ m = 1,\ n = 5,\ r = 3$ and $\beta = \arccos(n/(2r))/\pi$,

$$\frac{1}{p}\left[(4r)^{2p^m} \sum_{k=1}^{j} \cos^{4p^m}\left(\frac{(2k+1-\beta)\pi}{2j}\right) - 6r^2 j\right] = 97781053802265{\cdot}000\ . \tag{6.8}$$

Sums of Even Powers of Secants

Likewise, the congruences (6.4) become:

$$\left[\frac{(-1)^j 2r + n}{4}\right]^{p^m} \sum_{k=1}^{j} \sec^{2p^m}\left(\frac{(2k-1-\beta)\pi}{2j}\right) \equiv (-1)^j r j^2 \pmod{p},$$

$$\left[\frac{(-1)^j 2r + n}{4}\right]^{2p^m} \sum_{k=1}^{j} \sec^{4p^m}\left(\frac{(2k-1-\beta)\pi}{2j}\right)$$
$$\equiv r^2 j^4 - \frac{1}{6}(-1)^j r j^2 (j^2-1)\left[(-1)^j 2r + n\right] \pmod{p},$$

$$\left[\frac{(-1)^j 2r + n}{4}\right]^{3p^m} \sum_{k=1}^{j} \sec^{6p^m}\left(\frac{(2k-1-\beta)\pi}{2j}\right)$$
$$\equiv (-1)^j r^3 j^6 - \frac{1}{4} r^2 j^4 (j^2-1) \times$$
$$\times \left[(-1)^j 2r + n\right] + \frac{1}{120}(-1)^j r j^2 (j^2-1)(j^2-4)\left[(-1)^j 2r + n\right]^2 \pmod{p}, \tag{6.9}$$

et cetera.

For example, with $j = 3,\ p = 3,\ m = 1,\ n = 2,\ r = 3$ and $\beta = \arccos(n/(2r))/\pi$,

$$\frac{1}{p}\Bigg[\left(\frac{(-1)^j 2r + n}{4}\right)^{3p^m} \sum_{k=1}^{j} \sec^{6p^m}\left(\frac{(2k+1-\beta)\pi}{2j}\right)$$
$$-\left((-1)^j r^3 j^6 - \frac{1}{4} r^2 j^4 (j^2-1)\left[(-1)^j 2r + n\right]\right.$$
$$\left. + \frac{1}{120}(-1)^j r j^2 (j^2-1)(j^2-4)\left[(-1)^j 2r + n\right]^2\right)\Bigg]$$
$$= -912284487672{\cdot}000\ . \tag{6.10}$$

For both c and β to be rational, $c = 2, 1, 0, -1, -2$ with $\beta = 0, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, 1$. The case $c = 0$ has been treated in (3.8)–(4.20).

If $c = 2$ with even $d = 2j$ and $x = 2\cos\vartheta$, then (5.12) becomes

$$0 = 2(\cos(2j\vartheta) + 1) = 4\cos^2(j\vartheta), \tag{6.11}$$

so that the zeros of $C_{2j}(x) + 2$ are given by those of cos($j\vartheta$), each with multiplicity 2. Those zeros have been treated in (3.18)–(3.26). If $c = -2$ with even $d = 2j$ and $x = 2\cos\vartheta$, then (5.12) becomes

$$0 = 2(\cos(2j\vartheta) - 1) = -4\sin^2(j\vartheta), \tag{6.12}$$

so that the zeros of $C_{2j}(x) - 2$ are given by those of sin($j\vartheta$), each with multiplicity 2. Those zeros have been treated in (4.2)–(4.10).

Accordingly, the only new congruences of the forms (6.7) and (6.8) with cosines and secants of *rational* multiples of π are given by $c = \pm 1$.

As was noted after (6.4), the congruences (6.9) do not apply for $c = 2(-1)^{j+1}$ — that case is covered by (6.11) and (6.12).

6.1. **Roots of** $C_{2j}(x)+1 = 0$**.** Here $c = 1,\ n = 1,\ r = 1$ and $\beta = \frac{1}{3}$, so that the even equation $C_{2j}(x)+1 = 0$ has roots (in $\pm$ pairs), which are given in decreasing order (cf. (5.9) and (5.11)), as

$$\begin{gathered} \alpha = 2\cos\left(\frac{\left(2k-1\mp\frac{1}{3}\right)\pi}{2j}\right) = 2\cos\left(\frac{(6k-3\mp 1)\pi}{6j}\right) = 2\cos\left(\frac{\iota\pi}{3j}\right), \\ \iota = 1,2,\ 4,5,\ 7,8,\ ,\ldots,\ 3j-8,3j-7,\ 3j-5,3j-4,\ 3j-2,3j-1. \end{gathered} \tag{6.13}$$

6.1.1. *Sums of Even Powers of Cosines.* Hence, the congruences (6.7) become identities in prime p and integers $j > 0$ and $m \geq 0$:

$$\begin{aligned} 4^{p^m}\sum_{k=1}^{j}\cos^{2p^m}\left(\frac{(3k-2)\pi}{3j}\right) &\equiv 2j \pmod p \quad (j>1), \\ 16^{p^m}\sum_{k=1}^{j}\cos^{4p^m}\left(\frac{(3k-2)\pi}{3j}\right) &\equiv 6j \pmod p \quad (j>2), \\ 64^{p^m}\sum_{k=1}^{j}\cos^{6p^m}\left(\frac{(3k-2)\pi}{3j}\right) &\equiv 20j \pmod p \quad (j>3), \end{aligned} \tag{6.14}$$

et cetera.

For example, with $j = 5,\ p = 11$ and $m = 1$,

$$\frac{1}{p}\left[16^{p^m}\sum_{k=1}^{j}\cos^{4p^m}\left(\frac{(3k-2)\pi}{3j}\right) - 6j\right] = 634877783325{\cdot}000\ . \tag{6.15}$$

6.1.2. *Sums of Even Powers of Secants.* Substituting (6.13) in (6.9) with $n = r = 1$ and even $j = 2i$, for which $[(-1)^j 2r + n] = 3$, we get identities in prime p and integers $i > 0$ and

$m \geq 0$:

$$\left(\frac{3}{4}\right)^{p^m} \sum_{k=1}^{2i} \sec^{2p^m}\left(\frac{(3k-2)\pi}{6i}\right) \equiv 4i^2 \pmod{p},$$

$$\left(\frac{3}{4}\right)^{2p^m} \sum_{k=1}^{2i} \sec^{4p^m}\left(\frac{(3k-2)\pi}{6i}\right) \equiv 2i^2(4i^2+1) \pmod{p},$$

$$\left(\frac{3}{4}\right)^{3p^m} \sum_{k=1}^{2i} \sec^{6p^m}\left(\frac{(3k-2)\pi}{6i}\right) \equiv \frac{2}{5}i^2\left(52i^4+15i^2+3\right) \pmod{p}, \tag{6.16}$$

et cetera.

For example, with $i=3,\ p=3$ and $m=1$,

$$\frac{1}{p}\left[\left(\frac{3}{4}\right)^{p^m} \sum_{k=1}^{2i} \sec^{2p^m}\left(\frac{(3k-2)\pi}{6i}\right) - 4i^2\right] = 1214417894460{\cdot}000\ . \tag{6.17}$$

Substituting (6.13) in (6.8) with $n=r=1$ and odd $j=2i+1$, for which $[(-1)^j 2r+n] = -1$, we get identities in prime p and integers $i \geq 0$ and $m \geq 0$:

$$\left(\frac{1}{4}\right)^{p^m} \sum_{k=1}^{2i+1} \sec^{2p^m}\left(\frac{(3k-2)\pi}{6i+3}\right) \equiv (2i+1)^2 \pmod{p},$$

$$\left(\frac{1}{16}\right)^{p^m} \sum_{k=1}^{2i+1} \sec^{4p^m}\left(\frac{(3k-2)\pi}{6i+3}\right) \equiv \frac{1}{6}(2i+1)^2\left(5(2i+1)^2+1\right) \pmod{p},$$

$$\left(\frac{1}{64}\right)^{p^m} \sum_{k=1}^{2i+1} \sec^{6p^m}\left(\frac{(3k-2)\pi}{6i+3}\right) \equiv$$

$$\frac{1}{120}(2i+1)^2\left(91(2i+1)^4+25(2i+1)^2+4\right) \pmod{p}, \quad et\ cetera. \tag{6.18}$$

