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# EXISTENCE OF BOUNDED SOLUTIONS FOR A CLASS OF STRONGLY NONLINEAR ELLIPTIC EQUATIONS IN ORLICZ-SOBOLEV SPACES

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ABSTRACT. We prove, in the setting of Orlicz-Sobolev spaces, the existence of bounded solutions for some strongly nonlinear elliptic equations with operator of the principal part having degenerate coercivity and lower order terms not satisfying the sign condition. The data have a suitable summability and no $\Delta_2$-condition is needed for the considered N-functions.

*Key words and phrases:* Orlicz-Sobolev spaces, Strongly nonlinear problems, $L^\infty$-estimates, Rearrangements.

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## 1. INTRODUCTION

Let $\Omega$ be a bounded open set of $\mathbb{R}^N$, $N \geq 2$, and $M$ be an N-function. In this paper, we consider a class of strongly nonlinear elliptic equations whose prototype is

$$\begin{cases} A(u) + \beta(u)M(|\nabla u|) = f \ \text{ in } \mathcal{D}'(\Omega), \\ u \in W_0^1L_M(\Omega) \cap L^\infty(\Omega), \end{cases} \tag{1.1}$$

where $A(u) = -\text{div}\left(\overline{M}^{-1}(M(\frac{1}{(1+|u|)^\theta}))a(x,u)\overline{M}^{-1}M(|\nabla u|)\frac{\nabla u}{|\nabla u|}\right)$, $\theta$ is a real such that $0 \leq \theta \leq 1$ and $\beta$ is a positive continuous function which does not satisfy the sign condition (i.e. $\beta(s)s \geq 0$). Existence of bounded solutions for problem (1.1) has been obtained in [8] when $M(t) = t^2$ and $f \in L^m(\Omega)$ with $m > \frac{N}{2}$. We are interested in the more general problem

$$\begin{cases} A(u) + B(x,u,\nabla u) = f & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.2}$$

where

$$A(u) = -\text{div}\, a(x,u,\nabla u)$$

is a Leray-Lions operator defined on its domain $\mathcal{D}(A) \subset W_0^1L_M(\Omega)$ such that

$$a(x,s,\xi)\cdot\xi \geq \overline{M}^{-1}(M(h(|s|)))M(|\xi|), \tag{1.3}$$

and $B$ is a first order term which does not satisfy the sign condition but having only the following natural growth:

$$|B(x,s,\xi)| \leq \beta(s)M(|\xi|).$$

In the setting of Sobolev spaces $W_0^{1,p}(\Omega)$, the existence of bounded solutions for (1.1) when $\beta = 0$ and $M(t) = t^2$ has been proved first in [3] and [6] when

$$f \in L^m(\Omega) \ \text{ with } m > \frac{N}{2},$$

then for (1.2) under the condition (1.3), when $B \equiv 0$ and $M(t) = t^p$ with $1 < p < N$ in [2] when

$$f \in L^m(\Omega) \ \text{ with } m > \frac{N}{p}. \tag{1.4}$$

When the functions $h$ in (1.3) and $\beta$ are constants, the existence of bounded solutions for problems like (1.2) has been obtained in [7], when $f$ can be replaced by $f - \text{div} g$ where $f \in L^m(\Omega)$ with $m > \sup(1, \frac{N}{p})$ and $g \in (L^q(\Omega))^N$ with $q = \frac{pm}{p-1}$.
In this framework, existence of bounded solutions for problems of the type (1.1), when $\theta = 0$, has been proved in [10] with $f - \text{div} g$ as data, when $f$ belongs to $L^m(\Omega)$ with $m > \frac{N}{p}$ and $g$ belongs to $(L^q(\Omega))^N$ with $q > \frac{N}{p-1}$. In this paper, the result has been obtained when $f$ and $g$ are assumed to satisfy a smallness condition.
In the case where $h$ is not necessarily constant, the existence of a bounded solution of problems like (1.2) has been proved in [20] when $f \in L^m(\Omega)$ with $m > \max(1, \frac{N}{p})$.

