The Australian Journal of Mathematical Analysis and Applications

http://ajmaa.org

Volume 5, Issue 2, Article 1, pp. 1-11, 2008

EXISTENCE RESULTS FOR PERTURBED FRACTIONAL DIFFERENTIAL INCLUSIONS

YONG-KUI CHANG

Received 2 March, 2007; accepted 14 December, 2007; published 2 September, 2008.

DEPARTMENT OF MATHEMATICS, LANZHOU JIAOTONG UNIVERSITY, LANZHOU, GANSU 730070, PEOPLE'S REPUBLIC OF CHINA
lzchangyk@163.com

ABSTRACT. This paper is mainly concerned with the following fractional differential inclusions with boundary condition

$$\begin{cases} {}_0^C D_t^{\delta} y(t) \in F(t, y(t)) + G(t, y(t)), t \in J := [0,1], \delta \in (1,2), \\ y(0) = \alpha, y(1) = \beta, \alpha, \beta \neq 0. \end{cases}$$

A sufficient condition is established for the existence of solutions of the above problem by using a fixed point theorem for multivalued maps due to Dhage. Our result is proved under the mixed generalized Lipschitz and Carathéodory conditions.

Key words and phrases: Perturbed fractional differential inclusions, Boundary conditions, Fixed point .

2000 *Mathematics Subject Classification.* 34B15.

ISSN (electronic): 1449-5910

Supported by "Qing Lan" Talent Engineering Funds (QL-05-16A) by Lanzhou Jiaotong University, the Scientific Research Fund of Gansu Provincial Education Department (0704-14).

1. INTRODUCTION

Recently, great attention has been paid to the existence results for fractional differential equation due to its wide applications in engineering, economics and other fields, see for instance [1, 5, 6, 7, 8, 12, 13, 14, 15, 18, 19, 20, 21, 22, 23, 24] and references therein. In particular, Zhang [24] studied the existence of solutions for equation

$$\begin{aligned} {}_0^C D_t^\delta u(t) &= g(t, u(t)), t \in J := [0,1], \delta \in (1,2), \\ u(0) &= \alpha, u(1) = \beta, \alpha, \beta \neq 0 \end{aligned}$$

by using Schauder fixed point theorem, where ${}_0^C D_t^\delta y(t)$ denotes the Caputo's derivative, $g : J \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

On the other hand, realistic problems arising from economics, optimal control and so on can be modeled as differential inclusions, so differential inclusions are widely investigated by many authors, see [3, 4, 9, 10] and references therein.

Motivated by [3, 4, 13, 15, 24], in this paper, we shall consider the existence of solutions for the following perturbed fractional differential inclusions with boundary conditions

$$\begin{aligned} &(1.1) \qquad {}_0^C D_t^\delta y(t) \in F(t, y(t)) + G(t, y(t)), t \in J := [0,1], \delta \in (1,2), \\ &(1.2) \qquad y(0) = \alpha, y(1) = \beta, \alpha, \beta \neq 0, \end{aligned}$$

where ${}_0^C D_t^\delta y(t)$ is the Caputo's derivative, $F, G : J \times \mathbb{R} \to 2^{\mathbb{R}} \backslash \{\emptyset\}$ are multivalued maps. A sufficient condition is established for the existence results of the above problem by using a recent fixed point theorem due to Dhage [11]. Our result is proved under the mixed generalized Lipschitz and Carathéodory conditions.

2. PRELIMINARIES

In this section, we shall introduce some basic definitions, notations and lemmas which are used throughout this paper.

Let $C(J)$ denote a Banach space of continuous functions from J into $\mathbb{R}$ with the norm $\|y\| = \sup_{t \in J}\{|y(t)|\}$.

Let $L^1(J, \mathbb{R})$ be the Banach space of functions $y : J \to \mathbb{R}$ which are Lebesgue integrable and normed by

$$||y||_{L^1} = \int_0^1 |y(t)|dt, \text{ for all } y \in L^1(J, \mathbb{R}).$$

Let $(X, |\cdot|)$ be a Banach space. Then a multivalued map $\Theta : X \to 2^X$ is convex (closed) valued if $\Theta(x)$ is convex (closed) for all $x \in X$. Θ is bounded on bounded sets if $\Theta(\mathbb{B}) = \cup_{x \in \mathbb{B}} \Theta(x)$ is bounded in X for any bounded set $\mathbb{B}$ of X (i.e. $\sup_{x \in \mathbb{B}}\{\sup\{|y| : y \in \Theta(x)\}\} < \infty$).