For example, with $i=3,\ p=3$ and $m=1$,

$$\frac{1}{p}\Bigg[\left(\frac{1}{64}\right)^{p^m} \sum_{k=1}^{2i+1} \sec^{6p^m}\left(\frac{(3k-2)\pi}{6i+3}\right) - \frac{1}{120}(2i+1)^2\left(91(2i+1)^4+25(2i+1)^2+4\right)\Bigg] = 240668556344496{\cdot}001\ . \tag{6.19}$$

6.2. **Roots of** $C_{2j}(x) - 1 = 0$**.** Here $c=1,\ n=-1,\ r=1$ and $\beta=\frac{2}{3}$, so that the even equation $C_{2j}(x)-1=0$ has roots (in $\pm$ pairs), which are given in decreasing order (cf. (5.9) and (5.10)), as

$$\alpha = 2\cos\left(\frac{(2k-1\mp\frac{2}{3})\pi}{2j}\right) = 2\cos\left(\frac{(6k-3\mp 2)\pi}{6j}\right)$$
$$= 2\cos\left(\frac{\iota\pi}{6j}\right), \qquad \iota = 1,\ 5, 7,\ 11, 13,\ ,\ldots$$
$$\ldots\ ,\ 6j-13, 6j-11,\ 6j-7, 6j-5,\ 6j-1. \tag{6.20}$$

6.2.1. *Sums of Even Powers of Cosines.* Hence, the congruences (6.7) become identities in prime p and integers $j > 0$ and $m \geq 0$:

$$4^{p^m} \sum_{k=1}^{j} \cos^{2p^m}\left(\frac{(6k-5)\pi}{6j}\right) \equiv 2j \pmod{p} \quad (j > 1),$$

$$16^{p^m} \sum_{k=1}^{j} \cos^{4p^m}\left(\frac{(6k-5)\pi}{6j}\right) \equiv 6j \pmod{p} \quad (j > 2),$$

$$64^{p^m} \sum_{k=1}^{j} \cos^{6p^m}\left(\frac{(6k-5)\pi}{6j}\right) \equiv 20j \pmod{p} \quad (j > 3), \tag{6.21}$$

et cetera.

For example, with $j = 5,\ p = 7$ and $m = 1$,

$$\frac{1}{p}\left[64^{p^m} \sum_{k=1}^{j} \cos^{6p^m}\left(\frac{(6k-5)\pi}{6j}\right) - 20j\right] = 500340387880{\cdot}000\ . \tag{6.22}$$

6.2.2. *Sums of Even Powers of Secants.* Substituting (6.20) in (6.9) with $n = -1,\ r = 1$ and even $j = 2i$, for which $[(-1)^j 2r + n] = 1$, we get identities in prime p and integers $i > 0$ and $m \geq 0$:

$$\left(\frac{1}{4}\right)^{p^m} \sum_{k=1}^{2i} \sec^{2p^m}\left(\frac{(6k-5)\pi}{12i}\right) \equiv 4i^2 \pmod{p},$$

$$\left(\frac{1}{16}\right)^{p^m} \sum_{k=1}^{2i} \sec^{4p^m}\left(\frac{(6k-5)\pi}{12i}\right) \equiv \frac{2}{3}i^2(20i^2+1) \pmod{p},$$

$$\left(\frac{1}{64}\right)^{p^m} \sum_{k=1}^{2i} \sec^{6p^m}\left(\frac{(6k-5)\pi}{12i}\right) \equiv \frac{2}{15}i^2\left(364i^4+25i^2+1\right) \pmod{p}, \tag{6.23}$$

et cetera.

For example, with $i = 2,\ p = 5$ and $m = 1$,

$$\frac{1}{p}\left[\left(\frac{1}{16}\right)^{p^m} \sum_{k=1}^{2i} \sec^{4p^m}\left(\frac{(6k-5)\pi}{12i}\right) - \frac{2}{3}i^2(20i^2+1)\right] = 92571373536{\cdot}000\ . \tag{6.24}$$

Substituting (6.20) in (6.9) with $n = -1,\ r = 1$ and odd $j = 2i+1$, for which $[(-1)^j 2r + n] = -3$, we get identities in prime p and integers $i \geq 0$ and $m \geq 0$:

$$\left(\frac{3}{4}\right)^{p^m} \sum_{k=1}^{2i+1} \sec^{2p^m}\left(\frac{(6k-5)\pi}{12i+6}\right) \equiv (2i+1)^2 \pmod{p},$$

$$\left(\frac{3}{4}\right)^{2p^m} \sum_{k=1}^{2i+1} \sec^{4p^m}\left(\frac{(6k-5)\pi}{12i+6}\right) \equiv (2i+1)^2\left(2i^2+2i+1\right) \pmod{p},$$

$$\left(\frac{3}{4}\right)^{3p^m} \sum_{k=1}^{2i+1} \sec^{6p^m}\left(\frac{(6k-5)\pi}{12i+6}\right) \equiv$$

$$\frac{1}{40}(2i+1)^2\left(13(2i+1)^4+15(2i+1)^2+12\right) \pmod{p}, \tag{6.25}$$

et cetera.

For example, with $i = 3,\ p = 3$ and $m = 2$,

$$\begin{aligned}&\frac{1}{p}\left[\left(\frac{3}{4}\right)^{p^m}\sum_{k=1}^{2i+1}\sec^{2p^m}\left(\frac{(6k-5)\pi}{12i+6}\right) \ - \ (2i+1)^2\right]\\ (6.26)\qquad & \qquad\qquad = \ 19004748528710{\cdot}000\ .\end{aligned}$$

7. ZEROS OF $C_d(x) \pm C_e(x)$

The significant trigonometric congruences (each in triples) which have been derived are (3.22), (4.13), (5.18), (5.24), (5.29), (6.7), (6.14) and (6.21) for integer sums of powers of cosines, and (3.26), (3.28), (4.17), 4.19), (5.20), (5.26),(5.31), (6.9), (6.16), (6.18), (6.23) and (6.25) for integer sums of powers of secants, each followed by a numerical example.

For integers $d > e \geq 0$, with $x = 2\ \cos\vartheta$,

$$\begin{aligned}C_d(x) + C_e(x) \ &= \ 2(\cos(d\vartheta) + \cos(e\vartheta))\\ (7.1)\qquad &= \ 4\ \cos\left(\frac{1}{2}(d+e)\vartheta\right)\cos\left(\frac{1}{2}(d-e)\vartheta\right),\end{aligned}$$

so that S_k is the sum of kth powers of cosines corresponding to the zeros of $\cos\left(\frac{1}{2}(d+e)\vartheta\right)$, plus those corresponding to the zeros of $\cos\left(\frac{1}{2}(d-e)\vartheta\right)$.

If d and e are both odd or both even, then

$$(7.2)\qquad C_d(x) + C_e(x) \ = \ 4\ \cos(f\vartheta)\cos(g\vartheta) \ = \ C_f(x)C_g(x)$$

for positive integers $f = (d+e) \div 2$ and $g = (d-e) \div 2$, and the zeros of $C_k(x)$ for integer k have been treated in (3.8)–(4.20).

Otherwise, with d and e of different parity,

$$(7.3)\qquad C_d(x) + C_e(x) \ = \ 4\ \cos\left(\left(h - \frac{1}{2}\right)\vartheta\right)\cos\left(\left(i - \frac{1}{2}\right)\vartheta\right)$$

for positive integers $h = (d+e+1) \div 2$ and $i = (d-e+1) \div 2$, and the zeros of $\cos\left(\left(n - \frac{1}{2}\right)\vartheta\right)$ for integer n have been treated in (5.3) and (5.4) (with $\gamma = 0$ in (5.5)).

