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STABILITY OF A MIXED ADDITIVE, QUADRATIC AND CUBIC FUNCTIONAL EQUATION IN QUASI–BANACH SPACES

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ABSTRACT. In this paper we establish the general solution of a mixed additive, quadratic and cubic functional equation and investigate the Hyers–Ulam–Rassias stability of this equation in quasi-Banach spaces. The concept of Hyers-Ulam-Rassias stability originated from Th. M. Rassias' stability theorem that appeared in his paper: On the stability of the linear mapping in Banach spaces, *Proc. Amer. Math. Soc.* **72** (1978), 297–300.

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1. INTRODUCTION

In 1940, S. M. Ulam [18] gave a talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems. Among these was the following question concerning the stability of homomorphisms.

Let $(G_1, *)$ *be a group and let* $(G_2, \diamond, d)$ *be a metric group with the metric* $d(\cdot, \cdot)$. *Given* $\epsilon > 0$, *does there exist a* $\delta(\epsilon) > 0$ *such that if a function* $h : G_1 \to G_2$ *satisfies the inequality*

$$d(h(x * y), h(x) \diamond h(y)) < \delta$$

for all $x, y \in G_1$, *then there is a homomorphism* $H : G_1 \to G_2$ *with*

$$d(h(x), H(x)) < \epsilon$$

for all $x \in G_1$?

In 1941, D. H. Hyers [8] considered the case of approximately additive functions $f : E \to E'$, where E and E' are Banach spaces and f satisfies *Hyers inequality*

$$\|f(x + y) - f(x) - f(y)\| \le \epsilon$$

for all $x, y \in E$. It was shown that the limit

$$L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

exists for all $x \in E$ and that $L : E \to E'$ is the unique additive function satisfying

$$\|f(x) - L(x)\| \le \epsilon.$$

In 1978, Th. M. Rassias [15] provided a generalization of Hyers' Theorem which allows the *Cauchy difference to be unbounded.*

Quadratic functional equation was used to characterize inner product spaces [1, 2, 9]. Several other functional equations were also to characterize inner product spaces. A square norm on an inner product space satisfies the important parallelogram equality

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2).$$

The functional equation

$$f(x + y) + f(x - y) = 2f(x) + 2f(y) \tag{1.1}$$

is related to a symmetric bi-additive function [1, 12]. It is natural that the equation 1.1 is called a quadratic functional equation. In particular, every solution of the quadratic equation (1.1) is said to be a quadratic function. It is well known that a function f between real vector spaces is quadratic if and only if there exists a unique symmetric biadditive function B such that $f(x) = B(x, x)$ for all x (see [1, 12]). The biadditive function B is given by

$$B(x, y) = \frac{1}{4}\Big(f(x + y) - f(x - y)\Big). \tag{1.2}$$

A Hyers–Ulam stability problem for the quadratic functional equation (1.1) was proved by Skof for functions $f : E_1 \to E_2$, where E_1 is a normed space and E_2 a Banach space (see [17]). Cholewa [4] noticed that the theorem of Skof is still true if the relevant domain E_1 is replaced by an Abelian group. In the paper [5], Czerwik proved the Hyers–Ulam–Rassias stability of the quadratic functional equation (1.1). Grabiec [7] has generalized these results mentioned above. Jun and Lee [10] proved the Hyers–Ulam–Rassias stability of the pexiderized quadratic equation (1.1).

Jun and Kim [11] introduced the following cubic functional equation

$$f(2x + y) + f(2x - y) = 2f(x + y) + 2f(x - y) + 12f(x) \tag{1.3}$$

and they established the general solution and the generalized Hyers–Ulam–Rassias stability problem for the functional equation (1.3). They proved that a function $f : E_1 \to E_2$ satisfies the functional equation (1.3) if and only if there exists a function $B : E_1 \times E_1 \times E_1 \to E_2$ such that $f(x) = B(x, x, x)$ for all $x \in E_1$, and B is symmetric for each fixed one variable and additive for each fixed two variables. The function B is given by

$$B(x,y,z) = \frac{1}{24}[f(x+y+z) + f(x-y-z) - f(x+y-z) - f(x-y+z)]$$

for all $x, y, z \in E_1$.

It is easy to see that the function $f(x) = cx^3$ is a solution of the functional equation (1.3). Thus, it is natural that (1.3) is called a *cubic functional equation* and every solution of the cubic functional equation (1.3) is said to be a *cubic function*.

In this paper, we deal with the following functional equation deriving from cubic, quadratic and additive functions:

$$f\Big(\sum_{i=1}^{4} x_i\Big) + \sum_{1\le i<j\le 4} f(x_i + x_j) = \sum_{i=1}^{4} f(x_i) + \sum_{1\le i<j<k\le 4} f(x_i + x_j + x_k). \tag{1.4}$$

It is easy to see that the function $f(x) = ax^3 + bx^2 + cx$ is a solution of the functional equation (1.4). For some results concerning the functional equation (1.4), we refer the reader to [13].

The main purpose of this paper is to establish the general solution of Eq. (1.4) and investigate the Hyers–Ulam–Rassias stability for Eq. (1.4).

We recall some basic facts concerning quasi-Banach spaces and some preliminary results.

Definition 1.1. $[3, 16]$ Let X be a real linear space. A *quasi-norm* is a real-valued function on X satisfying the following:

(i) $\|x\| \ge 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$.
(ii) $\|\lambda x\| = |\lambda|\|x\|$ for all $\lambda \in \mathbb{R}$ and all $x \in X$.
(iii) There is a constant $K \ge 1$ such that $\|x + y\| \le K(\|x\| + \|y\|)$ for all $x, y \in X$.

It follows from condition (iii) that

$$\Big\|\sum_{i=1}^{2n} x_i\Big\| \le K^n \sum_{i=1}^{2n} \|x_i\|, \quad \Big\|\sum_{i=1}^{2n+1} x_i\Big\| \le K^{n+1} \sum_{i=1}^{2n+1} \|x_i\|$$

for all integers $n \ge 1$ and all $x_1, x_2, \dots, x_{2n+1} \in X$.

The pair $(X, \|.\|)$ is called a *quasi-normed space* if $\|.\|$ is a quasi-norm on X. The smallest possible K is called the *modulus of concavity* of $\|.\|$. A *quasi-Banach space* is a complete quasi-normed space.

A quasi-norm $\|.\|$ is called a p-*norm* $(0 < p \le 1)$ if

$$\|x + y\|^p \le \|x\|^p + \|y\|^p$$

for all $x, y \in X$. In this case, a quasi-Banach space is called a p-*Banach space.*

By the Aoki–Rolewicz theorem [16] (see also [3]), each quasi-norm is equivalent to some p-norm. Since it is much easier to work with p-norms than quasi-norms, henceforth we restrict our attention mainly to p-norms.

2. SOLUTIONS OF EQ. (1.4)

Throughout this section, X and Y will be real vector spaces. Before proceeding the proof of Theorem 2.4 which is the main result in this section, we shall need the following lemmas.

Lemma 2.1. *If an even function $f : X \to Y$ satisfies (1.4), then f is quadratic.*

Proof. Note that, in view of the evenness of f, we have $f(-x) = f(x)$ for all $x \in X$. Putting $x_1 = x_2 = x_3 = x_4 = 0$ in (1.4), we get that $f(0) = 0$. Letting $x_1 = x_2 = x$ and $x_3 = x_4 = y$ in (1.4), we get

$$\begin{aligned} &f(2x+2y) + 4f(x+y) + f(2x) + f(2y) \\ &= 2f(2x+y) + 2f(x+2y) + 2f(x) + 2f(y) \end{aligned} \tag{2.1}$$

for all $x, y \in X$. Letting $y = -x$ in (2.1) and using the evenness of f, we get that

$$f(2x) = 4f(x) \tag{2.2}$$

for all $x \in X$. Therefore it follows from (2.1) and (2.2) that

$$f(2x+y) + f(x+2y) = 4f(x+y) + f(x) + f(y) \tag{2.3}$$

for all $x, y \in X$. Replacing y by $y - x$ in (2.3) and using the evenness of f, we get

$$f(x-2y) = f(x-y) - f(x+y) + f(x) + 4f(y) \tag{2.4}$$

for all $x, y \in X$. Replacing y by $-y$ in (2.4) and using the evenness of f, we get

$$f(x+2y) = f(x+y) - f(x-y) + f(x) + 4f(y) \tag{2.5}$$

for all $x, y \in X$. Replacing x and y by y and x in (2.5), respectively, and using the evenness of f, we get

$$f(2x+y) = f(x+y) - f(x-y) + f(y) + 4f(x) \tag{2.6}$$

for all $x, y \in X$. Adding (2.5) to (2.6) and using (2.3), we get that

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

for all $x, y \in X$. Therefore the function $f : X \to Y$ is quadratic. ∎

Lemma 2.2. *If an odd function $f : X \to Y$ satisfies (1.4), then the function $g : X \to Y$ defined by $g(x) = f(2x) - 8f(x)$ is additive.*