Θ is called upper semicontinuous (u.s.c.) on X if for each $x_0 \in X$, the set $\Theta(x_0)$ is a nonempty closed subset of X, and if for each open set $\mathbb{B}$ of X containing $\Theta(x_0)$, there exists an open neighborhood $\mathcal{N}$ of x_0 such that $\Theta(\mathcal{N}) \subseteq \mathbb{B}$.

Θ is said to be completely continuous if $\Theta(\mathbb{B})$ is relatively compact for every bounded subset $\mathbb{B}$ of X.

If the multivalued map Θ is completely continuous with nonempty compact values, then Θ is u.s.c. if and only if Θ has a closed graph, i.e.,

$$x_n \to x_*, y_n \to y_*, y_n \in \Theta(x_n) \text{ imply } y_* \in \Theta(x_*).$$

Let $P_{b,cl}(X)$ and $P_{cp,cv}(X)$ denote respectively the classes of all bounded-closed and compact-convex subsets of X. Similarly, $P_{bcc}(X)$ denotes the classes of all bounded, closed and convex subsets of X. For $x \in X$ and $Y, Z \in P_{b,cl}(X)$, we denote by $D(x, Y) = \inf\{\|x - y\| : y \in Y\}$, and $\rho(Y, Z) = \sup_{a \in Y} D(a, Z)$.

Define the function $H : P_{b,cl}(X) \times P_{b,cl}(X) \to \mathbb{R}_+$ by

$$H(Y, Z) = \max\{\rho(Y, Z), \rho(Z, Y)\}.$$

The function H is called a Hausdorff metric on $P_{b,cl}(X)$.

Θ has a fixed point if there is an $x \in X$ such that $x \in \Theta(x)$. For more details on multivalued maps see the books of Deimling [9] and Hu and Papageorgious [16].

Definition 2.1. [10] $\Theta : X \to P_{b,cl}(X)$ be a multi-valued map. Then Θ is called a multi-valued contraction if there exists a constant $k \in (0, 1)$ such that for each $x, y \in X$, we have

$$H(\Theta(x), \Theta(y)) \leq k|x - y|.$$

The constant k is called a contraction constant of N.

Definition 2.2. [24] Caputo's derivative for a function $f : [0, \infty) \to \mathbb{R}$ can be written as

$${}_0^C D_x^s f(x) = \frac{1}{\Gamma(n-s)} \int_0^x \frac{f^n(t)}{(x-t)^{s+1-n}} dt, n - 1 < s < n,$$

where Γ is the gamma function.

Definition 2.3. [18] The Riemann-Liouville fractional integral of order s for a function f is defined as

$$I^s f(x) = \frac{1}{\Gamma(s)} \int_0^x \frac{f(t)}{(x-t)^{1-s}} dt, x > 0, s > 0$$

provided the right side is pointwise defined on $(0, \infty)$.

From the above definitions, we can see that

$${}_0^C D_x^s f(x) = \frac{1}{\Gamma(n-s)} \int_0^x \frac{f^n(t)}{(x-t)^{s+1-n}} dt = I^{n-s} f(x).$$

Definition 2.4. [13, 24] A function $y \in C(J)$ is said to be a solution of (1.1)-(1.2) if there exists $f, g \in L^1(J, \mathbb{R})$ such that $f(t) \in F(t, y(t)), g(t) \in G(t, y(t))$ and

$$\begin{aligned} y(t) &= \alpha + (\beta - \alpha) t + \frac{1}{\Gamma(\delta)} \int_0^t (t-s)^{\delta-1} f(s) ds \\ &\quad - \frac{t}{\Gamma(\delta)} \int_0^1 (1-s)^{\delta-1} f(s) ds \\ &\quad + \frac{1}{\Gamma(\delta)} \int_0^t (t-s)^{\delta-1} g(s) ds \\ &\quad - \frac{t}{\Gamma(\delta)} \int_0^1 (1-s)^{\delta-1} g(s) ds. \end{aligned}$$

The following properties are well known (see [1, 5, 18, 24]).

Lemma 2.1. *Let ϵ, ε be two positive real numbers, then*
(i) $I^\epsilon : L^1(J, \mathbb{R}) \to L^1(J, \mathbb{R})$.
(ii) $I^\epsilon I^\varepsilon f(x) = I^{\epsilon+\varepsilon} f(x), f \in L^1(J, \mathbb{R})$.
(iii) $\lim_{\epsilon \to n} I^\epsilon f(x) = I^n f(x), n = 1, 2, \cdots, I^1 f(x) = \int_0^x f(t) dt$.