Likewise,

$$\begin{aligned}C_d(x) - C_e(x) \ &= \ 2(\cos(d\vartheta) - \cos(e\vartheta))\\ (7.4)\qquad &= \ -4\ \sin\left(\frac{1}{2}(d+e)\vartheta\right)\sin\left(\frac{1}{2}(d-e)\vartheta\right),\end{aligned}$$

so that S_k is the sum of kth powers of cosines corresponding to the zeros of $\sin\left(\frac{1}{2}(d+e)\vartheta\right)$, plus those corresponding to the zeros of $\sin\left(\frac{1}{2}(d-e)\vartheta\right)$.

If d and e are both odd or both even, then

$$(7.5)\qquad C_d(x) - C_e(x) \ = \ -4\ \sin(f\vartheta)\sin(g\vartheta)$$

for positive integers f and g, and the zeros of sin($k\vartheta$) for integer k have been treated in (4.10).

Otherwise, if d is even and e is odd then

$$(7.6)\qquad C_d(x) - C_e(x) \ = \ C_d(-x) + C_e(-x);$$

but if d is odd and e is even then

$$(7.7)\qquad C_d(x) - C_e(x) \ = \ -\big(C_d(-x) + C_e(-x)\big).$$

Thus, for d and e of opposite parity, the zeros of $C_d(x) - C_e(x)$ are -1 times the zeros of $C_d(x) + C_e(x)$, as in (7.3), Thus, $C_d(x) \pm C_e(x)$ is a polynomial of degree d with integer coefficients whose zeros are known explicitly, but the congruences for sums of powers of its zeros do not yield any new result.

And similarly for $S_d(x) \pm S_e(x)$, with Chebyshev polynomials of the second type.

The transformation of $T_d(x)$ to a monic polynomial, as in Corollary 2.2, involves scaling with $b = 2$ in (2.33), to get $C_d(z)$ as in (3.11). Accordingly, the relation (2.39) for σ_p for the non–monic polynomial T_d just reproduces the congruences (3.22) for the monic polynomial C_d.

Application of (2.49) to (3.4) yields congruences which can be obtained by multiplying (3.26) by 4^{p^m} (or 16^{p^m}, 64^{p^m},...), bearing in mind that $4^{p^m} \equiv 4$ (mod p) (or $16^{p^m} \equiv 16$ (mod p) *et cetera*), by Fermat's Little Theorem. Likewise, application of (2.49) to (3.5) yields congruences which can be obtained by multiplying (3.28) by 2^{p^m} (or 4^{p^m}, 8^{p^m}, *et cetera*). The inverse inferences are not so straightforward.

8. Tangents of Multiple Angles

For complex $z = x + iy$ and positive integer n.

$$z^n = (x+iy)^n = \sum_{j=0}^{n} \binom{n}{j} i^j x^{n-j} y^j = H_n^{(0)}(x,y) + iH_n^{(1)}(x,y) \,, \tag{8.1}$$

where the real and imaginary parts of z^n are given by the harmonic polynomials (Lyusternik *et alia* [7], Appendix 1 §**2.1**):

$$H_n^{(0)}(x,y) = \sum_{k=0}^{n\div 2} (-1)^k \binom{n}{2k} x^{n-2k} y^{2k} \tag{8.2}$$

$$H_n^{(1)}(x,y) = \sum_{k=0}^{(n-1)\div 2} (-1)^k \binom{n}{2k+1} x^{n-2k-1} y^{2k+1} \,. \tag{8.3}$$

For all $\psi \in \mathbb{C}$,

$$\cos\psi + i\sin\psi = e^{i\psi} \tag{8.4}$$

(by Cotes's Theorem); and hence (De Moivre's Theorem) for integer $n \geq 0$

$$\cos(n\psi) + i\sin(n\psi) = e^{in\psi} = (\cos\psi + i\sin\psi)^n. \tag{8.5}$$

Therefore

$$\begin{aligned} \cos(n\psi) + i\sin(n\psi) &= (\cos\psi + i\sin\psi)^n \\ &= H_n^{(0)}(\cos\psi, \sin\psi) + iH_n^{(1)}(\cos\psi, \sin\psi) \,. \end{aligned} \tag{8.6}$$

Equating real and imaginary parts (for real ψ), we get that

$$\cos(n\psi) = H_n^{(0)}(\cos\psi, \sin\psi), \quad \sin(n\psi) = H_n^{(1)}(\cos\psi, \sin\psi); \tag{8.7}$$

and hence

$$\tan(n\psi) = \frac{\sin(n\psi)}{\cos(n\psi)} = \frac{H_n^{(1)}(\cos\psi, \sin\psi)}{H_n^{(0)}(\cos\psi, \sin\psi)} \,. \tag{8.8}$$

Dividing numerator and denominator in (8.8) by $\cos^n \psi$, we get (Lyusternik *et alia* [7], p. 172) $\tan(n\psi)$ as a rational function of $\tan\,\psi$, with integer coefficients:

$$\tan(n\psi) \;=\; \frac{H_n^{(1)}(1, \tan\,\psi)}{H_n^{(0)}(1, \tan\,\psi)} \;=\; \frac{\sum_{k=0}^{(n-1)\div 2} \binom{n}{2k+1} (-1)^k \tan^{2k+1}\psi}{\sum_{k=0}^{n\div 2} \binom{n}{2k} (-1)^k \tan^{2k}\psi}\,. \tag{8.9}$$

Denote

$$t \;\overset{\text{def}}{=}\; \tan\,\psi, \qquad z \;\overset{\text{def}}{=}\; -t^2 \;=\; -\tan^2\psi. \tag{8.10}$$

Then, it follows from (8.8) that

$$\tan(n\psi) \;=\; g_n(\tan\,\psi), \tag{8.11}$$

where $g_n(t)$ is a rational function with positive integer coefficients

$$g_n(t) \;=\; \frac{H_n^{(1)}(1,t)}{H_n^{(0)}(1,t)} \;=\; t\,\frac{A_n(z)}{B_n(z)}, \tag{8.12}$$

(except that $A_0(t) = 0$), with :

$$A_n(z) \;=\; \sum_{k=0}^{(n-1)\div 2} \binom{n}{2k+1} z^k, \qquad B_n(z) \;=\; \sum_{k=0}^{n\div 2} \binom{n}{2k} z^k, \tag{8.13}$$

and $B_n(0) = 1$.

In more detail, for odd $n = 2j+1$:

$$\begin{aligned} g_{2j+1}(t) \;&=\; t\,\frac{A_{2j+1}(z)}{B_{2j+1}(z)} \\ &=\; \frac{t\left(2j+1+\binom{2j+1}{3}z+\binom{2j+1}{5}z^2+\cdots\binom{2j+1}{2}z^{j-1}+z^j\right)}{1+\binom{2j+1}{2}z+\binom{2j+1}{4}z^2+\cdots+\binom{2j+1}{3}z^{j-1}+(2j+1)z^j}, \end{aligned} \tag{8.14}$$

where the polynomials A_{2j+1} and B_{2j+1} are mutually reciprocal:

$$z^j A_{2j+1}\left(\frac{1}{z}\right) \;=\; B_{2j+1}(z), \qquad z^j B_{2j+1}\left(\frac{1}{z}\right) \;=\; A_{2j+1}(z) \tag{8.15}$$

for all $z \neq 0$. For even $n = 2j$:

$$\begin{aligned} g_{2j}(t) \;&=\; t\,\frac{A_{2j}(z)}{B_{2j}(z)} \\ &=\; t\,\frac{2j+\binom{2j}{3}z+\binom{2j}{5}z^2+\cdots+\binom{2j}{3}z^{j-3}+2j\,z^{j-1}}{1+\binom{2j}{2}z+\binom{2j}{4}z^2+\cdots+\binom{2j}{2}z^{j-1}+z^j}, \end{aligned} \tag{8.16}$$

where both A_{2j} and B_{2j} are self–reciprocal polynomials:

$$A_{2j}(z) \;=\; z^{j-1}A_{2j}\left(\frac{1}{z}\right), \qquad B_{2j}(z) \;=\; z^j B_{2j}\left(\frac{1}{z}\right), \tag{8.17}$$

for all $z \neq 0$.