Proof. Note that, in view of the oddness of f, we have $f(-x) = -f(x)$ for all $x \in X$. So $f(0) = 0$. Replacing y by $y - x$ in (2.1) and using the oddness of f, we get

$$\begin{aligned} &f(2y-2x) + f(2x) + f(2y) + 4f(y) \\ &= 2f(2y-x) + 2f(x+y) - 2f(x-y) + 2f(x) \end{aligned} \tag{2.7}$$

for all $x, y \in X$. Replacing x by $-x$ in (2.7) and using the oddness of f, we get

$$\begin{aligned} &f(2x+2y) - f(2x) + f(2y) + 4f(y) \\ &= 2f(x+2y) + 2f(x+y) - 2f(x-y) - 2f(x) \end{aligned} \tag{2.8}$$

for all $x, y \in X$. Replacing x and y by y and x in (2.8), respectively, and using the oddness of f, we get

$$\begin{aligned} &f(2x+2y) - f(2y) + f(2x) + 4f(x) \\ &= 2f(2x+y) + 2f(x+y) + 2f(x-y) - 2f(y) \end{aligned} \tag{2.9}$$

for all $x, y \in X$. Adding (2.8) to (2.9) and using the oddness of f, we get

$$2f(2x+y) + 2f(x+2y) = 2f(2x+2y) - 4f(x+y) + 6f(x) + 6f(y) \tag{2.10}$$

for all $x, y \in X$. It follows from (2.1) and (2.10) that

$$f(2x+2y) - 8f(x+y) = f(2x) + f(2y) - 8f(x) - 8f(y)$$

for all $x, y \in X$. So by the definition of g, we have

$$g(x+y) = g(x) + g(y)$$

for all $x, y \in X$. Therefore the function $g : X \to Y$ is additive.
∎

Lemma 2.3. *If an odd function $f : X \to Y$ satisfies (1.4), then the function $h : X \to Y$ defined by $h(x) = f(2x) - 2f(x)$ is cubic.*

Proof. It is clear that $f(0) = 0$. Let $g : X \to Y$ be a function defined by $g(x) = f(2x) - 8f(x)$ for all $x \in X$. By Lemma 2.2, the function g, is additive. It is clear that

$$h(x) = g(x) + 6f(x), \quad f(2x) = g(x) + 8f(x) \tag{2.11}$$

for all $x \in X$. Therefore the functional equation (2.9) means

$$\begin{aligned} &g(x+y) + 8f(x+y) + g(x) + 12f(x) - f(2y) \\ &= 2f(2x+y) + 2f(x+y) + 2f(x-y) - 2f(y) \end{aligned} \tag{2.12}$$

for all $x, y \in X$. Replacing y by $-y$ in (2.12) and using the oddness of f, we get

$$\begin{aligned} &g(x-y) + 8f(x-y) + g(x) + 12f(x) + f(2y) \\ &= 2f(2x-y) + 2f(x-y) + 2f(x+y) + 2f(y) \end{aligned} \tag{2.13}$$

for all $x, y \in X$. Adding (2.12) to (2.13) and using the additivity of g, we get

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 12f(x) + 2g(x) \tag{2.14}$$

for all $x, y \in X$. So it follows from (2.11) and (2.14) that

$$\begin{aligned} &h(2x+y) + h(2x-y) - [g(2x+y) + g(2x-y)] \\ &\qquad = 2[h(x+y) + h(x-y)] + 12h(x) - 2[g(x+y) + g(x-y)] \end{aligned} \tag{2.15}$$

for all $x, y \in X$. Since g is additive, then (2.15) implies that

$$h(2x+y) + h(2x-y) = 2[h(x+y) + h(x-y)] + 12h(x)$$

for all $x, y \in X$. Therefore the function h is cubic. ∎

Theorem 2.4. *A function $f : X \to Y$ satisfies the functional equation (1.4) if and only if there exist functions $C : X \times X \times X \to Y$, $B : X \times X \to Y$ and $A : X \to Y$ such that*

$$f(x) = C(x,x,x) + B(x,x) + A(x)$$

for all $x \in X$, where the function C is symmetric for each fixed one variable and additive for fixed two variables, the function B is symmetric bi-additive and the function A is additive.

Proof. We first assume that f is a solution of the functional equation (1.4). We decompose f into the even part and the odd part by putting

$$f_e(x) = \frac{f(x) + f(-x)}{2} \quad \text{and} \quad f_o(x) = \frac{f(x) - f(-x)}{2}$$

for all $x \in X$. It is clear that $f(x) = f_e(x) + f_o(x)$ for all $x \in X$. It is easy to show that each of the functions f_e and f_o satisfies (1.4). Hence by Lemmas 2.1, 2.2 and 2.3 we achieve that the functions $h, f_e, g : X \to Y$ are cubic, quadratic and additive, respectively, where

$$h(x) = f_o(2x) - 2f_o(x), \quad g(x) = f_o(2x) - 8f_o(x)$$

for all $x \in X$. Therefore by Theorem [11, Theorem 2.1] there exists a function $C : X \times X \times X \to Y$ such that $h(x) = 6C(x,x,x)$ for all $x \in X$, and C is symmetric for each fixed one variable and is additive for fixed two variables. Also there exists a symmetric bi-additive function $B : X \times X \to Y$ such that $f_e(x) = B(x,x)$ for all $x \in X$ (see [1, 12]). So

$$f(x) = C(x,x,x) + B(x,x) + A(x)$$

for all $x \in X$, where $A(x) = -\frac{1}{6}g(x)$ for all $x \in X$.

Conversely, let

$$f(x) = C(x, x, x) + B(x, x) + A(x)$$

for all $x \in X$, where the function C is symmetric for each fixed one variable and additive for fixed two variables, the function B is symmetric bi-additive and the function A is additive. By a simple computation one can show that the function f satisfies the equation (1.4). ∎

3. HYERS–ULAM–RASSIAS STABILITY OF EQ. (1.4)

Throughout this paper, assume that X is a quasi-normed space with quasi-norm $\|.\|_X$ and that Y is a p-Banach space with p-norm $\|.\|_Y$. Let K be the modulus of concavity of $\|.\|_Y$.

In this section we have four parts. In each part, using an idea of Găvruta [6] we prove the stability of Eq. (1.4) in the spirit of Hyers, Ulam and Rassias. For convenience, we use the following abbreviation for a given function $f : X \times X \times X \times X \to Y$:

$$\begin{aligned} Df(x_1, x_2, x_3, x_4) := f\Big(\sum_{i=1}^{4} x_i\Big) + \sum_{1\le i<j\le 4} f(x_i + x_j) - \sum_{i=1}^{4} f(x_i) \\ - \sum_{1\le i<j<k\le 4} f(x_i + x_j + x_k) \end{aligned}$$

for all $x_1, x_2, x_3, x_4 \in X$.

We will use the following lemma in this section.

Lemma 3.1. [14] *Let $0 < p \le 1$ and let $x_1, x_2, \dots, x_n$ be non-negative real numbers. Then*

$$\Big(\sum_{i=1}^{n} x_i\Big)^p \le \sum_{i=1}^{n} x_i^p. \tag{3.1}$$

3.1. **Part I.** In this part, we find some conditions that there exists a true quadratic function near an approximately quadratic function.

Theorem 3.2. *Let $\varphi : X^4 \to [0, \infty)$ be a function such that*

$$\lim_{n\to\infty} 4^n \varphi\Big(\frac{x_1}{2^n}, \frac{x_2}{2^n}, \frac{x_3}{2^n}, \frac{x_4}{2^n}\Big) = 0 \tag{3.2}$$

and

$$\widetilde{\varphi_e}(x) := \sum_{i=1}^{\infty} 4^{ip} \varphi^p\Big(\frac{x}{2^i}, \frac{x}{2^i}, -\frac{x}{2^i}, -\frac{x}{2^i}\Big) < \infty \tag{3.3}$$

for all $x, x_1, x_2, x_3, x_4 \in X$. Suppose that an even function $f : X \to Y$ satisfies the inequality

$$\|Df(x_1, x_2, x_3, x_4)\|_Y \le \varphi(x_1, x_2, x_3, x_4) \tag{3.4}$$

for all $x_1, x_2, x_3, x_4 \in X$. Then the limit

$$Q(x) := \lim_{n\to\infty} 4^n f\Big(\frac{x}{2^n}\Big) \tag{3.5}$$

exists for all $x \in X$ and $Q : X \to Y$ is a unique quadratic function satisfying

$$\|f(x) - Q(x)\|_Y \le \frac{1}{8}[\widetilde{\varphi_e}(x)]^{\frac{1}{p}} \tag{3.6}$$

for all $x \in X$.

Proof. It follows from (3.2) that $\varphi(0,0,0,0)=0$. So by letting $x_1=x_2=x_3=x_4=0$ in (3.4), we get that $f(0)=0$. Letting $x_1=x_2=x$ and $x_3=x_4=-x$ in (3.4), we get

$$\|f(2x)-4f(x)\|_Y \le \frac{1}{2}\varphi(x,x,-x,-x) \tag{3.7}$$

for all $x\in X$. If we replace x in (3.7) by $\frac{x}{2^{n+1}}$ and multiply both sides of (3.7) to 4^n, then we have

$$\left\|4^{n+1}f\Big(\frac{x}{2^{n+1}}\Big)-4^nf\Big(\frac{x}{2^n}\Big)\right\|_Y \le \frac{4^n}{2}\varphi\Big(\frac{x}{2^{n+1}},\frac{x}{2^{n+1}},-\frac{x}{2^{n+1}},-\frac{x}{2^{n+1}}\Big) \tag{3.8}$$

for all $x\in X$ and all non-negative integers n. Since Y is a p-Banach space, we have

$$\begin{aligned}
&\left\|4^{n+1}f\Big(\frac{x}{2^{n+1}}\Big)-4^mf\Big(\frac{x}{2^m}\Big)\right\|_Y^p\\
&\le \sum_{i=m}^{n}\left\|4^{i+1}f\Big(\frac{x}{2^{i+1}}\Big)-4^if\Big(\frac{x}{2^i}\Big)\right\|_Y^p\\
&\le 2^{-p}\sum_{i=m}^{n}4^{ip}\varphi^p\Big(\frac{x}{2^{i+1}},\frac{x}{2^{i+1}},-\frac{x}{2^{i+1}},-\frac{x}{2^{i+1}}\Big)
\end{aligned} \tag{3.9}$$