Let us list the following hypotheses:

(H1) $F : J \times \mathbb{R} \to P_{bcc}(\mathbb{R})\,; (t, y) \to F(t, y)$ is measurable with respect to t for each $y \in \mathbb{R}$, u.s.c. with respect to y for a.e. $t \in J$, and for each fixed $y \in \mathbb{R}$ the set

$$S_{F,y} := \left\{ f \in L^1(J, \mathbb{R}) : f(t) \in F(t, y) \text{ for a.e. } t \in J \right\}$$

is nonempty.

(H2) There exists a Carathédory function $Q : J \times \mathbb{R}_+ \to \mathbb{R}_+$ which is nondecreasing with respect to its second argument such that

$$\|F(t, y)\| := \sup \{|v| : v(t) \in F(t, y)\} \le Q(t, |y|) \text{ for a.e. } t \in J \text{ and } y \in \mathbb{R}.$$

(H3) The multi-valued map $t \mapsto G(t, y)$ is measurable for each $y \in \mathbb{R}$ and integrally bounded, i.e. there exists a function $M \in L^1(J, \mathbb{R}_+)$ such that

$$\|G(t, y)\| := \sup \{|g| : g(t) \in G(t, y)\} \le M(t) \text{ for a.e. } t \in J \text{ and } y \in \mathbb{R}.$$

(H4) $G : J \times \mathbb{R} \to P_{bcc}(\mathbb{R})$ and there exists a function $l \in L^1(J, \mathbb{R})$ such that

$$H(G(t, x), G(t, y)) \le l(t)|x - y|, t \in J$$

for all $x, y \in \mathbb{R}$ with

$$\frac{2}{\Gamma(\delta)} \|l\|_{L^1} < 1.$$

(H5) There exists a real number $r > 0$ such that

$$\frac{r}{|\alpha| + |\beta - \alpha| + \frac{2}{\Gamma(\delta)} \left[\int_0^1 Q(t, r)dt + \|M\|_{L^1} \right]} > 1.$$

Remark 2.1. It is known from Hu and Papageorgiou [16] that if G is integrablly bounded, then the set $S_{G,y} = \{g \in L^1(J, \mathbb{R}) : g(t) \in G(t, y)\}$ of all integrable selections of G is closed and nonempty.

The following lemmas are of great importance in the proof of our main results.

Lemma 2.2. [11] *Let $B(0, r)$ and $B[0, r]$ denote respectively the open and closed balls in a Banach space E centered at origin and of radius r and let $A : E \to P_{bcc}(E)$ and $B : B[0, r] \to P_{cp,cv}(E)$ be two multi-valued operators satisfying:*

(i) A is a multi-valued contraction, and

(ii) B is upper semicontinuous and completely continuous.

Then either

(a) the operator inclusion $x \in A(x) + B(x)$ has a solution in $B[0, r]$, or

(b) there exists a $u \in E$ with $\|u\| = r$ such that $\lambda u \in A(u) + B(u)$ for some $\lambda > 1$.

Lemma 2.3. [17] *Let I be a compact real interval. Let F be a multivalued map satisfying (H1) and let $\digamma$ be a linear continuous from $L^1(I, \mathbb{R}) \to C(I)$, then the operator*

$$\digamma \circ S_F : C(I) \to P_{bcc}(C(I)), y \mapsto (\digamma \circ S_F)(y) = \digamma(S_{F,y}),$$

is a closed graph operator in $C(I) \times C(I)$.

3. EXISTENCE RESULTS

In this section, we shall present and prove our main result.

In view of Ref. [24], if $y \in C(J)$ is a solution of the problem (1.1)-(1.2), then y satisfies the following inclusions

$$\begin{aligned} y(t) \in\ & \alpha + (\beta - \alpha) t + \frac{1}{\Gamma(\delta)} \int_0^t (t-s)^{\delta-1} F(s, y(s))\, ds \\ & - \frac{t}{\Gamma(\delta)} \int_0^1 (1-s)^{\delta-1} F(s, y(s))\, ds \\ & + \frac{1}{\Gamma(\delta)} \int_0^t (t-s)^{\delta-1} G(s, y(s))\, ds \\ & - \frac{t}{\Gamma(\delta)} \int_0^1 (1-s)^{\delta-1} G(s, y(s))\, ds, t \in J. \end{aligned}$$