In the present paper, our main goal is to prove the existence of bounded solutions, in a sense that we will define later, for the problem (1.2) by extending such results obtained in [8, 20] (and also in [2, 3, 6] when $B \equiv 0$) to the setting of the Orlicz-Sobolev spaces when the datum $f$ satisfies a summability condition recovering (1.4) in the case of power growth. For this, we judge important to list some difficulties that we have found in dealing with problem (1.2). First of all, the operator considered in (1.2) does not satisfy the Leray-Lions conditions in the setting of Orlicz spaces (see [14]), this is due to the hypothesis (1.3) and the fact that no bounds are assumed on the function $h$, consequently, classical methods can not be applied. To get rid of this difficulty, we will consider approximate equations in which we introduce a truncation. The second difficulty concerns the lower order term which does not satisfy the well known sign condition (i.e. $B(x,s,\xi)s \geq 0$), and so appears the problem of getting the a priori estimates. To overcome this hindrance, we will use test functions of exponential type and a comparison result.

Our paper is organized as follows. After listing some preliminaries in Section 2, we give the precise assumptions and state the main result in Section 3. In order to get $L^\infty$-estimates for solutions of approximate equations, we need to prove some auxiliary lemmas which will be proved in Section 4. Finally, Section 5 is devoted to the proof of the main result.

## 2. PREREQUISITES

**2.1** Let $M : \mathbb{R}^+ \to \mathbb{R}^+$ be an N-function, ie. M is continuous, convex, with $M(t) > 0$ for $t > 0$, $\frac{M(t)}{t} \to 0$ as $t \to 0$ and $\frac{M(t)}{t} \to \infty$ as $t \to \infty$. The N-function conjugate to $M$ is defined as $\overline{M}(t) = \sup\{st - M(t), s \geq 0\}$. We will extend these N-functions into even functions on all $\mathbb{R}$. We recall that (see [1])

(2.1) $$M(t) \leq t\overline{M}^{-1}(M(t)) \leq 2M(t) \quad \text{for all } t \geq 0$$

and the Young's inequality: for all $s,t \geq 0$, $st \leq \overline{M}(s) + M(t)$ and . If for some $k > 0$,

(2.2) $$M(2t) \leq kM(t) \qquad \text{for all } \ t \geq 0,$$

we said that $M$ satisfies the $\Delta_2$-condition, and if (2.2) holds only for $t$ greater than or equal to $t_0 \geq 0$, then M is said to satisfy the $\Delta_2$-condition near infinity.

Let $P$ and $Q$ be two N-functions. The notation $P \ll Q$ means that $P$ grows essentially less rapidly than $Q$, i.e.

$$\text{for all } \epsilon > 0, \quad \frac{P(t)}{Q(\epsilon t)} \to 0 \quad \text{as } t \to \infty,$$

that is the case if and only if

$$\frac{Q^{-1}(t)}{P^{-1}(t)} \to 0 \quad \text{as } t \to \infty.$$

**2.2** Let $\Omega$ be an open subset of $\mathbb{R}^N$. The Orlicz class $K_M(\Omega)$ ( resp. the Orlicz space $L_M(\Omega)$) is defined as the set of (equivalence class of) real-valued measurable functions $u$ on $\Omega$ such that:

$$\int_\Omega M(u(x))dx < \infty \quad \left(\text{resp.} \int_\Omega M\left(\frac{u(x)}{\lambda}\right) dx < \infty \ \text{ for some } \lambda > 0\right).$$

Endowed with the norm

$$\|u\|_M = \inf\left\{\lambda > 0 : \int_\Omega M\left(\frac{u(x)}{\lambda}\right) dx < \infty\right\},$$

$L_M(\Omega)$ is a Banach space and $K_M(\Omega)$ is a convex subset of $L_M(\Omega)$. We define the Orlicz norm $\|u\|_{(M)}$ by

$$\|u\|_{(M)} = \sup \int_\Omega u(x)v(x)dx,$$

where the supremum is taken over all $v \in E_{\overline{M}}(\Omega)$ such that $\|v\|_{\overline{M}} \leq 1$, for which

$$\|u\|_M \leq \|u\|_{(M)} \leq 2\|u\|_M$$

holds for all $u \in L_M(\Omega)$ (see [16]). The closure in $L_M(\Omega)$ of the set of bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_M(\Omega)$.

**2.3** The Orlicz-Sobolev space $W^1L_M(\Omega)$ (resp. $W^1E_M(\Omega)$) is the space of functions $u$ such that $u$ and its distributional derivatives up to order $1$ lie in $L_M(\Omega)$ (resp. $E_M(\Omega)$). It is a Banach space under the norm

$$\|u\|_{1,M} = \sum_{|\alpha|\leq 1} \|D^\alpha u\|_M.$$

Thus, $W^1L_M(\Omega)$ and $W^1E_M(\Omega)$ can be identified with subspaces of the product of $(N+1)$ copies of $L_M(\Omega)$. Denoting this product by $\Pi L_M$, we will use the weak topologies $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ and $\sigma(\Pi L_M, \Pi L_{\overline{M}})$. The space $W_0^1E_M(\Omega)$ is defined as the norm closure of the Schwartz space $\mathcal{D}(\Omega)$ in $W^1E_M(\Omega)$ and the space $W_0^1L_M(\Omega)$ as the $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ closure of $\mathcal{D}(\Omega)$ in $W^1L_M(\Omega)$.