for all $x\in X$ and all non-negative integers m and n with $n\ge m$. Therefore we conclude from (3.3) and (3.9) that the sequence $\{4^nf(\frac{x}{2^n})\}$ is a Cauchy sequence in Y for all $x\in X$. Since Y is complete, the sequence $\{4^nf(\frac{x}{2^n})\}$ converges in Y for all $x\in X$. So one can define the function $Q:X\to Y$ by (3.5) for all $x\in X$. Letting $m=0$ and passing the limit $n\to\infty$ in (3.9), we get

$$\begin{aligned}
\|f(x)-Q(x)\|_Y^p &\le 2^{-p}\sum_{i=0}^{\infty}4^{ip}\varphi^p\Big(\frac{x}{2^{i+1}},\frac{x}{2^{i+1}},-\frac{x}{2^{i+1}},-\frac{x}{2^{i+1}}\Big)\\
&=\frac{1}{8^p}\sum_{i=1}^{\infty}4^{ip}\varphi^p\Big(\frac{x}{2^i},\frac{x}{2^i},-\frac{x}{2^i},-\frac{x}{2^i}\Big)
\end{aligned} \tag{3.10}$$

for all $x\in X$. Therefore we obtain (3.6). Now, we show that Q is quadratic. It follows from (3.2), (3.4) and (3.5),

$$\begin{aligned}
\|DQ(x_1,x_2,x_3,x_4)\|_Y &= \lim_{n\to\infty}4^n\left\|Df\Big(\frac{x_1}{2^n},\frac{x_2}{2^n},\frac{x_3}{2^n},\frac{x_4}{2^n}\Big)\right\|_Y\\
&\le \lim_{n\to\infty}4^n\varphi\Big(\frac{x_1}{2^n},\frac{x_2}{2^n},\frac{x_3}{2^n},\frac{x_4}{2^n}\Big)=0
\end{aligned}$$

for all $x_1,x_2,x_3,x_4\in X$. Therefore the function $Q:X\to Y$ satisfies (1.4). Since f is even, then Q is even. So by Lemma 2.1 we get that the function $Q:X\to Y$ is quadratic.

To prove the uniqueness of Q, let $T:X\to Y$ be another quadratic function satisfying (3.6). Since

$$\begin{aligned}
\lim_{n\to\infty}4^{np}\widetilde{\varphi_e}\Big(\frac{x}{2^n}\Big) &= \lim_{n\to\infty}4^{np}\sum_{i=1}^{\infty}4^{ip}\varphi^p\Big(\frac{x}{2^{n+i}},\frac{x}{2^{n+i}},-\frac{x}{2^{n+i}},-\frac{x}{2^{n+i}}\Big)\\
&=\lim_{n\to\infty}\sum_{i=n+1}^{\infty}4^{ip}\varphi^p\Big(\frac{x}{2^i},\frac{x}{2^i},-\frac{x}{2^i},-\frac{x}{2^i}\Big)=0
\end{aligned}$$

for all $x\in X$, then it follows from (3.6) that

$$\begin{aligned}
\|Q(x)-T(x)\|_Y^p &= \lim_{n\to\infty}4^{np}\left\|f\Big(\frac{x}{2^n}\Big)-T\Big(\frac{x}{2^n}\Big)\right\|_Y^p\\
&\le \frac{1}{8^p}\lim_{n\to\infty}4^{np}\widetilde{\varphi_e}\Big(\frac{x}{2^n}\Big)=0
\end{aligned}$$

for all $x \in X$. So $Q = T$. ∎

Theorem 3.3. *Let* $\Phi : X^4 \to [0, \infty)$ *be a function such that*

$$\lim_{n\to\infty} \frac{1}{4^n}\Phi(2^n x_1, 2^n x_2, 2^n x_3, 2^n x_4) = 0 \tag{3.11}$$

and

$$\widetilde{\Phi_e}(x) := \sum_{i=0}^{\infty} \frac{1}{4^{ip}}\Phi^p(2^i x, 2^i x, -2^i x, -2^i x) < \infty \tag{3.12}$$

for all $x, x_1, x_2, x_3, x_4 \in X$. *Suppose that an even function* $f : X \to Y$ *satisfies the inequality*

$$\|Df(x_1, x_2, x_3, x_4)\|_Y \le \Phi(x_1, x_2, x_3, x_4) \tag{3.13}$$

for all $x_1, x_2, x_3, x_4 \in X$. *Then the limit*

$$Q(x) := \lim_{n\to\infty} \frac{1}{4^n} f(2^n x) \tag{3.14}$$

exists for all $x \in X$ *and* $Q : X \to Y$ *is a unique quadratic function satisfying*

$$\left\| f(x) - Q(x) + \frac{5}{6} f(0)\right\|_Y \le \frac{1}{8}[\widetilde{\Phi_e}(x)]^{\frac{1}{p}} \tag{3.15}$$

for all $x \in X$.

Proof. Letting $x_1 = x_2 = x$ and $x_3 = x_4 = -x$ in (3.4), we get

$$\left\| f(2x) - 4f(x) + \frac{5}{2} f(0)\right\|_Y \le \frac{1}{2}\Phi(x, x, -x, -x) \tag{3.16}$$

for all $x \in X$. If we replace x in (3.16) by $2^n x$ and divide both side of (3.16) by 4^{n+1}, then we have

$$\begin{aligned} &\left\| \frac{1}{4^{n+1}} f(2^{n+1}x) - \frac{1}{4^n} f(2^n x) + \frac{5}{2\times 4^{n+1}} f(0)\right\|_Y \\ &\le \frac{1}{2\times 4^{n+1}}\Phi(2^n x, 2^n x, -2^n x, -2^n x) \end{aligned} \tag{3.17}$$

for all $x \in X$ and all non-negative integers n. Since Y is a p-Banach space,

$$\begin{aligned} &\left\| \frac{1}{4^{n+1}} f(2^{n+1}x) - \frac{1}{4^m} f(2^m x) + \frac{1}{2}\sum_{i=m}^{n} \frac{5}{4^{i+1}} f(0)\right\|_Y^p \\ &\le \sum_{i=m}^{n} \left\| \frac{1}{4^{i+1}} f(2^{i+1}x) - \frac{1}{4^i} f(2^i x) + \frac{5}{2\times 4^{i+1}} f(0)\right\|_Y^p \\ &\le \frac{1}{8^p}\sum_{i=m}^{n} \frac{1}{4^{ip}}\Phi^p(2^i x, 2^i x, -2^i x, -2^i x) \end{aligned} \tag{3.18}$$

for all $x \in X$ and all non-negative integers m and n with $n \ge m$. Since $\sum_{i=0}^{\infty} \frac{1}{4^i}$ converges, then it follows from (3.12) and (3.18) that the sequence $\{ \frac{1}{4^n} f(2^n x) \}$ is a Cauchy sequence in Y for all $x \in X$. Since Y is complete, the sequence $\{ \frac{1}{4^n} f(2^n x) \}$ converges in Y for all $x \in X$. So one can define the function $Q : X \to Y$ by (3.14).

The rest of the proof is similar to the proof of Theorem 3.2. ∎

Corollary 3.4. *Let θ be a non-negative real number. Suppose that an even function $f : X \to Y$ satisfies the inequality*

$$\|Df(x_1, x_2, x_3, x_4)\|_Y \le \theta \tag{3.19}$$

for all $x_1, x_2, x_3, x_4 \in X$. Then there exists a unique quadratic function $Q : X \to Y$ satisfies

$$\|f(x) - Q(x)\|_Y \le \frac{K\theta}{2}\Big[\frac{1}{(4^p - 1)^{\frac{1}{p}}} + \frac{5}{3}\Big]$$

for all $x \in X$.

Proof. It follows from (3.19) that $\|f(0)\|_Y \le \theta$. Hence the result follows by Theorem 3.3. ∎

Corollary 3.5. *Let $\theta, \{r_i\}_{i\in J}$ be non-negative real numbers such that $r_i > 2\,(0 < r_i < 2)$ for all $i \in J$, where J is a non-empty subset of $\{1, 2, 3, 4\}$. Suppose that an even function $f : X \to Y$ satisfies the inequality*

$$\|Df(x_1, x_2, x_3, x_4)\|_Y \le \theta \sum_{i\in J} \|x_i\|_X^{r_i} \tag{3.20}$$

for all $x_1, x_2, x_3, x_4 \in X$. Then there exists a unique quadratic function $Q : X \to Y$ satisfies

$$\|f(x) - Q(x)\|_Y \le \frac{\theta}{2}\Big\{\sum_{i\in J} \frac{1}{|2^{pr_i} - 4^p|}\|x\|_X^{pr_i}\Big\}^{\frac{1}{p}}$$

for all $x \in X$.

Proof. It follows from (3.20) that $f(0) = 0$. Hence the result follows by Theorems 3.2 and 3.3. ∎

Corollary 3.6. *Let $\theta, \{r_i\}_{i\in J}$ be non-negative real numbers such that $\lambda = \sum_{i\in J} r_i \in (0, 2) \cup (2, +\infty)$, where J is a non-empty subset of $\{1, 2, 3, 4\}$. Suppose that an even function $f : X \to Y$ satisfies the inequality*

$$\|Df(x_1, x_2, x_3, x_4)\|_Y \le \theta \prod_{i\in J} \|x_i\|_X^{r_i} \tag{3.21}$$

for all $x_1, x_2, x_3, x_4 \in X$. Then there exists a unique quadratic function $Q : X \to Y$ satisfies

$$\|f(x) - Q(x)\|_Y \le \frac{\theta}{2|2^{\lambda p} - 4^p|^{\frac{1}{p}}}\|x\|_X^{\lambda}$$

for all $x \in X$.