Now we define the multivalued maps A and B as follows

$$\begin{aligned} A(y) &= \left\{ h \in C(J) : h(t) = \frac{1}{\Gamma(\delta)} \int_0^t (t-s)^{\delta-1} g(s)\, ds \right. \\ &\quad \left. - \frac{t}{\Gamma(\delta)} \int_0^1 (1-s)^{\delta-1} g(s)\, ds, g \in S_{G,y} \right\}, \end{aligned} \tag{3.1}$$

$$\begin{aligned} B(y) &= \left\{ z \in C(J) : z(t) = \alpha + (\beta - \alpha) t + \frac{1}{\Gamma(\delta)} \int_0^t (t-s)^{\delta-1} f(s)\, ds \right. \\ &\quad \left. - \frac{t}{\Gamma(\delta)} \int_0^1 (1-s)^{\delta-1} f(s)\, ds, f \in S_{F,y} \right\}. \end{aligned} \tag{3.2}$$

We shall prove that the operators A and B satisfy all the conditions of Lemma {refl2.2.

Lemma 3.1. *Assume that (H3)-(H4) hold. Then the operator A defined by (3.1) has bounded, closed and convex values on $C(J)$.*

Proof. From Remark 2.1 and condition (H3), the operator A has closed values. Next we show that A has convex and bounded values.

Step 1. A has convex values.

In fact, if h_1, h_2 belong to $A(y)$, then there exist $g_1, g_2 \in S_{G,y}$ such that, for each $t \in J$, we have

$$h_i(t) = \frac{1}{\Gamma(\delta)} \int_0^t (t-s)^{\delta-1} g_i(s)\, ds - \frac{t}{\Gamma(\delta)} \int_0^1 (1-s)^{\delta-1} g_i(s)\, ds, i = 1, 2.$$

Let $0 \leq \lambda \leq 1$. Then, for each $t \in J$, we have

$$\begin{aligned} & [\lambda h_1 + (1-\lambda) h_2](t) \\ = & \frac{1}{\Gamma(\delta)} \int_0^t (t-s)^{\delta-1} [\lambda g_1(s) + (1-\lambda) g_2(s)]\, ds \\ & - \frac{t}{\Gamma(\delta)} \int_0^1 (1-s)^{\delta-1} [\lambda g_1(s) + (1-\lambda) g_2(s)]\, ds. \end{aligned}$$

Since $S_{G,y}$ is convex (because G has convex values), we obtain $\lambda h_1 + (1-\lambda) h_2 \in A(y)$.

Step 2. A is bounded on bounded sets of $C(J)$.

Let $B_q = \{y \in C(J) : \|y\| \leq q\}$ be a bounded set in $C(J)$. Now, for each $y \in B_q, h \in A(y)$, there exists a function $g \in S_{G,y}$ such that for each $t \in J$,

$$h(t) = \frac{1}{\Gamma(\delta)} \int_0^t (t-s)^{\delta-1} g(s)\, ds - \frac{t}{\Gamma(\delta)} \int_0^1 (1-s)^{\delta-1} g(s)\, ds.$$

Then, by (H3) we have

$$|h(t)| \leq \frac{2}{\Gamma(\delta)} \int_0^1 M(t)dt.$$

Thus, for each $h \in A(B_q)$, we get

$$\|h\| \leq \frac{2}{\Gamma(\delta)} \int_0^1 M(t)dt.$$

i.e. A is bounded on bounded sets of $C(J)$. ∎

Lemma 3.2. *Suppose that (H3)and (H4) are satisfied. Then the operator defined by (3.1) is a contraction operator.*

Proof. Let $y, \overline{y} \in C(J)$ and $h \in A(y)$. Then there exists a function $g \in S_{G,y}$ such that for each $t \in J$,

$$h(t) = \frac{1}{\Gamma(\delta)} \int_0^t (t-s)^{\delta-1} g(s)\, ds - \frac{t}{\Gamma(\delta)} \int_0^1 (1-s)^{\delta-1} g(s)\, ds.$$

From (H4) it follows that

$$H\left(G(t,y), G(t,\overline{y})\right) \leq l(t)|y(t) - \overline{y}(t)|.$$

Hence there exists a function $w \in G(t, \overline{y})$ such that

$$|g(t) - w(t)| \leq l(t)|y(t) - \overline{y}(t)|.$$

Consider the operator $U(t) = S_{G,\overline{y}} \cap W(t)$, where

$$W(t) = \{w : |g(t) - w(t)| \leq l(t)|y(t) - \overline{y}(t)|\}.$$

Since the multivalued operator $U(t)$ is measurable (see [2, Proposition III. 4]), there exists a measurable selection function $\overline{g}(t)$ for U. Thus, $\overline{g}(t) \in G(t, \overline{y})$ and

$$|g(t) - \overline{g}(t)| \leq l(t)|y(t) - \overline{y}(t)|.$$

Define

$$\overline{h}(t) = \frac{1}{\Gamma(\delta)} \int_0^t (t-s)^{\delta-1} \overline{g}(s)\, ds - \frac{t}{\Gamma(\delta)} \int_0^1 (1-s)^{\delta-1} \overline{g}(s)\, ds.$$