For example,

$$g_0(t) = t\,\frac{0}{1} = 0\,, \qquad g_1(t) = t\,\frac{1}{1} = t\,,$$

$$g_2(t) = t\,\frac{2}{1+z}\,, \qquad g_3(t) = t\,\frac{3+z}{1+3z}\,,$$

$$g_4(t) = t\,\frac{4+4z}{1+6z+z^2}\,, \qquad g_5(t) = t\,\frac{5+10z+z^2}{1+10z+5z^2}\,, \tag{8.18}$$

$$g_6(t) = t\,\frac{6+20z+6z^2}{1+15z+15z^2+z^3}\,, \qquad g_7(t) = t\,\frac{7+35z+21z^2+z^3}{1+21z+35z^2+7z^3}\,,$$

$$g_8(t) = t\,\frac{8+56z+56z^2+8z^3}{1+28z+70z^2+28z^3+z^4}, \qquad et\ cetera.$$

For all integers $m \geq 0$ and $n \geq 0$ and complex t, with $\psi = \tan^{-1} t$

$$\begin{aligned} g_{m+n}(t) \;&=\; g_{m+n}(\tan\,\psi) \;=\; \tan((m+n)\psi) \;=\; \tan(m\psi + n\psi) \\ &= \frac{\tan(m\psi) + \tan(n\psi)}{1 \,-\, \tan(m\psi)\,\tan(n\psi)} \;=\; \frac{g_m(t) + g_n(t)}{1 - g_m(t)g_n(t)}. \end{aligned} \tag{8.19}$$

In particular,

$$g_{n+1}(t) \;=\; \frac{t + g_n(t)}{1 \,-\, t\,g_n(t)}. \tag{8.20}$$

This recurrence relation (8.20) could be used to define the sequence of rational functions $\{g_n\}$ by induction on n, starting with $g_0(t) = 0$.

8.1. **Even binomial coefficients.**

Theorem 8.1. *If $n > u > 0$, and $q \mid n$ but q and u are co–prime, then $q \mid \binom{n}{u}$*

Proof. For integer $k > 0$, the product of k consecutive integers is divisible by $k!$. Indeed, for $q \geq k$,

$$\frac{q(q-1)\dots(q-k+1)}{k!} \;=\; \binom{q}{k}, \tag{8.21}$$

which is an integer. For $-k < q < k$ those k consecutive integers include 0, so that $q(q-1)\dots(q-k+1) \;=\; k! \times 0$; and for $q \leq -k$ that product is $(-1)^k$ times the product of k consecutive positive integers, which is divisible by $k!$

Hence,

$$\binom{n}{u} \;=\; \frac{n(n-1)\dots(n-u+1)}{u!} \;=\; \frac{n(n-1)\dots(n-u+1)}{u(u-1)!} \;=\; \frac{na}{u}, \tag{8.22}$$

where $a = \binom{n-1}{u-1}$ is an integer. Now, $n = bq$ for integer b, and therefore

$$u\binom{n}{u} \;=\; abq, \tag{8.23}$$

so that q divides $u\binom{n}{u}$.

By hypothesis, $\gcd(u, q) = 1$, and therefore q does divide $\binom{n}{u}$. ∎

Corollary 8.2. *For integers $n > u > 0$, $\binom{n}{u}$ is divisible by $n \div gcd(n, u)$.*

Corollary 8.3. *If p is prime and $\mu > 0$ and p^μ divides n but p does not divide u, then p^μ divides the binomial coefficient $\binom{n}{u}$.*

Corollary 8.4. *If $2^\mu \mid n$ with $\mu > 0$ and u is odd, then $2^\mu \mid \binom{n}{u}$.*

Thus, if n is even and u is odd, then $\binom{n}{u}$ is even.

Hence, with $j = 2^\beta(2\alpha+1)$ where $\alpha, \beta \geq 0$ (cf. (8.16)), it follows from Corollary 8.4 that every coefficient in the polynomial A_{2j} (of degree $j-1$) is divisible by $2^{\beta+1}$. Therefore the polynomial

$$\begin{aligned}\left(\frac{1}{2}\right)^{\beta+1} A_{2j}(z) \;&=\; 2\alpha+1+\left(\frac{1}{2}\right)^{\beta+1}\binom{2j}{3}z+\left(\frac{1}{2}\right)^{\beta+1}\binom{2j}{5}z^2+\cdots \\ &\cdots+\left(\frac{1}{2}\right)^{\beta+1}\binom{2j}{5}z^{n-3}+\left(\frac{1}{2}\right)^{\beta+1}\binom{2j}{3}z^{j-2}+(2\alpha+1)z^{j-1}\end{aligned} \tag{8.24}$$

has integer coefficients. Thus, (8.18) may be rewritten as:

$$\begin{aligned}
&g_0(t) = t\,\frac{0}{1} = 0\,, && g_1(t) = t\,\frac{1}{1} = t\,,\\
&g_2(t) = 2t\,\frac{1}{1+z}\,, && g_3(t) = t\,\frac{3+z}{1+3z}\,,\\
&g_4(t) = 4t\,\frac{1+z}{1+6z+z^2}\,, && g_5(t) = t\,\frac{5+10z+z^2}{1+10z+5z^2}\,,\\
&g_6(t) = 2t\,\frac{3+10z+3z^2}{1+15z+15z^2+z^3}\,, && g_7(t) = t\,\frac{7+35z+21z^2+z^3}{1+21z+35z^2+7z^3}\,,\\
&g_8(t) = 8t\,\frac{1+7z+7z^2+z^3}{1+28z+70z^2+28z^3+z^4}\,, && \textit{et cetera.}
\end{aligned} \tag{8.25}$$

9. ZEROS OF $\tan(n\psi)$

The equation

$$\tan(n\psi) \;=\; 0 \tag{9.1}$$

has the solution

$$n\psi \;=\; k\pi \tag{9.2}$$

for integer k, and hence

$$\psi \;=\; \frac{k\pi}{n}. \tag{9.3}$$

From (8.12) and (8.13), $\tan(n\psi) = 0$ if and only if either $t = 0$, or $A_n(z) = 0$ or else $B_n(z) = \infty$ (which requires $t = \infty$). If $t = \infty$ then $\psi = (h+\frac{1}{2})\pi$ for integer h, so that $\tan(2j\psi) = \tan((2h+1)j\pi) = 0$ and $\tan((2j+1)\psi) = \tan\left((2jh+j+h+\frac{1}{2})\pi\right) = \infty$.

Hence, the roots of the rational equation $g_n(t) = 0$ are of the forms $t = 0$ and $t = \tan(k\pi/n)$ (the zeros of A_n); and for even $n = 2j$ there is also the root $t = \infty$.

Therefore, the polynomial equation $A_{2j+1}(z) = 0$ has j distinct roots, which are given by

$$u_k \;=\; -\tan^2\!\left(\frac{k\pi}{2j+1}\right) \qquad (k = 1, 2, \ldots, j), \tag{9.4}$$

with $0 > u_1 > u_2 > \cdots > u_{j-1} > u_j$; and the polynomial equation $A_{2j}(z) = 0$ has $j-1$ distinct roots, which are given by

$$z_k \;=\; -\tan^2\left(\frac{k\pi}{2j}\right) \qquad (k = 1, 2, \ldots, j-1), \tag{9.5}$$

with $0 > z_1 > z_2 > \cdots > z_{j-2} > z_{j-1}$.

9.1. **Positive Powers of Zeros of A_{2j+1}.** Apply Newton's Rules to the zeros u_k of the monic polynomial A_{2j+1} in (8.14) and it follows that, for all integers $j > 0$ and $q \geq 0$, $\sum_{k=1}^{j} \tan^{2q}(k\pi/(2j+1))$ has integer value.