Proof. The result follows by Theorems 3.2 and 3.3. ∎

3.2. **Part II.** In this part, we find some conditions that there exists a true additive function near an approximately additive function.

Theorem 3.7. *Let $\varphi : X^4 \to [0, \infty)$ be a function such that*

$$\lim_{n\to\infty} 2^n \varphi\Big(\frac{x_1}{2^n}, \frac{x_2}{2^n}, \frac{x_3}{2^n}, \frac{x_4}{2^n}\Big) = 0 \tag{3.22}$$

and

$$\sum_{i=1}^{\infty} 2^{ip}\varphi^p\Big(\frac{x}{2^i}, \frac{x}{2^i}, \frac{x}{2^i}, \frac{y}{2^i}\Big) < \infty \tag{3.23}$$

for all $x, x_1, x_2, x_3, x_4 \in X$ *and* $y = \pm x$. *Suppose that an odd function* $f : X \to Y$ *satisfies the inequality (3.4) for all* $x_1, x_2, x_3, x_4 \in X$. *Let* $g : X \to Y$ *be a function defined by* $g(x) = f(2x) - 8f(x)$ *for all* $x \in X$. *Then the limit*

$$A(x) := \lim_{n\to\infty} 2^n g\Big(\frac{x}{2^n}\Big) \tag{3.24}$$

exists for all $x \in X$ *and* $A : X \to Y$ *is a unique additive function satisfying*

$$\|f(2x) - 8f(x) - A(x)\|_Y \le \frac{K}{2}[\widetilde{\varphi_a}(x)]^{\frac{1}{p}} \tag{3.25}$$

for all $x \in X$, *where*

$$\widetilde{\varphi_a}(x) := \sum_{i=1}^{\infty} 2^{ip}\Big\{\varphi^p\Big(\frac{x}{2^i}, \frac{x}{2^i}, \frac{x}{2^i}, \frac{x}{2^i}\Big) + 4^p\varphi^p\Big(\frac{x}{2^i}, \frac{x}{2^i}, \frac{x}{2^i}, -\frac{x}{2^i}\Big)\Big\} \tag{3.26}$$

Proof. Letting $x_1 = x_2 = x_3 = x_4 = x$ in (3.4), we get

$$\|f(4x) - 4f(3x) + 6f(2x) - 4f(x)\|_Y \le \varphi(x, x, x, x) \tag{3.27}$$

for all $x \in X$. Putting $x_1 = x_2 = x_3 = x$ and $x_4 = -x$ in (3.4) and using the oddness of f, we have

$$\|f(3x) - 4f(2x) + 5f(x)\|_Y \le \varphi(x, x, x, -x) \tag{3.28}$$

for all $x \in X$. It follows form (3.27) and (3.28) that

$$\|f(4x) - 10f(2x) + 16f(x)\|_Y \le K\varphi_1(x) \tag{3.29}$$

for all $x \in X$, where

$$\varphi_1(x) = \varphi(x, x, x, x) + 4\varphi(x, x, x, -x). \tag{3.30}$$

It follows from (3.29) and the definition of g,

$$\|g(2x) - 2g(x)\|_Y \le K\varphi_1(x) \tag{3.31}$$

for all $x \in X$. If we replace x in (3.31) by $\dfrac{x}{2^{n+1}}$ and multiply both sides of (3.31) to 2^n, we get

$$\Big\|2^{n+1}g\Big(\frac{x}{2^{n+1}}\Big) - 2^n g\Big(\frac{x}{2^n}\Big)\Big\|_Y \le K2^n\varphi_1\Big(\frac{x}{2^{n+1}}\Big) \tag{3.32}$$

for all $x \in X$ and all non-negative integers n. Since Y is a p-Banach space,

$$\begin{aligned} \Big\|2^{n+1}g\Big(\frac{x}{2^{n+1}}\Big) - 2^m g\Big(\frac{x}{2^m}\Big)\Big\|_Y^p &\le \sum_{i=m}^{n}\Big\|2^{i+1}g\Big(\frac{x}{2^{i+1}}\Big) - 2^i g\Big(\frac{x}{2^i}\Big)\Big\|_Y^p \\ &\le K^p\sum_{i=m}^{n} 2^{ip}\varphi_1^p\Big(\frac{x}{2^{i+1}}\Big) \end{aligned} \tag{3.33}$$

for all $x \in X$ and all non-negative integers m and n with $n \ge m$. Since $0 < p \le 1$, then by Lemma 3.1, we get

$$\varphi_1^p(x) \le \varphi^p(x, x, x, x) + 4^p\varphi^p(x, x, x, -x) \tag{3.34}$$

for all $x \in X$. Therefore it follows from (3.22), (3.23) and (3.34) that

$$\sum_{i=1}^{\infty} 2^{ip}\varphi_1^p\Big(\frac{x}{2^i}\Big) < \infty, \qquad \lim_{n\to\infty} 2^n\varphi_1\Big(\frac{x}{2^n}\Big) = 0 \tag{3.35}$$

for all $x \in X$. Therefore we conclude from (3.33) and (3.35) that the sequence $\{2^n g(\frac{x}{2^n})\}$ is a Cauchy sequence in Y for all $x \in X$. Since Y is complete, the sequence $\{2^n g(\frac{x}{2^n})\}$ converges in Y for all $x \in X$. So one can define the function $A : X \to Y$ by (3.24) for all $x \in X$. Letting

$m = 0$ and passing the limit $n \to \infty$ in (3.33), and using (3.34), we get (3.25). Now, we show that A is additive. It follow from (3.24), (3.32) and (3.35) that

$$\begin{aligned}\|A(2x) - 2A(x)\| &= \lim_{n\to\infty} \left\| 2^n g\Big(\frac{x}{2^{n-1}}\Big) - 2^{n+1} g\Big(\frac{x}{2^n}\Big)\right\| \\ &= 2\lim_{n\to\infty} \left\| 2^{n-1} g\Big(\frac{x}{2^{n-1}}\Big) - 2^{n} g\Big(\frac{x}{2^n}\Big)\right\| \\ &\le K \lim_{n\to\infty} 2^n \varphi_1\Big(\frac{x}{2^n}\Big) = 0\end{aligned}$$

for all $x \in X$. Therefore

$$A(2x) = 2A(x) \tag{3.36}$$

for all $x \in X$. On the other hand it follows from (3.4), (3.22) and (3.24),

$$\begin{aligned}\|DA(x_1,x_2,x_3,x_4)\|_Y &= \lim_{n\to\infty} 2^n \left\| Dg\Big(\frac{x_1}{2^n},\frac{x_2}{2^n},\frac{x_3}{2^n},\frac{x_4}{2^n}\Big)\right\|_Y \\ &= \lim_{n\to\infty} 2^n \left\{ \left\| Df\Big(\frac{x_1}{2^{n-1}},\frac{x_2}{2^{n-1}},\frac{x_3}{2^{n-1}},\frac{x_4}{2^{n-1}}\Big) - 8Df\Big(\frac{x_1}{2^n},\frac{x_2}{2^n},\frac{x_3}{2},\frac{x_4}{2^n}\Big)\right\|_Y \right\} \\ &\le K \lim_{n\to\infty} 2^n \left\{ \left\| Df\Big(\frac{x_1}{2^{n-1}},\frac{x_2}{2^{n-1}},\frac{x_3}{2^{n-1}},\frac{x_4}{2^{n-1}}\Big)\right\|_Y + 8\left\| Df\Big(\frac{x_1}{2^n},\frac{x_2}{2^n},\frac{x_3}{2},\frac{x_4}{2^n}\Big)\right\|_Y \right\} \\ &\le K \lim_{n\to\infty} 2^n \left\{ \varphi\Big(\frac{x_1}{2^{n-1}},\frac{x_2}{2^{n-1}},\frac{x_3}{2^{n-1}},\frac{x_4}{2^{n-1}}\Big) + 8\varphi\Big(\frac{x_1}{2^n},\frac{x_2}{2^n},\frac{x_3}{2^n},\frac{x_4}{2^n}\Big)\right\} = 0\end{aligned}$$

for all $x_1, x_2, x_3, x_4 \in X$. Therefore the function $A : X \to Y$ satisfies (1.4). Since f is an odd function, then g is odd. So (3.24) implies that the function $A : X \to Y$ is odd. Therefore by Lemma 2.2, the function $x \mapsto A(2x) - 8A(x)$ is additive. So (3.36) implies that the function $A : X \to Y$ is additive.