It follows that $\overline{h} \in A(\overline{y})$ and

$$\begin{aligned}
&\left|h(t) - \overline{h}(t)\right| \\
\leq\ & \frac{1}{\Gamma(\delta)} \left| \int_0^t (t-s)^{\delta-1} \left[g(s) - \overline{g}(s)\right] ds \right| \\
&+ \frac{t}{\Gamma(\delta)} \left| \int_0^1 (1-s)^{\delta-1} \left[g(s) - \overline{g}(s)\right] ds \right| \\
\leq\ & \frac{2}{\Gamma(\delta)} \int_0^1 |g(s) - \overline{g}(s)|\, ds \\
\leq\ & \frac{2}{\Gamma(\delta)} \int_0^1 l(s)\, |y(s) - \overline{y}(s)|\, ds \\
\leq\ & \frac{2}{\Gamma(\delta)} \|l\|_{L^1} \|y - \overline{y}\|.
\end{aligned}$$

Then
$$\|h-\overline{h}\| \leq \frac{2}{\Gamma(\delta)}\|l\|_{L^1}\|y-\overline{y}\|.$$
From this and the analogous inequality obtained by interchanging the roles of y and $\overline{y}$, we get
$$H(A(y),A(\overline{y})) \leq \frac{2}{\Gamma(\delta)}\|l\|_{L^1}\|y-\overline{y}\|.$$
This shows that A is a multivalued contraction, since $\frac{2}{\Gamma(\delta)}\|l\|_{L^1} < 1$ by (H4). ∎

Lemma 3.3. *Assume that (H1)-(H2) hold. Then the operator defined by (3.2) is completely continuous with convex values.*

Proof. For the sake of convenience, we break the proof into several steps.

Step 1. $B(y)$ is convex for each $y \in C(J)$.

In fact, if z_1, z_2 belong to $B(y)$, then there exist $f_1, f_2 \in S_{F,y}$ such that, for each $t \in J$, we have
$$\begin{aligned} z_i(t) &= \alpha+(\beta-\alpha)t+\frac{1}{\Gamma(\delta)}\int_0^t (t-s)^{\delta-1} f_i(s)\,ds \\ &\quad -\frac{t}{\Gamma(\delta)}\int_0^1 (1-s)^{\delta-1} f_i(s)\,ds,\ i=1,2. \end{aligned}$$
Let $0 \leq \lambda \leq 1$. Then, for each $t \in J$, we have
$$\begin{aligned} &[\lambda z_1+(1-\lambda)z_2](t) \\ =\ & \alpha+(\beta-\alpha)t+\frac{1}{\Gamma(\delta)}\int_0^t (t-s)^{\delta-1}[\lambda f_1(s)+(1-\lambda)f_2(s)]\,ds \\ & -\frac{t}{\Gamma(\delta)}\int_0^1 (1-s)^{\delta-1}[\lambda f_1(s)+(1-\lambda)f_2(s)]\,ds. \end{aligned}$$
Since $S_{F,y}$ is convex (because F has convex values), we obtain $\lambda z_1+(1-\lambda)z_2 \in B(y)$.

Step 2. B is bounded on bounded sets of $C(J)$.

Let $B_q = \{y \in C(J) : \|y\| \leq q\}$ be a bounded set in $C(J)$. Now, for each $y \in B_q$, $z \in B(y)$, there exists a function $f \in S_{F,y}$ such that for each $t \in J$,
$$\begin{aligned} z(t) &= \alpha+(\beta-\alpha)t+\frac{1}{\Gamma(\delta)}\int_0^t (t-s)^{\delta-1} f(s)\,ds \\ &\quad -\frac{t}{\Gamma(\delta)}\int_0^1 (1-s)^{\delta-1} f(s)\,ds, \end{aligned}$$
Then, by (H2) we have
$$\begin{aligned} |z(t)| &\leq |\alpha|+|\beta-\alpha|+\frac{2}{\Gamma(\delta)}\int_0^1 Q(t,|y(t)|)dt \\ &\leq |\alpha|+|\beta-\alpha|+\frac{2}{\Gamma(\delta)}\int_0^1 Q(t,\|y\|)dt \end{aligned}$$
Thus, for each $z \in B(B_q)$, we get
$$\|z\| \leq |\alpha|+|\beta-\alpha|+\frac{2}{\Gamma(\delta)}\int_0^1 Q(t,q)dt.$$
i.e. B is bounded on bounded sets of $C(J)$.