We say that a sequence $\{u_n\}$ converges to $u$ for the modular convergence in $W^1L_M(\Omega)$ if, for some $\lambda > 0$,

$$\int_\Omega M\left(\frac{D^\alpha u_n - D^\alpha u}{\lambda}\right) dx \to 0 \quad \text{for all } |\alpha| \leq 1;$$

this implies the convergence for $\sigma(\Pi L_M, \Pi L_{\overline{M}})$.
If $M$ satisfies the $\Delta_2$-condition on $\mathbb{R}^+$ (near infinity only if $\Omega$ has finite measure), then the modular convergence coincides with norm convergence. Recall that the norm $\|Du\|_M$ defined on $W_0^1L_M(\Omega)$ is equivalent to $\|u\|_{1,M}$ (see [13]).

Let $W^{-1}L_M(\Omega)$ (resp. $W^{-1}E_M(\Omega)$) denotes the space of distributions on $\Omega$ which can be written as sums of derivatives of order $\leq 1$ of functions in $L_M(\Omega)$ (resp. $E_M(\Omega)$). It is a Banach space under the usual quotient norm. Recall that an open domain $\Omega \subset \mathbb{R}^N$ has the segment property (see [13] p.167) if there exist a locally finite open covering $\{O_i\}$ of the boundary $\partial\Omega$ of $\Omega$ and corresponding vectors $\{y_i\}$ such that if $x \in \overline{\Omega} \cap O_i$ for some $i$, then $x + ty_i \in \Omega$ for $0 < t < 1$. If the open $\Omega$ has the segment property then the space $\mathcal{D}(\Omega)$ is dense in $W_0^1L_M(\Omega)$ for the topology $\sigma(\Pi L_M, \Pi L_{\overline{M}})$ (see [13]). Consequently, the action of a distribution in $W^{-1}L_{\overline{M}}(\Omega)$ on an element of $W_0^1L_M(\Omega)$ is well defined.

For an exhaustive treatment one can see for example [1, 16].

**2.4** We will use the following lemma, (see[9]), which concerns operators of Nemytskii Type in Orlicz spaces. It is slightly different from the analogous one given in [16].

**Lemma 2.1.** *Let $\Omega$ be an open subset of $\mathbb{R}^N$ with finite measure. let $M$, $P$ and $Q$ be N-functions such that $Q \ll P$, and let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that, for a.e.$x \in \Omega$ and for all $s \in \mathbb{R}$,*

$$|f(x,s)| \leq c(x) + k_1 P^{-1} M(k_2|s|),$$

*where $k_1, k_2$ are real constants and $c(x) \in E_Q(\Omega)$. Then the Nemytskii operator $N_f$, defined by $N_f(u)(x) = f(x, u(x))$, is strongly continuous from $\mathcal{P}(E_M, \frac{1}{k_2}) = \{u \in L_M(\Omega) : d(u, E_M(\Omega)) < \frac{1}{k_2}\}$ into $E_Q(\Omega)$.*

We recall here the Orlicz version of the Poincaré's inequality (see lemma 5.7 in [13]).

**Lemma 2.2.** *Let $\Omega$ be a bounded open subset of $\mathbb{R}^N$. Then there exist two constants $\lambda_1$ and $\lambda_2$ such that*

$$\int_\Omega M(|u|)dx \leq \lambda_1 \int_\Omega M(\lambda_2|\nabla u|)dx \tag{2.3}$$

*for all $u \in W_0^1L_M(\Omega)$.*

We will also use the following technical lemma which can be found in [15] or in [9].