To prove the uniqueness of A, let $T : X \to Y$ be another additive function satisfying (3.25). Since

$$\begin{aligned}&\lim_{n\to\infty} 2^{np} \sum_{i=1}^{\infty} 2^{ip} \varphi^p\Big(\frac{x}{2^{n+i}},\frac{x}{2^{n+i}},\frac{x}{2^{n+i}},\frac{y}{2^{n+i}}\Big) \\ &= \lim_{n\to\infty} \sum_{i=n+1}^{\infty} 2^{ip} \varphi^p\Big(\frac{x}{2^i},\frac{x}{2^i},\frac{x}{2^i},\frac{y}{2^i}\Big) = 0\end{aligned}$$

for all $x \in X$ and $y \in \{x, -x\}$, then

$$\lim_{n\to\infty} 2^{np} \widetilde{\varphi_a}\Big(\frac{x}{2^n}\Big) = 0 \tag{3.37}$$

for all $x \in X$. It follows from (3.24), (3.25) and (3.37) that

$$\begin{aligned}\|A(x) - T(x)\|_Y^p &= \lim_{n\to\infty} 2^{np} \left\| g\Big(\frac{x}{2^n}\Big) - T\Big(\frac{x}{2^n}\Big)\right\|_Y^p \\ &\le \frac{K^p}{2^p} \lim_{n\to\infty} 2^{np} \widetilde{\varphi_a}\Big(\frac{x}{2^n}\Big) = 0\end{aligned}$$

for all $x \in X$. So $A = T$. ∎

Theorem 3.8. *Let* $\Phi : X^4 \to [0,\infty)$ *be a function such that*

$$\lim_{n\to\infty} \frac{1}{2^n} \Phi\Big(2^n x_1, 2^n x_2, 2^n x_3, 2^n x_4\Big) = 0 \tag{3.38}$$

and

$$\sum_{i=0}^{\infty} \frac{1}{2^{ip}} \Phi^p\Big(2^i x, 2^i x, 2^i x, 2^i y\Big) < \infty \tag{3.39}$$

for all $x, x_1, x_2, x_3, x_4 \in X$ and $y = \pm x$. Suppose that an odd function $f : X \to Y$ satisfies the inequality (3.13) for all $x_1, x_2, x_3, x_4 \in X$. Let $g : X \to Y$ be a function defined by $g(x) = f(2x) - 8f(x)$ for all $x \in X$. Then the limit

$$A(x) := \lim_{n\to\infty} \frac{1}{2^n} g\big(2^n x\big) \tag{3.40}$$

exists for all $x \in X$ and $A : X \to Y$ is a unique additive function satisfying

$$\|f(2x) - 8f(x) - A(x)\|_Y \le \frac{K}{2}[\widetilde{\Phi_a}(x)]^{\frac{1}{p}} \tag{3.41}$$

for all $x \in X$, where

$$\widetilde{\Phi_a}(x) := \sum_{i=0}^{\infty} \frac{1}{2^{ip}} \Big\{ \Phi^p\big(2^i x, 2^i x, 2^i x, 2^i x\big) + 4^p \Phi^p\big(2^i x, 2^i x, 2^i x, -2^i x\big) \Big\} \tag{3.42}$$

Proof. Similar to the proof of Theorem 3.7, we infer that

$$\Big\| \frac{1}{2^{n+1}} g\big(2^{n+1}x\big) - \frac{1}{2^n} g\big(2^n x\big) \Big\|_Y \le \frac{K}{2^{n+1}} \Phi_1\big(2^n x\big) \tag{3.43}$$

for all $x \in X$ and all non-negative integers n, where

$$\Phi_1(x) = \Phi(x, x, x, x) + 4\Phi(x, x, x, -x). \tag{3.44}$$

By Lemma 3.1, it follows from (3.38) and (3.39) that

$$\sum_{i=0}^{\infty} \frac{1}{2^{ip}} \Phi_1^p\big(2^i x\big) < \infty, \qquad \lim_{n\to\infty} \frac{1}{2^n} \Phi_1\big(2^n x\big) = 0 \tag{3.45}$$

for all $x \in X$. Since Y is a p-Banach space,

$$\begin{aligned} \Big\| \frac{1}{2^{n+1}} g\big(2^{n+1}x\big) - \frac{1}{2^m} g\big(2^m x\big) \Big\|_Y^p &\le \sum_{i=m}^{n} \Big\| \frac{1}{2^{i+1}} g\big(2^{i+1}x\big) - \frac{1}{2^i} g\big(2^i x\big) \Big\|_Y^p \\ &\le \Big(\frac{K}{2}\Big)^p \sum_{i=m}^{n} \frac{1}{2^{ip}} \Phi_1^p\big(2^i x\big) \end{aligned} \tag{3.46}$$

for all $x \in X$ and all non-negative integers m and n with $n \ge m$. Therefore we conclude from (3.45) and (3.46) that the sequence $\{\frac{1}{2^n} g(2^n x)\}$ is a Cauchy sequence in Y for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{2^n} g(2^n x)\}$ converges in Y for all $x \in X$. So one can define the function $A : X \to Y$ by (3.40) for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 3.7 ∎

Corollary 3.9. *Let θ be a non-negative real number. Suppose that an odd function $f : X \to Y$ satisfies the inequality (3.19) for all $x_1, x_2, x_3, x_4 \in X$. Then there exists a unique additive function $A : X \to Y$ satisfies*

$$\|f(2x) - 8f(x) - A(x)\|_Y \le K\theta \Big(\frac{4^p + 1}{2^p - 1}\Big)^{\frac{1}{p}}$$

for all $x \in X$.

Proof. The result follows by Theorem 3.8. ∎

Corollary 3.10. *Let $\theta, \{r_i\}_{i\in J}$ be non-negative real numbers such that $r_i > 1\,(0 < r_i < 1)$ for all $i \in J$, where J is a non-empty subset of $\{1, 2, 3, 4\}$. Suppose that an odd function*

$f : X \to Y$ satisfies the inequality (3.20) for all $x_1, x_2, x_3, x_4 \in X$. Then there exists a unique additive function $A : X \to Y$ satisfies

$$\|f(2x) - 8f(x) - A(x)\|_Y \le K\theta\Big\{\sum_{i\in J}\frac{4^p+1}{|2^{pr_i} - 2^p|}\|x\|_X^{pr_i}\Big\}^{\frac{1}{p}}$$

for all $x \in X$.

Proof. The result follows by Theorems 3.7 and 3.8. ∎

Corollary 3.11. *Let $\theta, \{r_i\}_{i\in J}$ be non-negative real numbers such that $\lambda = \sum_{i\in J} r_i \in (0,1) \cup (1,+\infty)$, where J is a non-empty subset of $\{1,2,3,4\}$. Suppose that an odd function $f : X \to Y$ satisfies the inequality (3.21) for all $x_1, x_2, x_3, x_4 \in X$. Then there exists a unique additive function $A : X \to Y$ satisfies*

$$\|f(2x) - 8f(x) - A(x)\|_Y \le K\theta\Big(\frac{4^p+1}{|2^{\lambda p} - 2^p|}\Big)^{\frac{1}{p}}\|x\|_X^{\lambda}$$

for all $x \in X$.

3.3. **Part III.** In this part, we find some conditions that there exists a true cubic function near an approximately cubic function.

Theorem 3.12. *Let $\psi : X^4 \to [0,\infty)$ be a function such that*

$$\lim_{n\to\infty} 8^n\psi\Big(\frac{x_1}{2^n}, \frac{x_2}{2^n}, \frac{x_3}{2^n}, \frac{x_4}{2^n}\Big) = 0 \tag{3.47}$$

and

$$\sum_{i=1}^{\infty} 8^{ip}\psi^p\Big(\frac{x}{2^i}, \frac{x}{2^i}, \frac{x}{2^i}, \frac{y}{2^i}\Big) < \infty \tag{3.48}$$

for all $x, x_1, x_2, x_3, x_4 \in X$ and $y = \pm x$. Suppose that an odd function $f : X \to Y$ satisfies the inequality

$$\|Df(x_1, x_2, x_3, x_4)\|_Y \le \psi(x_1, x_2, x_3, x_4) \tag{3.49}$$

for all $x_1, x_2, x_3, x_4 \in X$. Let $h : X \to Y$ be a function defined by $h(x) = f(2x) - 2f(x)$ for all $x \in X$. Then the limit

$$C(x) := \lim_{n\to\infty} 8^n h\Big(\frac{x}{2^n}\Big) \tag{3.50}$$

exists for all $x \in X$ and $C : X \to Y$ is a unique cubic function satisfying

$$\|f(2x) - 2f(x) - C(x)\|_Y \le \frac{K}{8}[\widetilde{\psi_c}(x)]^{\frac{1}{p}} \tag{3.51}$$

for all $x \in X$, where

$$\widetilde{\psi_c}(x) := \sum_{i=1}^{\infty} 8^{ip}\Big\{\psi^p\Big(\frac{x}{2^i}, \frac{x}{2^i}, \frac{x}{2^i}, \frac{x}{2^i}\Big) + 4^p\psi^p\Big(\frac{x}{2^i}, \frac{x}{2^i}, \frac{x}{2^i}, -\frac{x}{2^i}\Big)\Big\} \tag{3.52}$$