Step 3. B sends bounded sets into equicontinuous sets of $C(J)$.

Let $t', t'' \in J, t' < t''$ and $B_q = \{y \in C(J) : \|y\| \le q\}$ be a bounded set in $C(J)$. If $y \in B_q$ and $z \in B(y)$, then there exists a function $f \in S_{F,y}$ such that for each $t \in J$ we have

$$\begin{aligned} z(t) &= \alpha + (\beta - \alpha) t + \frac{1}{\Gamma(\delta)} \int_0^t (t-s)^{\delta-1} f(s)\, ds \\ &\quad - \frac{t}{\Gamma(\delta)} \int_0^1 (1-s)^{\delta-1} f(s)\, ds. \end{aligned}$$

Thus

$$\begin{aligned} & |z(t'') - z(t')| \\ \le\ & \Big| (\beta-\alpha)(t''-t') + I^\delta f(t'') - I^\delta f(t') \\ & - \frac{t''-t'}{\Gamma(\delta)} \int_0^1 (1-s)^{\delta-1} f(s)\, ds \Big| \\ \le\ & |\beta-\alpha| (t''-t') + \frac{t''-t'}{\Gamma(\delta)} \int_0^1 Q(s,q)\, ds \\ & + \left| \frac{1}{\Gamma(\delta)} \int_0^{t'} \left[(t''-s)^{\delta-1} - (t'-s)^{\delta-1} \right] f(s)\, ds \right| \\ & + \left| \frac{1}{\Gamma(\delta)} \int_{t'}^{t''} (t''-s)^{\delta-1} f(s)\, ds \right| \\ \le\ & |\beta-\alpha| (t''-t') + \frac{t''-t'}{\Gamma(\delta)} \int_0^1 Q(s,q)\, ds \\ & + \frac{1}{\Gamma(\delta)} \int_0^{t'} \left| (t''-s)^{\delta-1} - (t'-s)^{\delta-1} \right| Q(s,q)\, ds \\ & + \frac{1}{\Gamma(\delta)} \int_{t'}^{t''} Q(s,q)\, ds. \end{aligned}$$

The right hand side of the above inequality tends to zero independently of $y \in B_q$ as $t'' \to t'$.

As a consequence of Step 1 to Step 3 together with the Ascoli-Arzela theorem, we can conclude that B is a compact valued map.

Step 4. B has closed graph.

Let $y_n \to y_*, z_n \in B(y_n)$ and $z_n \to z_*$. We need to show that $z_* \in B(y_*)$. The relation $z_n \in B(z_n)$ means that there exists $f_n \in S_{F,y_n}$ such that for each $t \in J$,

$$\begin{aligned} z_n(t) &= \alpha + (\beta - \alpha) t + \frac{1}{\Gamma(\delta)} \int_0^t (t-s)^{\delta-1} f_n(s)\, ds \\ &\quad - \frac{t}{\Gamma(\delta)} \int_0^1 (1-s)^{\delta-1} f_n(s)\, ds. \end{aligned}$$

We must show that there exists $f_* \in S_{F,y_*}$ such that for each $t \in J$,

$$\begin{aligned} z_*(t) &= \alpha + (\beta - \alpha) t + \frac{1}{\Gamma(\delta)} \int_0^t (t-s)^{\delta-1} f_*(s)\, ds \\ &\quad - \frac{t}{\Gamma(\delta)} \int_0^1 (1-s)^{\delta-1} f_*(s)\, ds. \end{aligned}$$

Consider the continuous linear operator

$$\begin{aligned} \digamma &: L^1(J,\mathbb{R}) \to C(J) \\ f &\mapsto \digamma(f)(t) = \frac{1}{\Gamma(\delta)}\int_0^t (t-s)^{\delta-1} f(s)\,ds - \frac{t}{\Gamma(\delta)}\int_0^1 (1-s)^{\delta-1} f(s)\,ds. \end{aligned}$$

Clearly,

$$\|(z_n(t)-\alpha-(\beta-\alpha)t)-(z_*(t)-\alpha-(\beta-\alpha)t)\| \to 0, \text{ as } n\to\infty.$$

From Lemma 2.3 it follows that $\digamma \circ S_F$ is a closed graph operator. Moreover, we have

$$z_n(t)-\alpha-(\beta-\alpha)t \in \digamma(S_{F,y_n}).$$

Since $y_n \to y_*$, Lemma 2.3 implies that

$$z_*(t)-\alpha-(\beta-\alpha)t = \frac{1}{\Gamma(\delta)}\int_0^t (t-s)^{\delta-1} f_*(s)\,ds - \frac{t}{\Gamma(\delta)}\int_0^1 (1-s)^{\delta-1} f_*(s)\,ds$$

for some $f_* \in S_{F,y_*}$.