For example, with $j = 3$ and $q = 10$,

$$\sum_{k=1}^{j} \tan^{2q}\left(\frac{k\pi}{2j+1}\right) \;=\; 6792546291251{\cdot}001\,. \tag{9.6}$$

For all integers $j > 0,\ f \geq 0$ and prime p, the integer congruences (2.23), for positive powers of zeros of the monic polynomial A_{2j+1} in (8.14), yield the congruence identities:

$$\begin{aligned}
\sum_{k=1}^{j} u_k^{p^f} \;&\equiv\; -\binom{2j+1}{2} \qquad (\mathrm{mod}\ p),\\
\sum_{k=1}^{j} u_k^{2p^f} \;&\equiv\; \binom{2j+1}{2}^2 - 2\binom{2j+1}{4} \qquad (\mathrm{mod}\ p),\\
\sum_{k=1}^{j} u_k^{3p^f} \;&\equiv\; -\binom{2j+1}{2}^3 + 3\binom{2j+1}{2}\binom{2j+1}{4} - 3\binom{2j+1}{6}\\
&\qquad\qquad (\mathrm{mod}\ p), \quad \textit{et cetera}.
\end{aligned} \tag{9.7}$$

These simplify to give the following congruence identities:

$$\begin{aligned}
\sum_{k=1}^{j} \tan^{2p^f}\left(\frac{k\pi}{2j+1}\right) \;&\equiv\; j(2j+1) \qquad (\mathrm{mod}\ p),\\
\sum_{k=1}^{j} \tan^{4p^f}\left(\frac{k\pi}{2j+1}\right) \;&\equiv\; \frac{1}{3}j(2j+1)(4j^2+6j-1) \qquad (\mathrm{mod}\ p),\\
\sum_{k=1}^{j} \tan^{6p^f}\left(\frac{k\pi}{2j+1}\right) \;&\equiv\; \frac{1}{15}j(2j+1)(32j^4+80j^3+40j^2-20j+3)\\
&\qquad\qquad (\mathrm{mod}\ p), \quad \textit{et cetera}.
\end{aligned} \tag{9.8}$$

For example, with $j = 3,\ p = 3$ and $f = 2$,

$$\frac{1}{p}\left[\sum_{k=1}^{j} \tan^{2p^f}\left(\frac{k\pi}{2j+1}\right) - j(2j+1)\right] \;=\; 117952755648{\cdot}000\,. \tag{9.9}$$

9.2. **Positive Powers of Zeros of A_{2j}.** Apply Newton's Rules to the zeros z_k of $\left(\frac{1}{2}\right)^{\beta+1} A_{2j}$ in (2.38), and it follows that, for all positive integers q and $j = (2\alpha+1)2^\beta$ (with $\alpha, \beta \geq 0$), $(2\alpha+1)^q \sum_{k=1}^{j-1}(k\pi/(2j))$ has integer value.

For example, with $j = 40,\ \alpha = 2,\ \beta = 3,\ q = 3$,

$$(2\alpha+1)^q \sum_{k=1}^{j-1} \tan^{2q}\left(\frac{k\pi}{2j}\right) = 34561553975{\cdot}000\,. \tag{9.10}$$

In particular, for positive integers β, q (with $\alpha = 0$), $\sum_{j=1}^{2^\beta-1} \tan^{2q}\left(k\pi/2^{\beta+1}\right)$ has integer value. For example, with $\beta = 5$ and $q = 4$,

$$\sum_{j=1}^{2^\beta-1} \tan^{2q}\left(\frac{k\pi}{2^{(\beta+1)}}\right) = 29592340127{\cdot}000\,. \tag{9.11}$$

For all positive integers $j = 2^\beta(2\alpha+1)$, with $\alpha, \beta, f \geq 0$ and prime p, the integer congruences (2.23), for positive powers of zeros of the polynomial $\left(\frac{1}{2}\right)^{\beta+1} A_{2j}$ in (8.16), yield the congruence identities:

$$\begin{aligned}
\sum_{k=1}^{j-1}((2\alpha+1)z_k)^{p^f} &\equiv -\left(\frac{1}{2}\right)^{\beta+1}\binom{2j}{3} \qquad (\mathrm{mod}\ p),\\
\sum_{k=1}^{j-1}((2\alpha+1)z_k)^{2p^f} &\equiv \left(\frac{1}{2}\right)^{2\beta+2}\binom{2j}{3}^2 - 2(2\alpha+1)\left(\frac{1}{2}\right)^{\beta+1}\binom{2j}{5} \qquad (\mathrm{mod}\ p),\\
\sum_{k=1}^{j-1}((2\alpha+1)z_k)^{3p^f} &\equiv -\left(\frac{1}{2}\right)^{3\beta+3}\binom{2j}{3}^3 + 3(2\alpha+1)\left(\frac{1}{2}\right)^{2\beta+2}\binom{2j}{3}\binom{2j}{5}\\
&\quad -3(2\alpha+1)^2\left(\frac{1}{2}\right)^{\beta+1}\binom{2j}{7} \qquad (\mathrm{mod}\ p), \quad \textit{et cetera}.
\end{aligned} \tag{9.12}$$

For all positive integers $j = 2^\beta(2\alpha+1)$, with $\alpha, \beta, f \geq 0$ and prime p, these simplify to give the identities:

$$\begin{aligned}
(2\alpha+1)^{p^f}\sum_{k=1}^{j-1}\tan^{2p^f}\left(\frac{k\pi}{2j}\right) &\equiv \frac{1}{3}(2\alpha+1)(j-1)(2j-1) \qquad (\mathrm{mod}\ p),\\
(2\alpha+1)^{2p^f}\sum_{k=1}^{j-1}\tan^{4p^f}\left(\frac{k\pi}{2j}\right) &\equiv \frac{1}{45}(2\alpha+1)^2(j-1)(2j-1)(4j^2+6j-13)\\
&\qquad (\mathrm{mod}\ p),\\
(2\alpha+1)^{3p^f}\sum_{k=1}^{j-1}\tan^{6p^f}\left(\frac{k\pi}{2j}\right) &\equiv \frac{1}{945}(2\alpha+1)^3(j-1)(2j-1)\times\\
&\quad \times(32j^4+48j^3-112j^2-192j+251)\\
&\qquad (\mathrm{mod}\ p), \quad \textit{et cetera}.
\end{aligned} \tag{9.13}$$

For example, with $j = 6,\ \alpha = 1,\ \beta = 1,\ f = 1$ and prime $p = 3$,

$$\frac{1}{p}\left[(2\alpha+1)^{2p^f}\sum_{k=1}^{j-1}\tan^{4p^f}\left(\frac{k\pi}{2j}\right) - \frac{1}{45}(2\alpha+1)^2(j-1)(2j-1)(4j^2+6j-13)\right]$$

$$= \ 1774271664{\cdot}000\ . \tag{9.14}$$

With $\alpha = 0$, we get that for all integers $\beta > 0,\ f \geq 0$ and prime p,

$$\begin{aligned}
\sum_{k=1}^{2^\beta-1} \tan^{2p^f}\!\left(\frac{k\pi}{2^{(\beta+1)}}\right) &\equiv \frac{1}{3}(2^\beta-1)(2^{\beta+1}-1) \qquad \pmod p,\\
\sum_{k=1}^{2^\beta-1} \tan^{4p^f}\!\left(\frac{k\pi}{2^{(\beta+1)}}\right) &\equiv \frac{1}{45}(2^\beta-1)(2^{\beta+1}-1)(2^{2\beta+2}+3\times 2^{\beta+1}-13)\\
&\qquad\qquad \pmod p,\\
\sum_{k=1}^{2^\beta-1} \tan^{6p^f}\!\left(\frac{k\pi}{2^{(\beta+1)}}\right) &\equiv \frac{1}{945}(2^\beta-1)(2^{\beta+1}-1)\times\\
&\quad \times(2^{4\beta+5}+3\times 2^{3\beta+4}-7\times 2^{2\beta+4}-3\times 2^{\beta+6}+251)\\
&\qquad \pmod p, \quad \textit{et cetera}.
\end{aligned} \tag{9.15}$$

For example, with $\beta = 2,\ f = 2$ and prime $p = 3$,

$$\frac{1}{p}\left[\sum_{k=1}^{2^\beta-1} \tan^{p^f}\!\left(\frac{k\pi}{2^{(\beta+1)}}\right) - \frac{1}{45}(2^\beta-1)(2^{\beta+1}-1)(2^{2\beta+2}+3\times 2^{\beta+1}-13)\right]$$
$$= \ 20081836064256{\cdot}001\ . \tag{9.16}$$

Newton's Rules for negative powers, applied to the self-reciprocal polynomial A_{2j}, reproduce (9.14) and (9.15).