Proof. Similar to the proof of Theorem 3.7, we infer that

$$\|h(2x) - 8h(x)\|_Y \le K\psi_1(x) \tag{3.53}$$

for all $x \in X$, where

$$\psi_1(x) = \psi(x,x,x,x) + 4\psi(x,x,x,-x). \tag{3.54}$$

By Lemma 3.1, it follows from (3.47) and (3.48) that

$$\sum_{i=1}^{\infty} 8^{ip}\psi_1^p\Big(\frac{x}{2^i}\Big) < \infty, \qquad \lim_{n\to\infty} 8^n\psi_1\Big(\frac{x}{2^n}\Big) = 0 \tag{3.55}$$

for all $x \in X$. If we replace x in (3.53) by $\frac{x}{2^{n+1}}$ and multiply both sides of (3.53) by 8^n, we get

$$\Big\|8^{n+1}h\Big(\frac{x}{2^{n+1}}\Big) - 8^n h\Big(\frac{x}{2^n}\Big)\Big\|_Y \leq K8^n\psi_1\Big(\frac{x}{2^{n+1}}\Big) \tag{3.56}$$

for all $x \in X$ and all non-negative integers n. Since Y is a p-Banach space,

$$\begin{aligned}\Big\|8^{n+1}h\Big(\frac{x}{2^{n+1}}\Big) - 8^m h\Big(\frac{x}{2^m}\Big)\Big\|_Y^p &\leq \sum_{i=m}^{n}\Big\|8^{i+1}h\Big(\frac{x}{2^{i+1}}\Big) - 8^i h\Big(\frac{x}{2^i}\Big)\Big\|_Y^p \\ &\leq K^p\sum_{i=m}^{n} 8^{ip}\psi_1^p\Big(\frac{x}{2^{i+1}}\Big)\end{aligned} \tag{3.57}$$

for all $x \in X$ and all non-negative integers m and n with $n \geq m$. Therefore we conclude from (3.55) and (3.57) that the sequence $\{8^n h(\frac{x}{2^n})\}$ is a Cauchy sequence in Y for all $x \in X$. Since Y is complete, the sequence $\{8^n h(\frac{x}{2^n})\}$ converges in Y for all $x \in X$. So one can define the function $C : X \to Y$ by (3.50) for all $x \in X$. Letting $m = 0$ and passing the limit $n \to \infty$ in (3.57), we get (3.51). Now, we show that C is cubic. It follow from (3.50), (3.55) and (3.56) that

$$\begin{aligned}\|C(2x) - 8C(x)\| &= \lim_{n\to\infty}\Big\|8^n h\Big(\frac{x}{2^{n-1}}\Big) - 8^{n+1}h\Big(\frac{x}{2^n}\Big)\Big\| \\ &= 8\lim_{n\to\infty}\Big\|8^{n-1}h\Big(\frac{x}{2^{n-1}}\Big) - 8^n h\Big(\frac{x}{2^n}\Big)\Big\| \\ &\leq K\lim_{n\to\infty} 8^n\psi_1\Big(\frac{x}{2^n}\Big) = 0\end{aligned}$$

for all $x \in X$. Therefore

$$C(2x) = 8C(x) \tag{3.58}$$

for all $x \in X$. On the other hand it follows from (3.47), (3.49) and (3.50),

$$\begin{aligned}\|DC(x_1,x_2,x_3,x_4)\|_Y &= \lim_{n\to\infty} 8^n\Big\|Dh\Big(\frac{x_1}{2^n},\frac{x_2}{2^n},\frac{x_3}{2^n},\frac{x_4}{2^n}\Big)\Big\|_Y \\ &= \lim_{n\to\infty} 8^n\Big\{\Big\|Df\Big(\frac{x_1}{2^{n-1}},\frac{x_2}{2^{n-1}},\frac{x_3}{2^{n-1}},\frac{x_4}{2^{n-1}}\Big) - 2Df\Big(\frac{x_1}{2^n},\frac{x_2}{2^n},\frac{x_3}{2},\frac{x_4}{2^n}\Big)\Big\|_Y\Big\} \\ &\leq K\lim_{n\to\infty} 8^n\Big\{\Big\|Df\Big(\frac{x_1}{2^{n-1}},\frac{x_2}{2^{n-1}},\frac{x_3}{2^{n-1}},\frac{x_4}{2^{n-1}}\Big)\Big\|_Y + 2\Big\|Df\Big(\frac{x_1}{2^n},\frac{x_2}{2^n},\frac{x_3}{2},\frac{x_4}{2^n}\Big)\Big\|_Y\Big\} \\ &\leq K\lim_{n\to\infty} 8^n\Big\{\psi\Big(\frac{x_1}{2^{n-1}},\frac{x_2}{2^{n-1}},\frac{x_3}{2^{n-1}},\frac{x_4}{2^{n-1}}\Big) + 2\psi\Big(\frac{x_1}{2^n},\frac{x_2}{2^n},\frac{x_3}{2^n},\frac{x_4}{2^n}\Big)\Big\} = 0\end{aligned}$$

for all $x_1, x_2, x_3, x_4 \in X$. Therefore the function $C : X \to Y$ satisfies (1.4). Since f is an odd function, then h is odd. So (3.50) implies that the function $C : X \to Y$ is odd. Therefore by Lemma 2.3, the function $x \mapsto C(2x) - 2C(x)$ is cubic. So (3.58) implies that the function $C : X \to Y$ is cubic.

To prove the uniqueness of C, let $T : X \to Y$ be another cubic function satisfying (3.51). Since

$$\lim_{n\to\infty} 8^{np} \sum_{i=1}^{\infty} 8^{ip}\psi^p\Big(\frac{x}{2^{n+i}}, \frac{x}{2^{n+i}}, \frac{x}{2^{n+i}}, \frac{y}{2^{n+i}}\Big) = \lim_{n\to\infty} \sum_{i=n+1}^{\infty} 8^{ip}\psi^p\Big(\frac{x}{2^{i}}, \frac{x}{2^{i}}, \frac{x}{2^{i}}, \frac{y}{2^{i}}\Big) = 0$$

for all $x \in X$ and $y \in \{x, -x\}$, then

(3.59) $$\lim_{n\to\infty} 8^{np}\widetilde{\psi}_c\big(\frac{x}{2^n}\big) = 0$$

for all $x \in X$. It follows from (3.50), (3.51) and (3.59) that

$$\|C(x) - T(x)\|_Y^p = \lim_{n\to\infty} 8^{np}\Big\|h\Big(\frac{x}{2^n}\Big) - T\Big(\frac{x}{2^n}\Big)\Big\|_Y^p \leq \frac{K^p}{8^p}\lim_{n\to\infty} 8^{np}\widetilde{\psi}_c\Big(\frac{x}{2^n}\Big) = 0$$

for all $x \in X$. So $C = T$. ∎

Theorem 3.13. *Let* $\Psi : X^4 \to [0, \infty)$ *be a function such that*

(3.60) $$\lim_{n\to\infty} \frac{1}{8^n}\Psi(2^n x_1, 2^n x_2, 2^n x_3, 2^n x_4) = 0$$

and

(3.61) $$\sum_{i=0}^{\infty} \frac{1}{8^{ip}}\Psi^p(2^i x, 2^i x, 2^i x, 2^i y) < \infty$$

for all $x, x_1, x_2, x_3, x_4 \in X$ *and* $y = \pm x$. *Suppose that an odd function* $f : X \to Y$ *satisfies the inequality*

(3.62) $$\|Df(x_1, x_2, x_3, x_4)\|_Y \leq \Psi(x_1, x_2, x_3, x_4)$$

for all $x_1, x_2, x_3, x_4 \in X$. *Let* $h : X \to Y$ *be a function defined by* $h(x) = f(2x) - 2f(x)$ *for all* $x \in X$. *Then the limit*

(3.63) $$C(x) := \lim_{n\to\infty} \frac{1}{8^n}h(2^n x)$$

exists for all $x \in X$ *and* $C : X \to Y$ *is a unique cubic function satisfying*

(3.64) $$\|f(2x) - 2f(x) - C(x)\|_Y \leq \frac{K}{8}[\widetilde{\Psi}_c(x)]^{\frac{1}{p}}$$

for all $x \in X$, *where*

(3.65) $$\widetilde{\Psi}_c(x) := \sum_{i=0}^{\infty} \frac{1}{8^{ip}}\Big\{\Psi^p(2^i x, 2^i x, 2^i x, 2^i x) + 4^p\Psi^p(2^i x, 2^i x, 2^i x, -2^i x)\Big\}$$

Proof. Similar to the proof of Theorem 3.12, we infer that

(3.66) $$\Big\|\frac{1}{8^{n+1}}h(2^{n+1}x) - \frac{1}{8^n}h(2^n x)\Big\|_Y \leq \frac{K}{8^{n+1}}\Psi_1(2^n x)$$

for all $x \in X$ and all non-negative integers n, where

(3.67) $$\Psi_1(x) = \Psi(x, x, x, x) + 4\Psi(x, x, x, -x).$$

By Lemma 3.1, it follows from (3.60) and (3.61) that

$$\sum_{i=1}^{\infty} \frac{1}{8^{ip}} \Psi_1^p(2^i x) < \infty, \qquad \lim_{n\to\infty} \frac{1}{8^n} \Psi_1(2^n x) = 0 \tag{3.68}$$

for all $x \in X$. Since Y is a p-Banach space,

$$\begin{aligned} \Big\| \frac{1}{8^{n+1}} h(2^{n+1}x) - \frac{1}{8^m} h(2^m x) \Big\|_Y^p &\le \sum_{i=m}^{n} \Big\| \frac{1}{8^{i+1}} h(2^{i+1}x) - \frac{1}{8^i} h(2^i x) \Big\|_Y^p \\ &\le \Big(\frac{K}{8}\Big)^p \sum_{i=m}^{n} \frac{1}{8^{ip}} \Psi_1^p(2^i x) \end{aligned} \tag{3.69}$$

for all $x \in X$ and all non-negative integers m and n with $n \ge m$. Therefore we conclude from (3.68) and (3.69) that the sequence $\{\frac{1}{8^n} h(2^n x)\}$ is a Cauchy sequence in Y for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{8^n} h(2^n x)\}$ converges in Y for all $x \in X$. So one can define the function $C : X \to Y$ by (3.63) for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 3.12 ∎

Corollary 3.14. *Let θ be a non-negative real number. Suppose that an odd function $f : X \to Y$ satisfies the inequality (3.19) for all $x_1, x_2, x_3, x_4 \in X$. Then there exists a unique cubic function $C : X \to Y$ satisfies*

$$\|f(2x) - 2f(x) - C(x)\|_Y \le K\theta \Big(\frac{4^p + 1}{8^p - 1}\Big)^{\frac{1}{p}}$$

for all $x \in X$.