Therefore, B is a compact multivalued map, u.s.c. with convex closed values. ∎

Theorem 3.4. *Suppose that (H1)-(H5) are satisfied. Then the problem (1.1)-(1.2) admits at least one solution on J.*

Proof. Define an open ball $B(0,r)$ in $C(J)$, where the real number r satisfies the inequality given in condition (H5). As a consequence of Lemmas 3.1-3.3, we can see that the operator A and B satisfy all the conditions of Lemma 2.2. Now, we shall show that the second assertion of Lemma 2.2 is not true. Let $u \in C(J)$ be a possible solution of $\lambda u \in A(u)+B(u)$ for some $\lambda > 1$ with $\|u\| = r$. Then there exist $f_u \in S_{F,u}$ and $g_u \in S_{G,u}$ such that for each $t \in J$ we have

$$\begin{aligned} u(t) &= \frac{1}{\lambda}\left(\alpha+(\beta-\alpha)t+\frac{1}{\Gamma(\delta)}\int_0^t (t-s)^{\delta-1} f_u(s)\,ds - \frac{t}{\Gamma(\delta)}\int_0^1 (1-s)^{\delta-1} f_u(s)\,ds \right. \\ &\quad \left. +\frac{1}{\Gamma(\delta)}\int_0^t (t-s)^{\delta-1} g_u(s))\,ds - \frac{t}{\Gamma(\delta)}\int_0^1 (1-s)^{\delta-1} g_u(s)\,ds\right). \end{aligned}$$

In view of (H2)-(H3) we obtain

$$\begin{aligned} |u(t)| &\le |\alpha|+|\beta-\alpha|+\frac{2}{\Gamma(\delta)}\int_0^1 Q(t,|u|)\,dt+\frac{2}{\Gamma(\delta)}\int_0^1 M(t)\,dt \\ &\le |\alpha|+|\beta-\alpha|+\frac{2}{\Gamma(\delta)}\int_0^1 Q(t,\|u\|)\,dt+\frac{2}{\Gamma(\delta)}\|M\|_{L^1}. \end{aligned}$$

Thus we have

$$\|u\| \le |\alpha|+|\beta-\alpha|+\frac{2}{\Gamma(\delta)}\int_0^1 Q(t,\|u\|)\,dt+\frac{2}{\Gamma(\delta)}\|M\|_{L^1}.$$

Substituting $\|u\| = r$ in the above inequality yields

$$\frac{r}{|\alpha|+|\beta-\alpha|+\frac{2}{\Gamma(\delta)}\left(\int_0^1 Q(t,r)\,dt+\|M\|_{L^1}\right)} \le 1,$$

which contradicts (H5). As a result, the conclusion of (b) in Lemma 2.2 does not hold. Consequently, the conclusion of (a) in Lemma 2.2 implies that the problem (1.1)-(1.2) has at least one solution on J. This ends of the proof. ∎

Let $G \equiv 0$, then the problem (1.1)-(1.2) reduces to the following differential inclusions which was considered in [5]

$$
\begin{aligned}
&(3.3) \qquad & {}_0^C D_t^{\delta} y(t) &\in F(t, y(t)), t \in J := [0,1], \delta \in (1,2), \\
&(3.4) \qquad & y(0) &= \alpha, y(1) = \beta, \alpha\beta \neq 0.
\end{aligned}
$$

Now from Theorem 3.4, we can obtain the following corollary.

Corollary 3.5. *Assume that (H1)-(H2) hold. Suppose further that if there exists a real number* $r > 0$ *such that*

$$\frac{r}{|\alpha| + |\beta - \alpha| + \frac{2}{\Gamma(\delta)} \int_0^1 Q(t,r)dt} > 1.$$

Then the problem (3.3)-(3.4) has at least one solution on J.