**Lemma 2.3.** *If $\{f_n\} \subset L^1(\Omega)$ with $f_n \to f \in L^1(\Omega)$ a.e. in $\Omega$, $f_n, f \geq 0$ a.e. in $\Omega$ and $\int_\Omega f_n(x)dx \to \int_\Omega f(x)dx$, then $f_n \to f$ in $L^1(\Omega)$.*

**2.5** We recall the definition of decreasing rearrangement of a measurable function $w : \Omega \to \mathbb{R}$. If one denotes by $|E|$ the Lebesgue measure of a set $E$, one can define the distribution function $\mu_w(t)$ of $w$ as:

$$\mu_w(t) = |\{x \in \Omega : |w(x)| > t\}|, \qquad t \geq 0.$$

The decreasing rearrangement $w^*$ of $w$ is defined as the generalized inverse function of $\mu_w$:

$$w^*(\sigma) = \inf\{t \geq 0 : \mu_w(t) \leq \sigma\}, \qquad \sigma \in (0, |\Omega|).$$

For every $t \geq 0$ we have

$$w^*(\mu_w(t)) \leq t, \tag{2.4}$$

with equality (see [19] p.935) when $w^*$ is restricted to the range of $\mu_w$ and $\mu_w$ is restricted to the interval $[0, \operatorname{ess\,sup} |w|]$.

More details can be found for example in [5, 17, 18, 19].

**2.6 An abstract existence result.** Let $(Y, Y_0; Z, Z_0)$ be a complementary system i.e. $Y$ and $Z$ are real Banach spaces in duality with respect to a continuous pairing $\langle\, , \rangle$ and $Y_0$ and $Z_0$ are closed subspaces of $Y$ and $Z$ respectively such that, by means of $\langle\, , \rangle$, the dual of $Y_0$ can be identified to $Z$ and that of $Z_0$ to $Y$. We consider a mapping $T$ from $D(T) \subset Y$ into $Z$ which satisfies the following conditions, with respect to some elements $\overline{u} \in Y_0$ and $f \in Z_0$ :

(i)- (Finite continuity) $Y_0 \subset D(T)$ and $T$ is continuous from each finite dimensional subspace of $Y_0$ into $Z$ for $\sigma(Z, Y_0)$,

(ii)- (Sequential pseudo-monotonicity) For any sequence $u_n \in D(T)$ such that $u_n \rightharpoonup u \in Y$ for $\sigma(Y, Z_0)$, $Tu_n \rightharpoonup \chi$ for $\sigma(Z, Y_0)$ and $\limsup \langle Tu_n, u_n\rangle \leq \langle \chi, u\rangle$, it follows that $u \in D(T)$, $Tu = \chi$ and $\langle Tu_n, u_n\rangle \to \langle \chi, u\rangle$.

(iii)- $Tu$ remains bounded in $Z$ whenever $u \in D(T)$ remains bounded in $Y$ and $\langle Tu, u - \overline{u}\rangle$ remains bounded from above,

(iv)- $\langle Tu - f, u - \overline{u}\rangle > 0$ when $u \in D(T)$ has sufficiently large norm in $Y$.

Given a convex set $K \subset Y$ and an element $f \in Z_0$, we are interested in finding a solution $u$ of the variational inequality:

$$(P) \qquad \begin{cases} u \in K \cap D(T), \\ \langle Tu, u - v\rangle \leq \langle f, u - v\rangle \ \ \forall v \in K. \end{cases}$$

Recall the following existence result (see [14, Proposition 1])

**Proposition 2.4.** *Let $(Y, Y_0; Z, Z_0)$ be a complementary system, with $Y_0$ and $Z_0$ separable. Let $K \subset Y$ be a convex, sequentially closed and such that $K \cap Y_0$ is $\sigma(Y, Z)$ dense in $K$. Let $f \in Z_0$. Let $T : D(T) \subset Y \to Z$ satisfy* (i), (ii), (iii) *and* (iv) *with respect to some $\overline{u} \in K \cap Y_0$ and the given $f$. Then, the variational inequality (P) has at least one solution $u$.*

**Remark 2.1. 1.** Notice that when $K \equiv Y$, Proposition 2.4 applies to the solvability of the equation:

$$(P) \qquad \begin{cases} u \in D(T), \\ \langle Tu, v\rangle = \langle f, v\rangle \ \ \forall v \in Y. \end{cases}$$

with $f$ given in $Z_0$.

**2.** It is shown in [14] that if $\Omega$ has the segment property, then

$$(W_0^1 L_M(\Omega), W_0^1 E_M(\Omega), W^{-1} L_{\overline{M}}(\Omega), W^{-1} E_{\overline{M}}(\Omega))$$

constitutes a complementary system.

**3.** Recall that if a bounded subset $\Omega$ of $\mathbb{R}^N$ has a locally Lipschitzian boundary (that is, that each $x$ on the boundary $\partial\Omega$ of $\Omega$ should have a neighborhood $\mathcal{U}_x$ such that $\partial\Omega\cap\mathcal{U}_x$ is the graph of a Lipschitz continuous function) then, $\Omega$ has the segment property (see [1, p.67]).