Proof. The result follows by Theorem 3.13. ∎

Corollary 3.15. *Let $\theta, \{r_i\}_{i\in J}$ be non-negative real numbers such that $r_i > 3\,(0 < r_i < 3)$ for all $i \in J$, where J is a non-empty subset of $\{1, 2, 3, 4\}$. Suppose that an odd function $f : X \to Y$ satisfies the inequality (3.20) for all $x_1, x_2, x_3, x_4 \in X$. Then there exists a unique cubic function $C : X \to Y$ satisfies*

$$\|f(2x) - 2f(x) - C(x)\|_Y \le K\theta \Big\{ \sum_{i\in J} \frac{4^p + 1}{|2^{pr_i} - 8^p|} \|x\|_X^{pr_i} \Big\}^{\frac{1}{p}}$$

for all $x \in X$.

Proof. The result follows by Theorems 3.12 and 3.13. ∎

Corollary 3.16. *Let $\theta, \{r_i\}_{i\in J}$ be non-negative real numbers such that $\lambda = \sum_{i\in J} r_i \in (0, 3) \cup (3, +\infty)$, where J is a non-empty subset of $\{1, 2, 3, 4\}$. Suppose that an odd function $f : X \to Y$ satisfies the inequality (3.21) for all $x_1, x_2, x_3, x_4 \in X$. Then there exists a unique cubic function $C : X \to Y$ satisfies*

$$\|f(2x) - 2f(x) - C(x)\|_Y \le K\theta \Big(\frac{4^p + 1}{|2^{\lambda p} - 8^p|}\Big)^{\frac{1}{p}} \|x\|_X^{\lambda}$$

for all $x \in X$.

3.4. **Part IV.** In this part, we give our main results. We find some conditions that there exist a true cubic function, a true quadratic function, and a true additive function near an approximately linear combination of cubic, quadratic and additive functions.

Theorem 3.17. *Let* $\Theta : X^4 \to [0,\infty)$ *be a function such that*

$$\lim_{n\to\infty} 8^n \Theta\Big(\frac{x_1}{2^n}, \frac{x_2}{2^n}, \frac{x_3}{2^n}, \frac{x_4}{2^n}\Big) = 0 \tag{3.70}$$

and

$$\sum_{i=1}^{\infty} 8^{ip}\Theta^p\Big(\frac{x}{2^i}, \frac{x}{2^i}, \frac{x}{2^i}, \frac{y}{2^i}\Big) < \infty, \qquad \sum_{i=1}^{\infty} 8^{ip}\Theta^p\Big(\frac{x}{2^i}, \frac{x}{2^i}, \frac{-x}{2^i}, \frac{-x}{2^i}\Big) < \infty \tag{3.71}$$

for all $x, x_1, x_2, x_3, x_4 \in X$ *and* $y = \pm x$. *Suppose that a function* $f : X \to Y$ *satisfies the inequality*

$$\|Df(x_1,x_2,x_3,x_4)\|_Y \le \Theta(x_1,x_2,x_3,x_4) \tag{3.72}$$

for all $x_1, x_2, x_3, x_4 \in X$. *Then there exist a unique cubic function* $C : X \to Y$, *a unique quadratic function* $Q : X \to Y$, *and a unique additive function* $A : X \to Y$ *such that*

$$\begin{aligned} &\|f(x) - C(x) - Q(x) - A(x)\|_Y \\ &\le \frac{K^2}{96}\Big\{K^2[L(x)]^{\frac{1}{p}} + 4K^2[M(x)]^{\frac{1}{p}} + 6[N(x)]^{\frac{1}{p}}\Big\} \end{aligned} \tag{3.73}$$

for all $x \in X$, *where*

$$\begin{aligned} \Gamma(x_1,x_2,x_3,x_4) &:= \Theta^p(x_1,x_2,x_3,x_4) + \Theta^p(-x_1,-x_2,-x_3,-x_4) \\ L(x) &:= \sum_{i=1}^{\infty} 8^{ip}\Big\{\Gamma\big(\frac{x}{2^i},\frac{x}{2^i},\frac{x}{2^i},\frac{x}{2^i}\big) + 4^p\Gamma\big(\frac{x}{2^i},\frac{x}{2^i},\frac{x}{2^i},-\frac{x}{2^i}\big)\Big\} \\ M(x) &:= \sum_{i=1}^{\infty} 2^{ip}\Big\{\Gamma\big(\frac{x}{2^i},\frac{x}{2^i},\frac{x}{2^i},\frac{x}{2^i}\big) + 4^p\Gamma\big(\frac{x}{2^i},\frac{x}{2^i},\frac{x}{2^i},-\frac{x}{2^i}\big)\Big\} \\ N(x) &:= \sum_{i=1}^{\infty} 4^{ip}\Gamma\big(\frac{x}{2^i},\frac{x}{2^i},-\frac{x}{2^i},-\frac{x}{2^i}\big) \end{aligned}$$

for all $x, x_1, x_2, x_3, x_4 \in X$.

Proof. Let f_e and f_0 be the even and the odd part of f, respectively. It follows from (3.72) that

$$\|Df_e(x_1,x_2,x_3,x_4)\|_Y \le \frac{K}{2}\Big[\Theta(x_1,x_2,x_3,x_4) + \Theta(-x_1,-x_2,-x_3,-x_4)\Big] \tag{3.74}$$

$$\|Df_o(x_1,x_2,x_3,x_4)\|_Y \le \frac{K}{2}\Big[\Theta(x_1,x_2,x_3,x_4) + \Theta(-x_1,-x_2,-x_3,-x_4)\Big] \tag{3.75}$$

for all $x_1, x_2, x_3, x_4 \in X$. For convenience, let

$$\Lambda(x_1,x_2,x_3,x_4) := \frac{K}{2}\Big[\Theta(x_1,x_2,x_3,x_4) + \Theta(-x_1,-x_2,-x_3,-x_4)\Big]$$

for all $x_1, x_2, x_3, x_4 \in X$. By Lemma 3.1, it follows from (3.70) and (3.71) that

$$\lim_{n\to\infty} 8^n \Lambda\Big(\frac{x_1}{2^n}, \frac{x_2}{2^n}, \frac{x_3}{2^n}, \frac{x_4}{2^n}\Big) = 0 \tag{3.76}$$

and

$$\sum_{i=1}^{\infty} 8^{ip}\Lambda^p\Big(\frac{x}{2^i}, \frac{x}{2^i}, \frac{x}{2^i}, \frac{y}{2^i}\Big) < \infty, \qquad \sum_{i=1}^{\infty} 4^{ip}\Lambda^p\Big(\frac{x}{2^i}, \frac{x}{2^i}, -\frac{x}{2^i}, -\frac{x}{2^i}\Big) < \infty \tag{3.77}$$

for all $x, x_1, x_2, x_3, x_4 \in X$ and $y = \pm x$. Therefore by Theorems 3.2, 3.7 and 3.12, there exist a unique quadratic function $Q : X \to Y$, a unique additive function $A_1 : X \to Y$, and a unique cubic function $C_1 : X \to Y$ such that

$$A_1(x) = \lim_{n\to\infty} 2^n g\Big(\frac{x}{2^n}\Big), \quad Q(x) = \lim_{n\to\infty} 4^n f\Big(\frac{x}{2^n}\Big), \quad C_1(x) = \lim_{n\to\infty} 8^n h\Big(\frac{x}{2^n}\Big),$$

$$\|f_e(x) - Q(x)\|_Y \le \frac{1}{8}[\widetilde{\Lambda_e}(x)]^{\frac{1}{p}} \tag{3.78}$$

$$\|g(x) - A_1(x)\|_Y \le \frac{K}{2}[\widetilde{\Lambda_a}(x)]^{\frac{1}{p}} \tag{3.79}$$

$$\|h(x) - C_1(x)\|_Y \le \frac{K}{8}[\widetilde{\Lambda_c}(x)]^{\frac{1}{p}} \tag{3.80}$$

for all $x \in X$, where

$$g(x) = f_o(2x) - 8f_o(x), \qquad h(x) = f_o(2x) - 2f_o(x),$$

$$\widetilde{\Lambda_e}(x) := \sum_{i=1}^{\infty} 4^{ip}\Lambda^p\Big(\frac{x}{2^i}, \frac{x}{2^i}, -\frac{x}{2^i}, -\frac{x}{2^i}\Big) \tag{3.81}$$

$$\widetilde{\Lambda_a}(x) := \sum_{i=1}^{\infty} 2^{ip}\Big\{\Lambda^p\Big(\frac{x}{2^i}, \frac{x}{2^i}, \frac{x}{2^i}, \frac{x}{2^i}\Big) + 4^p\Lambda^p\Big(\frac{x}{2^i}, \frac{x}{2^i}, \frac{x}{2^i}, -\frac{x}{2^i}\Big)\Big\} \tag{3.82}$$

$$\widetilde{\Lambda_c}(x) := \sum_{i=1}^{\infty} 8^{ip}\Big\{\Lambda^p\Big(\frac{x}{2^i}, \frac{x}{2^i}, \frac{x}{2^i}, \frac{x}{2^i}\Big) + 4^p\Lambda^p\Big(\frac{x}{2^i}, \frac{x}{2^i}, \frac{x}{2^i}, -\frac{x}{2^i}\Big)\Big\}. \tag{3.83}$$

It follows from (3.78), (3.79) and (3.80) that

$$\begin{aligned} &\Big\|f(x) - \frac{1}{6}C_1(x) - Q(x) + \frac{1}{6}A_1(x)\Big\|_Y \\ &\le \frac{K}{48}\Big\{6[\widetilde{\Lambda_e}(x)]^{\frac{1}{p}} + 4K^2[\widetilde{\Lambda_a}(x)]^{\frac{1}{p}} + K^2[\widetilde{\Lambda_c}(x)]^{\frac{1}{p}}\Big\} \end{aligned} \tag{3.84}$$

for all $x \in X$. Therefore we obtain (3.73) by Lemma 3.1 and letting $C(x) = \frac{1}{6}C_1(x)$ and $A(x) = -\frac{1}{6}A_1(x)$ for all $x \in X$.