Remark 3.1. Let $F(t, y) = \{f(t, y)\}$ in the problem (3.3)-(3.4), where $f : J \times \mathbb{R} \to \mathbb{R}$ is a continuous function. Then Corollary 3.5 gives a new sufficient condition for the corresponding single-valued problem in [24]. And also this corollary presents a new existence theorem for the problem discussed in [5]

REFERENCES

[1] A. BAHAKHANI and V. DAFTARDAR-GEJJI, Existence of positive solutions of nonlinear fractional differential equations, *J. Math. Anal. Appl.*, **278** (2003), pp. 434-442.

[2] C. CASTAING and M. VALADIER, Convex analysis and measurable multifunctions, *Lecture Notes in Mathematics*, Vol. 580, Springer Verlag, Berlin-Heidelberg-New York, 1977.

[3] Y. K. CHANG and W. T. LI, Existence results for second order impulsive functional differential inclusions, *J. Math. Anal. Appl.*, **301** (2005), pp. 477-490.

[4] Y. K. CHANG and L. M. QI, Existence results for second order impulsive functional differential inclusions, *J. Appl. Math. Stochastic Anal.*, **2006** (2006), pp. 1-12.

[5] Y. K. CHANG and JUAN. J. NITEO, Some new existence results for fractional differential inclusions with boundary conditions, *Math. Comput. Modelling*, in press.

[6] V. DAFTARDAR-GEJJI and A. BABAKHANI, Analysis of a system of fractional differential equations, *J. Math. Anal. Appl.*, **293** (2004), pp. 511-522.

[7] V. DAFTARDAR-GEJJI, Positive solutions of a system of non-autonomous fractional differential equations, *J. Math. Anal. Appl.*, **302** (2005), pp. 56-64.

[8] D. DELBOSCO and L. RODINO, Existence and uniqueness for a nonlinear fractional differential equations, *J. Math. Anal. Appl.*, **204** (1996), pp. 609-625.

[9] K. DEIMLING, *Multivalued Differential Equations*, De Gruyter, Berlin, 1992.

[10] B. C. DHAGE, A. BOUCHERIF and S. K. NTOUYAS, On periodic boundary value problems of first-order perturbed impulsive differential inclusions, *Electron. J. Differential Equations*, **2004** (84) (2004), pp. 1-9.

[11] B. C. DHAGE, Multi-valued mappings and fixed points II, *Tamkang J. Math.*, **37** (2006), pp. 27-46.

[12] A. M. A. EI-SAYED, On the fractional differential equations, *Appl. Math. Comput.*, **49** (1992), pp. 205-213.

[13] A. M. A EI-SAYED and A.-G. IBRAHIM, Multivalued fractional differential equations, *Appl. Math. Comput.*, **68** (1995), pp. 15-25.

[14] A. M. A EI-SAYED, Nonlinear functional differential equations of arbitrary order, *Nonlinear Anal. TMA*, **33** (1998), pp. 181-186.

[15] A. M. A EI-SAYED and A.-G. IBRAHIM, Set-valued integral equations of fractional-orders, *Appl. Math. Comput.*, **118** (2001), pp. 113-121.

[16] S. HU and N. PAPAGEORGIOU, *Handbook of Multivalued Analysis*, Kluwer, Dordrecht, Boston,1997.

[17] A. LASOTA and Z. OPIAL, An application of the Kakutani-Ky Fan theorem in the theory of ordinary differential equations, *Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys.*, **13** (1965), pp. 781-786.

[18] J. P. SUN and Y. H. ZHAO, Multiplicity of positive solutions of a class of nonlinear fractional differential equations, *Comput. Math. Appl.*, **49** (2005), pp. 73-80.

[19] C. YU and G. GAO, Existence of fractional differential equations, *J. Math. Anal. Appl.*, **310** (2005), pp. 26-29.

[20] C. YU and G. GAO, On the solution of nonlinear fractional order differential equation, *Nonlinear Anal.: Theory, Methods & Applications*, **63** (2005), pp. 971-976.

[21] S. ZHANG, The existence of a positive solution for a nonlinear fractional differential equation, *J. Math. Anal. Appl.*, **252** (2000), pp. 804-812.

[22] S. ZHANG, Existence of positive solution for some class of nonlinear fractional differential equations, *J. Math. Anal. Appl.*, **278** (2003), pp. 136-148.

[23] S. ZHANG, Existence of positive solutions for fractional differential equations with Riemann-Liouville left-sided and right-hand fractional derivatives, *Electron. J. Differential Equations*, **2004** (2004), pp. 1-12.

[24] S. ZHANG, Existence of solution for a boundary value problem of fractional order, *Acta Math. Scientia.*, **26B** (2) (2006) pp. 220-228.