9.3. **Negative Powers of Zeros of** A_{2j+1}**.** Apply Newton's Rules to the zeros $1/u_k$ of the polynomial (i.e. B_{2j+1}) which is reciprocal to A_{2j+1}. It follows that, for all integers $j \geq 0$ and $q \geq 0$,
$(2j+1)^q \sum_{k=1}^{j} \cot^{2q}(k\pi/(2j+1))$ has integer value.

For example, with $j = 4$ and $q = 4$,

$$(2j+1)^q \sum_{k=1}^{j} \cot^{2q}\!\left(\frac{k\pi}{2j+1}\right) \ = \ 36269685127325{\cdot}999\ . \tag{9.17}$$

For all integers $j \geq 0,\ f \geq 0$ and prime p,

$$\begin{aligned}
(2j+1)^{p^f} \sum_{k=1}^{j} \cot^{2p^f}\!\left(\frac{k\pi}{2j+1}\right) &\equiv \frac{1}{3}j(2j+1)(2j-1) \qquad \pmod p,\\
(2j+1)^{2p^f} \sum_{k=1}^{j} \cot^{4p^f}\!\left(\frac{k\pi}{2j+1}\right) &\equiv \frac{1}{45}j(2j+1)^2(2j-1)(4j^2+10j-9)\\
&\qquad\qquad \pmod p,\\
(2j+1)^{3p^f} \sum_{k=1}^{j} \cot^{6p^f}\!\left(\frac{k\pi}{2j+1}\right) &\equiv \frac{1}{945}j(2j+1)^3(2j-1)\times\\
&\quad \times(32j^4+112j^3+8j^2-252j+135)\\
&\qquad \pmod p, \quad \textit{et cetera}.
\end{aligned} \tag{9.18}$$

For all integers $j \geq 0,\ f \geq 0$ and prime p the integer congruences (2.48), for negative powers of the zeros u_k of the integer polynomial A_{2j+1}, yield the congruence identities:

$$\begin{aligned}
\sum_{k=1}^{j}\left(\frac{2j+1}{v_k}\right)^{pf} &\equiv -\binom{2j+1}{3} \pmod{p},\\
\sum_{k=1}^{j}\left(\frac{2j+1}{v_k}\right)^{2pf} &\equiv \binom{2j+1}{3}^2 - 2(2j+1)\binom{2j+1}{5} \pmod{p},\\
\sum_{k=1}^{j}\left(\frac{2j+1}{v_k}\right)^{3pf} &\equiv -\binom{2j+1}{3}^3 + 3(2j+1)\binom{2j+1}{3}\binom{2j+1}{5}\\
&\quad -3(2j+1)^2\binom{2j+1}{7} \pmod{p}, \quad \textit{et cetera}.
\end{aligned} \tag{9.19}$$

These simplify to give the following congruence identities:

$$\begin{aligned}
(2j+1)^{pf}\sum_{k=1}^{j}\cot^{2pf}\left(\frac{k\pi}{2j+1}\right) &\equiv \frac{1}{3}j(2j+1)(2j-1) \pmod{p},\\
(2j+1)^{2pf}\sum_{k=1}^{j}\cot^{4pf}\left(\frac{k\pi}{2j+1}\right) &\equiv \frac{1}{45}j(2j+1)^2(2j-1)(4j^2+10j-9)\\
&\qquad \pmod{p},\\
(2j+1)^{3pf}\sum_{k=1}^{j}\cot^{6pf}\left(\frac{k\pi}{2j+1}\right) &\equiv \frac{1}{945}j(2j+1)^3(2j-1)\times\\
&\quad \times(32j^4+112j^3+8j^2-252j+135)\\
&\qquad \pmod{p}, \quad \textit{et cetera}.
\end{aligned} \tag{9.20}$$

For example, with $j=2,\ p=7$ and $f=1$,

$$\frac{1}{p}\left[(2j+1)^{2pf}\sum_{k=1}^{j}\cot^{4pf}\left(\frac{k\pi}{2j+1}\right) - \frac{1}{45}j(2j+1)^2(2j-1)(4j^2+10j-9)\right]$$

$$= 6686100200879{\cdot}999\,. \tag{9.21}$$

9.4. **Zeros of** $\cot(n\psi)$**.** The equation

$$\cot(n\psi) = 0 \tag{9.22}$$

has the solution

$$n\psi = \left(k-\tfrac{1}{2}\right)\pi \tag{9.23}$$

for integer k, and hence

$$\psi = \frac{(2k-1)\pi}{2n}, \tag{9.24}$$

From (8.13) and (2.9),

$$\cot(n\psi) = \frac{B_n(z)}{t\,A_n(z)}, \tag{9.25}$$

and hence $\cot(n\psi) = 0$ if and only if either $B_n(z) = 0$, or else $t = \infty$, or else $A_n(z) = \infty$ (which requires $t = \infty$). If $t = \infty$ then $\psi = (h + \frac{1}{2})\pi$ for integer h, so that $\cot(2j\psi) = \cot((2h+1)j\pi) = \infty$ and $\cot((2j+1)\psi) = \cot\left(\left(2jh + j + h + \frac{1}{2}\right)\pi\right) = 0$.

Hence, the roots of the rational equation $1/g_n(t) = 0$ are of the form $t = \tan((2k-1)\pi/(2n))$ (the zeros of the polynomial B_n); and for odd $n = 2j+1$ there is also the root $t = \infty$.

Therefore, the polynomial equation $B_{2j}(z) = 0$ has j distinct roots, which are given by

$$v_k = -\tan^2\left(\frac{(2k-1)\pi}{4j}\right) \qquad (k = 1, \ldots, j) \tag{9.26}$$

with $0 > v_1 > v_2 > \cdots > v_{j-1} > v_j$; and the polynomial equation $B_{2j+1}(z) = 0$ has j distinct roots, which are given by

$$w_k = -\tan^2\left(\frac{(2k-1)\pi}{4j+2}\right) \qquad (k = 1, \ldots, j) \tag{9.27}$$

with $0 > w_1 > w_2 > \cdots > w_{j-1} > w_j$.

9.4.1. *Positive Powers of Zeros of* B_{2j}. Apply Newton's Rules to the zeros v_k of the monic polynomial B_{2j} in (8.16), and it follows that for integers $j > 0$ and $q \geq 0$, $\sum_{k=1}^{j} \tan^{2q}((2k-1)\pi/(4j))$ has integer value.

For example, with $j = 4$ and $q = 9$,

$$\sum_{k=1}^{j} \tan^{2q}\left(\frac{(2k-1)\pi}{4j}\right) = 4208117405212{\cdot}000\,. \tag{9.28}$$

The zeros of B_{2j+1} are the inverses of the zeros of A_{2j+1}, and those inverses have been dealt with in (9.20).

For all integers $j > 0$, $f \geq 0$ and prime p, the integer congruences (2.23) for positive powers of the zeros v_k of the monic polynomial B_{2j}, yield the congruence identities:

$$\begin{aligned}
\sum_{k=1}^{j} v_k^{p^f} &\equiv -\binom{2j}{2} \qquad \pmod p, \\
\sum_{k=1}^{j} v_k^{2p^f} &\equiv \binom{2j}{2}^2 - 2\binom{2j}{4} \qquad \pmod p, \\
\sum_{k=1}^{j} v_k^{3p^f} &\equiv -\binom{2j}{2}^3 + 3\binom{2j}{4}\binom{2j}{2} - 3\binom{2j}{6} \qquad \pmod p, \quad \textit{et cetera}.
\end{aligned} \tag{9.29}$$

These simplify to give the following congruence identities:

$$\begin{aligned}
\sum_{k=1}^{j} \tan^{2p^f}\left(\frac{(2k-1)\pi}{4j}\right) &\equiv j(2j-1) \qquad \pmod p, \\
\sum_{k=1}^{j} \tan^{4p^f}\left(\frac{(2k-1)\pi}{4j}\right) &\equiv \frac{1}{3}j(2j-1)(4j^2+2j-3) \qquad \pmod p, \\
\sum_{k=1}^{j} \tan^{6p^f}\left(\frac{(2k-1)\pi}{4j}\right) &\equiv \frac{1}{15}j(2j-1)(32j^4+16j^3-32j^2-16j+15) \\
&\qquad \pmod p, \quad \textit{et cetera}.
\end{aligned} \tag{9.30}$$

For example, with $j = 2,\ p = 5$ and $f = 1$,

$$\frac{1}{p}\left[\sum_{k=1}^{j}\tan^{6p^f}\!\left(\frac{(2k-1)\pi}{4j}\right) - \frac{1}{15}j(2j-1)(32j^4+16j^3-32j^2-16j+15)\right]$$

(9.31) $$= 60855600960{\cdot}000\ .$$

The congruences (2.40) for negative powers, applied to the self-reciprocal polynomial B_{2j}, yield exactly the same congruences as in (9.30). The zeros of B_{2j+1} are the inverses of the zeros of A_{2j+1}, which have been dealt with in (9.20).