To prove the uniqueness of C, Q, A, let $C_0, Q_0, A_0 : X \to Y$ be another cubic, quadratic and additive functions, respectively, satisfying (3.73). It follows from (3.71) that

$$\lim_{n\to\infty} 8^{np}L\big(\frac{x}{2^n}\big) = \lim_{n\to\infty} 2^{np}M\big(\frac{x}{2^n}\big) = \lim_{n\to\infty} 4^{np}N\big(\frac{x}{2^n}\big) = \lim_{n\to\infty} 8^{np}N_1\big(\frac{x}{2^n}\big) = 0$$

for all $x \in X$, where

$$N_1(x) := \sum_{i=1}^{\infty} 8^{ip}\Gamma\big(\frac{x}{2^i}, \frac{x}{2^i}, -\frac{x}{2^i}, -\frac{x}{2^i}\big).$$

Let $C' = C - C_0$, $Q' = Q - Q_0$, and $A' = A - A_0$. Therefore we have

$$\begin{aligned} \|C'(x) + Q'(x) + A'(x)\|_Y &\le K\Big\{\|f(x) - C(x) - Q(x) - A(x)\|_Y \\ &\quad + \|f(x) - C_0(x) - Q_0(x) - A_0(x)\|_Y\Big\} \\ &\le \frac{K^3}{48}\Big\{K^2[L(x)]^{\frac{1}{p}} + 4K^2[M(x)]^{\frac{1}{p}} + 6[N(x)]^{\frac{1}{p}}\Big\} \end{aligned} \tag{3.85}$$

for all $x \in X$. Hence

$$\begin{aligned}\lim_{n\to\infty} 2^n \Big\| C'\big(\frac{x}{2^n}\big) + Q'\big(\frac{x}{2^n}\big) + A'\big(\frac{x}{2^n}\big) \Big\|_Y &= 0 \\ \lim_{n\to\infty} 4^n \Big\| C'\big(\frac{x}{2^n}\big) + Q'\big(\frac{x}{2^n}\big) + A'\big(\frac{x}{2^n}\big) \Big\|_Y &= 0\end{aligned}$$

for all $x \in X$. Since A', Q' and C' are additive, quadratic and cubic functions, respectively, then it follows from the last relations that $A' = Q' = 0$. Therefore it follows from (3.85) that

$$\|C'(x)\|_Y \le \frac{K^3}{48}\Big\{5K^2[L(x)]^{\frac{1}{p}} + 6[N_1(x)]^{\frac{1}{p}}\Big\}$$

for all $x \in X$. Since C' is cubic, then $C' = 0$. This proves the uniqueness of A, Q and C. ∎

Theorem 3.18. *Let $\Delta : X^4 \to [0,\infty)$ be a function such that*

$$\lim_{n\to\infty} \frac{1}{2^n}\Delta\big(2^n x_1, 2^n x_2, 2^n x_3, 2^n x_4\big) = 0$$

and

$$\sum_{i=0}^{\infty} \frac{1}{2^{ip}}\Delta^p\big(2^i x, 2^i x, 2^i x, 2^i y\big) < \infty, \quad \sum_{i=0}^{\infty} \frac{1}{2^{ip}}\Delta^p\big(2^i x, 2^i x, -2^i x, -2^i x\big) < \infty$$

for all $x, x_1, x_2, x_3, x_4 \in X$ and $y = \pm x$. Suppose that a function $f : X \to Y$ satisfies the inequality

$$\|Df(x_1, x_2, x_3, x_4)\|_Y \le \Delta(x_1, x_2, x_3, x_4)$$

for all $x_1, x_2, x_3, x_4 \in X$. Then there exist a unique cubic function $C : X \to Y$, a unique quadratic function $Q : X \to Y$, and a unique additive function $A : X \to Y$ such that

$$\begin{aligned}&\Big\| f(x) + \frac{5}{6}f(0) - C(x) - Q(x) - A(x)\Big\|_Y \\ &\le \frac{K^2}{96}\Big\{K^2[\mathbf{L}(x)]^{\frac{1}{p}} + 4K^2[\mathbf{M}(x)]^{\frac{1}{p}} + 6[\mathbf{N}(x)]^{\frac{1}{p}}\Big\}\end{aligned}$$

for all $x \in X$, where

$$\begin{aligned}\Upsilon(x_1, x_2, x_3, x_4) &:= \Delta^p(x_1, x_2, x_3, x_4) + \Delta^p(-x_1, -x_2, -x_3, -x_4) \\ \mathbf{L}(x) &:= \sum_{i=0}^{\infty} \frac{1}{8^{ip}}\Big\{\Upsilon(2^i x, 2^i x, 2^i x, 2^i x) + 4^p\Upsilon(2^i x, 2^i x, 2^i x, -2^i x)\Big\} \\ \mathbf{M}(x) &:= \sum_{i=0}^{\infty} \frac{1}{2^{ip}}\Big\{\Upsilon(2^i x, 2^i x, 2^i x, 2^i x) + 4^p\Upsilon(2^i x, 2^i x, 2^i x, -2^i x)\Big\} \\ \mathbf{N}(x) &:= \sum_{i=0}^{\infty} \frac{1}{4^{ip}}\Upsilon(2^i x, 2^i x, -2^i x, -2^i x)\end{aligned}$$

for all $x, x_1, x_2, x_3, x_4 \in X$.

Proof. Similar to the proof of Theorem 3.17, the result follows from Theorems 3.3, 3.8 and 3.13. ∎

Corollary 3.19. *Let θ be a non-negative real number. Suppose that a function $f : X \to Y$ satisfies the inequality (3.19) for all $x_1, x_2, x_3, x_4 \in X$. Then there exist a unique cubic function*

$C : X \to Y$ and a unique quadratic function $Q : X \to Y$ and a unique additive function $A : X \to Y$ satisfies

$$\|f(x) - C(x) - Q(x) - A(x)\|_Y \\ \leq \frac{K^3\theta}{12}\Big\{K^2\Big(\frac{2(4^p+1)}{8^p-1}\Big)^{\frac{1}{p}} + K^2\Big(\frac{2(4^p+1)}{2^p-1}\Big)^{\frac{1}{p}} + 3\Big(\frac{2}{4^p-1}\Big)^{\frac{1}{p}}\Big\} + \frac{5}{6}K\theta$$

for all $x \in X$.

Proof. It follows from (3.19) that $\|f(0)\|_Y \leq \theta$. Hence the result follows by Theorem 3.18. ∎

Corollary 3.20. *Let $\theta, \{r_i\}_{i\in J}$ be non-negative real numbers such that $r_i > 3\,(0 < r_i < 1)$ for all $i \in J$, where J is a non-empty subset of $\{1,2,3,4\}$. Suppose that a function $f : X \to Y$ satisfies (3.20) for all $x_1, x_2, x_3, x_4 \in X$. Then there exist a unique cubic function $C : X \to Y$ and a unique quadratic function $Q : X \to Y$ and a unique additive function $A : X \to Y$ satisfies*

$$\|f(x) - C(x) - Q(x) - A(x)\|_Y \\ \leq \frac{K^2\theta}{12}\Big\{K^2\Big[\sum_{i\in J}\frac{2(4^p+1)}{|2^{pr_i}-8^p|}\|x\|_X^{pr_i}\Big]^{\frac{1}{p}} + K^2\Big[\sum_{i\in J}\frac{2(4^p+1)}{|2^{pr_i}-2^p|}\|x\|_X^{pr_i}\Big]^{\frac{1}{p}} \\ + 3\Big[\sum_{i\in J}\frac{2}{|2^{pr_i}-4^p|}\|x\|_X^{pr_i}\Big]^{\frac{1}{p}}\Big\}$$

for all $x \in X$.

Proof. It follows from (3.20) that $f(0) = 0$. Hence the result follows by Theorems 3.17 and 3.18. ∎

Corollary 3.21. *Let $\theta, \{r_i\}_{i\in J}$ be non-negative real numbers such that $\lambda = \sum_{i\in J} r_i \in (0,1) \cup (3, +\infty)$, where J is a non-empty subset of $\{1,2,3,4\}$. Suppose that a function $f : X \to Y$ satisfies the inequality (3.21) for all $x_1, x_2, x_3, x_4 \in X$. Then there exist a unique cubic function $C : X \to Y$ and a unique quadratic function $Q : X \to Y$ and a unique additive function $A : X \to Y$ satisfies*

$$\|f(x) - C(x) - Q(x) - A(x)\|_Y \\ \leq \frac{K^2\theta}{12}\Big\{K^2\Big(\frac{2(4^p+1)}{|2^{\lambda p}-8^p|}\Big)^{\frac{1}{p}} + K^2\Big(\frac{2(4^p+1)}{|2^{\lambda p}-2^p|}\Big)^{\frac{1}{p}} + 3\Big(\frac{2}{|2^{\lambda p}-4^p|}\Big)^{\frac{1}{p}}\Big\}\|x\|_X^{\lambda}$$

for all $x \in X$.

Proof. The result follows by Theorems 3.17 and 3.18. ∎

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