10. ZEROS OF $\tan(n\psi) - c$

For real t, denote the principal value of $\tan^{-1} t$ by $\arctan\ t$,with

(10.1) $$-\tfrac{1}{2}\pi < \arctan\ t < \tfrac{1}{2}\pi.$$

For real c, the equation

(10.2) $$\tan(n\psi) - c \ = \ 0$$

has the solution

(10.3) $$n\psi \ = \ k\pi + \zeta,$$

for integer k, where

(10.4) $$\zeta \ = \ \arctan\ c.$$

Denote

(10.5) $$\lambda = \frac{\zeta}{\pi},$$

so that

(10.6) $$-\tfrac{1}{2} < \lambda < \tfrac{1}{2},$$

and

(10.7) $$n\psi \ = \ (k+\lambda)\pi.$$

Thus, the equation (10.2) has the general solution

(10.8) $$\psi \ = \ \frac{k+\lambda}{n}\pi.$$

Thus, for all ψ satisfying the equation (10.2), $t = \tan\ \psi$ has one of the n distinct values

(10.9) $$t_k \ = \ \tan\left(\frac{(k+\lambda)\pi}{n}\right), \qquad (k = 1, 2, \ldots, n)$$

Reduction to Polynomial Equations

The function tan is an odd function, and so changing the sign of c changes the sign of the solution (10.9) of (10.2). Hence, for our purposes it is sufficient to consider $c \geq 0$; and since the case $c = 0$ has been dealt with in (9.1) to (9.31), we need only consider $c > 0$ and $0 < \lambda < \frac{1}{2}$.

We shall consider rational $c = m/r$, with m and r co–prime. The only positive rational c giving rational λ is $1/1 = \tan(\pi/4)$.

In terms of the rational function g_n, (10.2) becomes

(10.10) $$\frac{m}{r} = c = g_n(t) = t\,\frac{A_n(z)}{B_n(z)},$$

and that reduces to the polynomial equation in $t = \tan\psi$, with integer coefficients:

$$rtA_n(z) \;=\; mB_n(z). \tag{10.11}$$

For odd $n = 2j+1$, this gives the integer polynomial equation of degree $2j+1$ in t:

$$(-1)^{j-1}\times$$
$$\times\left[rt^{2j+1} - m(2j+1)t^{2j} - rj(2j+1)t^{2j-1} + \frac{1}{3}mj(2j+1)(2j-1)t^{2j-2} + \cdots\right]$$
$$\cdots + \frac{1}{3}rj(2j+1)(2j-1)t^3 - mj(2j+1)t^2 - r(2j+1)t + m \;=\; 0. \tag{10.12}$$

For even $n = 2j$, this gives the polynomial equation of degree $2j$ in t:

$$(-1)^{j-1}\left[mt^{2j} + 2rjt^{2j-1} - mj(2j-1)t^{2j-2} - \frac{2}{3}rj(j-1)(2j-1)t^{2j-3} + \cdots\right]$$
$$\cdots + \frac{2}{3}rj(j-1)(2j-1)t^3 - mj(2j-1)t^2 - 2rjt + m \;=\; 0\,. \tag{10.13}$$

For all $n > 0$, the low–order terms of the polynomial equation (10.11) are of the form

$$\cdots + r\binom{n}{3}t^3 - m\binom{n}{2}t^2 - r\binom{n}{1}t + m \;=\; 0\,. \tag{10.14}$$

The roots t_k of the polynomial equations (10.12) and (10.13) are given by (10.9), with

$$\lambda \;=\; \frac{\arctan(m/r)}{\pi}. \tag{10.15}$$

10.1. **Sums of Powers of Tangents.** Apply Newton's Rules for positive powers of the $t_k$to the integer polynomial equations (10.12) and (10.13). It follows that, for all positive integers m, n, q, r, if n is even then $m^q\sum_{k=1}^{n}\tan^q((k+\lambda)\pi)/n$ has integer value; and if n is odd then $r^q\sum_{k=1}^{n}\tan^q((k+\lambda)\pi/n)$ has integer value, where $\lambda = \arctan(m/r)\pi$.

For example, with even $n = 4,\ q = 5,\ m = 5$ and $r = 3$,

$$m^q\sum_{k=1}^{n}\tan^q\left(\frac{(k+\lambda)\pi}{n}\right) \;=\; -2416332{\cdot}000\,; \tag{10.16}$$

and with odd $n = 3,\ q = 5,\ m = 5$ and $r = 3$,

$$r^q\sum_{k=1}^{n}\tan^q\left(\frac{(k+\lambda)\pi}{n}\right) \;=\; 1212975{\cdot}000\,. \tag{10.17}$$

With $m = \pm 1,\ r = 1$ and $\lambda = \pm\frac{1}{4}$, this shews that both

$$\sum_{k=1}^{n}\tan^q\left(\frac{(4k+1)\pi}{4n}\right) \quad\text{and}\quad \sum_{k=1}^{n}\tan^q\left(\frac{(4k-1)\pi}{4n}\right) \tag{10.18}$$

have integer values for all positive integers q and n. Indeed, for even q those sums are equal, and for odd q each is the negative of the other.

For example, with $n = 18$ and $q = 9$,

$$\sum_{k=1}^{n}\tan^q\left(\frac{(4k+1)\pi}{4n}\right) \;=\; -\sum_{k=1}^{n}\tan^q\left(\frac{(4k-1)\pi}{4n}\right) \;=\; -1734367456242{\cdot}000\,. \tag{10.19}$$

10.1.1. *Zeros of* $\tan((2j+1)\psi - c$. For all integers $f \geq 0,\ j \geq 0,\ m > 0,\ r > 0$ and prime p, the congruences (2.31) for positive powers of the roots t_k of the integer polynomial equation (10.11) (scaled by $(-1)^{j-1}$) yield the congruences:

$$\begin{aligned}
\sum_{k=1}^{2j+1}(rt_k)^{p^f} &\equiv m(2j+1) \qquad (\mathrm{mod}\ p),\\
\sum_{k=1}^{2j+1}(rt_k)^{2p^f} &\equiv (m(2j+1))^2 + 2r^2j(2j+1) \qquad (\mathrm{mod}\ p),\\
\sum_{k=1}^{2j+1}(rt_k)^{3p^f} &\equiv (m(2j+1))^3 + 3mr^2j(2j+1)^2 - 3r^2\tfrac{1}{3}mj(2j+1)(2j-1)\\
&\qquad (\mathrm{mod}\ p), \quad et\ cetera.
\end{aligned} \tag{10.20}$$

These simplify to give the following congruence identities:

$$\begin{aligned}
r^{p^f}\sum_{k=1}^{2j+1}\tan^{p^f}\!\left(\frac{(k+\lambda)\pi}{2j+1}\right) &\equiv m(2j+1) \qquad (\mathrm{mod}\ p),\\
r^{2p^f}\sum_{k=1}^{2j+1}\tan^{2p^f}\!\left(\frac{(k+\lambda)\pi}{2j+1}\right) &\equiv (2j+1)(2(m^2+r^2)j+m^2) \qquad (\mathrm{mod}\ p),\\
r^{3p^f}\sum_{k=1}^{2j+1}\tan^{3p^f}\!\left(\frac{(k+\lambda)\pi}{2j+1}\right) &\equiv m(2j+1)(4(m^2+r^2)j(j+1)+m^2)\\
&\qquad (\mathrm{mod}\ p), \quad et\ cetera.
\end{aligned} \tag{10.21}$$

For example, with $m = 1,\ r = 2,\ j = 3,\ f = 1$ and $p = 3$,

$$\begin{aligned}
\frac{1}{p}\left[r^{3p^f}\sum_{k=1}^{2j+1}\tan^{3p^f}\!\left(\frac{(k+\lambda)\pi}{2j+1}\right) - m(2j+1)(4(m^2+r^2)j(j+1)+m^2)\right]&\\
= 2546108880{\cdot}000\ .&
\end{aligned} \tag{10.22}$$

10.1.2. *Zeros of* $\tan(2j\psi) - c$. For all integers $f \geq 0,\ j > 0,\ m > 0,\ r > 0$ and prime p, the congruences (2.31) for positive powers of the roots t_k of the integer polynomial equation (10.13) (scaled by $(-1)^{j-1}$) yield the congruences:

$$\begin{aligned}
\sum_{k=1}^{2j}(mt_k)^{p^f} &\equiv -2rj \qquad (\mathrm{mod}\ p),\\
\sum_{k=1}^{2j}(mt_k)^{2p^f} &\equiv (2rj)^2 + 2m^2j(2j-1) \qquad (\mathrm{mod}\ p),\\
\sum_{k=1}^{2j}(mt_k)^{3p^f} &\equiv -(2rj)^3 - 6m^2rj^2(2j-1) + 3m^2\tfrac{2}{3}rj(j-1)(2j-1)\\
&\qquad (\mathrm{mod}\ p) \quad et\ cetera.
\end{aligned} \tag{10.23}$$

These simplify to give the following congruence identities:

$$m^{pf}\sum_{k=1}^{2j}\tan^{pf}\left(\frac{(k+\lambda)\pi}{2j}\right) \equiv -2rj \qquad (\mathrm{mod}\ p),$$

$$m^{2pf}\sum_{k=1}^{2j}\tan^{2pf}\left(\frac{(k+\lambda)\pi}{2j}\right) \equiv 2j(2(m^2+r^2)j-m^2) \qquad (\mathrm{mod}\ p),$$

$$m^{3pf}\sum_{k=1}^{2j}\tan^{3pf}\left(\frac{(k+\lambda)\pi}{2j}\right) \equiv 2rj(m^2-4(m^2+r^2)j^2) \ (\mathrm{mod}\ p), \tag{10.24}$$

et cetera.

For example, with $m=2,\ r=1,\ j=3,\ f=1$ and $p=13$,

$$\frac{1}{p}\left[m^{pf}\sum_{k=1}^{2j}\tan^{pf}\left(\frac{(k+\lambda)\pi}{2j}\right)+2rj\right] = -1888127879010{\cdot}000\ . \tag{10.25}$$

10.2. **Sums of Powers of Cotangents.** Apply Newton's Rules to the zeros $1/t_k$ of the polynomial which is reciprocal to (10.11). It follows that for all positive integers $n,\ m,\ r$ and q, $m^q\sum_{k=1}^{n}\cot^q((k+\lambda)\pi/n)$ has integer value, with $\lambda=\arctan(m/r)/\pi$.

For example, with $n=9,\ q=3,\ m=5$ and $r=3$,

$$m^q\sum_{k=1}^{n}\cot^q\left(\frac{(k+\lambda)\pi}{n}\right) = 6844029489{\cdot}000\ . \tag{10.26}$$

For all integers $n>0.\ f\geq 0,\ m>0,\ r>0$ and prime p, the congruences (2.32) for negative powers of the roots t_k of the integer polynomial equation (10.11) yield the congruences:

$$\sum_{k=1}^{n}\left(\frac{m}{t_k}\right)^{pf} \equiv rn \qquad (\mathrm{mod}\ p),$$

$$\sum_{k=1}^{n}\left(\frac{m}{t_k}\right)^{2pf} \equiv r^2n^2+2m^2\binom{n}{2} \qquad (\mathrm{mod}\ p),$$

$$\sum_{k=1}^{n}\left(\frac{m}{t_k}\right)^{3pf} \equiv r^3n^3+3rm^2n\binom{n}{2}-3m^2r\binom{n}{3} \qquad (\mathrm{mod}\ p),\ \textit{et cetera}. \tag{10.27}$$

These simplify to give the following congruence identities:

$$m^{pf}\sum_{k=1}^{n}\cot^{pf}\left(\frac{(k+\lambda)\pi}{n}\right) \equiv rn \qquad (\mathrm{mod}\ p),$$

$$m^{2pf}\sum_{k=1}^{n}\cot^{2pf}\left(\frac{(k+\lambda)\pi}{n}\right) \equiv n((m^2+r^2)n-m^2) \qquad (\mathrm{mod}\ p),$$

$$m^{3pf}\sum_{k=1}^{n}\cot^{3pf}\left(\frac{(k+\lambda)\pi}{n}\right) \equiv rn((m^2+r^2)n^2-m^2) \quad (\mathrm{mod}\ p), \tag{10.28}$$

et cetera.

For example, with $m=3,\ r=2,\ n=4,\ f=2$ and $p=3$,

$$\frac{1}{p}\left[m^{pf}\sum_{k=1}^{n}\cot^{pf}\left(\frac{(k+\lambda)\pi}{n}\right) - rn\right] = 1672622592{\cdot}000\ . \tag{10.29}$$

With $m = \pm 1,\ r = 1$ and $\lambda = \pm\frac{1}{4}$, this shews that for integers $n > 0,\ f \geq 0$ and prime p,

$$\sum_{k=1}^{n} \cot^{p^f}\!\left(\frac{(4k \pm 1)\pi}{4n}\right) \equiv \pm n \pmod{p},$$

$$\sum_{k=1}^{n} \cot^{2p^f}\!\left(\frac{(4k \pm 1)\pi}{4n}\right) \equiv n(2n-1) \pmod{p},$$

$$\sum_{k=1}^{n} \cot^{3p^f}\!\left(\frac{(4k \pm 1)\pi}{4n}\right) \equiv \pm n(2n^2-1) \pmod{p}, \quad \textit{et cetera}. \tag{10.30}$$

For example, with $n = 3,\ f = 1$ and $p = 7$,

$$\frac{1}{p}\left[\sum_{k=1}^{n} \cot^{3p^f}\!\left(\frac{(4k \pm 1)\pi}{4n}\right) \mp n(2n^2-1)\right] = \pm 146487463200{\cdot}000\ . \tag{10.31}$$

With $m = \pm 1,\ r = 1$ and $\lambda = \pm\frac{1}{4}$,

$$\sum_{k=1}^{n} \cot^{q}\!\left(\frac{(4k \pm 1)\pi}{4n}\right) = \sum_{k=1}^{n} \tan^{q}\!\left(\frac{\pi}{2} - \frac{(4k \pm 1)\pi}{4n}\right)$$

$$= \sum_{k=1}^{n} \tan^{q}\!\left(\frac{(4j + 2 \mp 1 - 4k)\pi}{4n}\right) = \sum_{h=1}^{n} \tan^{q}\!\left(\frac{(4h \pm 1)\pi}{4n}\right). \tag{10.32}$$

Hence, for this case (10.26) gives sums of powers of tangents which are equal to the sums of powers of cotangents in (10.28) for odd $n = 2j + 1$, (10.23) gives sums of powers of tangents which are equal to the sums of powers of cotangents in (10.32) for even $n = 2j$, and (10.26) gives the same sums as in (10.24).

The many congruence identities, derived in this paper for integer sums of powers of the trigonometric functions, provide highly sensitive tests for the accuracy of software for evaluation of trigonometric and inverse trigonometric functions, as in the numerical examples computed in this paper. In each case the result must have integer value, within the errors for computation with rounded arithmetic and finite approximations to those functions.

Similar identities have been constructed for integer sums of powers of Jacobian elliptic functions (Tee [13]), and for the Weierstraß elliptic function $\wp$ (Tee [14]).

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