## 3. ASSUMPTIONS AND MAIN RESULT

Let $\Omega$ be a bounded open subset of $\mathbb{R}^N$, $N\geq 2$, with locally Lipschitzian boundary and $M$ is an N-function twice continuously differentiable and strictly increasing, and $P$ is an N-function such that $P\ll M$. Let $a:\Omega\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}^N$ and $B:\Omega\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}$ be Carathéodory functions satisfying, for a.e. $x\in\Omega$, and for all $s\in\mathbb{R}$ and all $\xi,\eta\in\mathbb{R}^N$, $\xi\neq\eta$,

$$a(x,s,\xi).\xi\geq\overline{M}^{-1}(M(h(|s|)))M(|\xi|) \tag{3.1}$$

where $h:\mathbb{R}^+\to]0,+\infty[$ is a continuous decreasing function such that : $h(0)\leq 1$ and its primitive $H(s)=\displaystyle\int_0^s h(t)dt$ is unbounded,

$$|a(x,s,\xi)|\leq a_0(x)+k_1\overline{P}^{-1}M(k_2|s|)+k_3\overline{M}^{-1}M(k_4|\xi|) \tag{3.2}$$

where $a_0(x)$ belongs to $E_{\overline{M}}(\Omega)$ and $k_1,k_2,k_3,k_4$ to $\mathbb{R}^*_+$,

$$(a(x,s,\xi)-a(x,s,\eta))\cdot(\xi-\eta)>0 \tag{3.3}$$

and

$$|B(x,s,\xi)|\ \leq\ \beta(s)M(|\xi|), \tag{3.4}$$

where $\beta:\mathbb{R}\to\mathbb{R}^+$ is a continuous function such that $t\to\dfrac{\beta(t)}{\overline{M}^{-1}(M(h(|t|)))}$ belongs to $L^1(\mathbb{R})$. So by defining

$$\gamma(s)=\int_0^s\frac{\beta(t)}{\overline{M}^{-1}(M(h(|t|)))}dt$$

for all $s\in\mathbb{R}$, the function $\gamma$ is bounded.
Finally, we assume one of the following two assumptions: Either

$$f\in L^N(\Omega), \tag{3.5}$$

or

$$\begin{cases} f\in L^m(\Omega)\ \text{ with }\ m=\frac{rN}{r+1}\ \text{ for some }\ r>0,\\ \text{and }\ \displaystyle\int_{\cdot}^{+\infty}\left(\frac{t}{M(t)}\right)^r dt<+\infty.\end{cases} \tag{3.6}$$

Let $A$: $D(A)\subset W_0^1L_M(\Omega)\to W^{-1}L_{\overline{M}}(\Omega)$ be a mapping (non-everywhere defined) given by

$$A(u)=-\mathrm{div}\,a(x,u,\nabla u).$$

In this paper, we are interested in proving the existence of bounded solutions to the strongly nonlinear problem:

$$\begin{cases} A(u)+B(x,u,\nabla u)=f & \text{in}\quad\Omega,\\ u=0 & \text{on}\quad\partial\Omega.\end{cases} \tag{3.7}$$

For that, we will use the following concept of solutions

**Definition 3.1.** Let $f\in L^1(\Omega)$, a function $u \in W_0^1L_M(\Omega)$ is said to be a weak solution of problem (3.7), if

$$\int_\Omega a(x,u,\nabla u)\nabla v dx + \int_\Omega B(x,u,\nabla u)v dx = \int_\Omega f v dx \tag{3.8}$$

holds for all $v \in W_0^1L_M(\Omega)\cap L^\infty(\Omega)$.

Now, we state the following theorem which contains the main result of this paper.

**Theorem 3.1.** *Let us assume that (3.1), (3.2), (3.3), (3.4) and either (3.5) or (3.6) hold true. Then, there exists a weak solution $u$ for problem (3.7) in the sense of (3.8), such that $u \in L^\infty(\Omega)$.*

**Remark 3.1.** Notice that when $M(t) = t^p$ with $1 < p < N$, the conditions (3.5) and (3.6) are reduced to

$$f \in L^m(\Omega) \text{ with } m > \frac{N}{p}.$$

Hence, our result is an extension to the Orlicz setting of those in [8] and [20].

**Remark 3.2.** Our result is an extension to strongly nonlinear elliptic equations of that obtained in [21] when $B \equiv 0$.

## 4. A PRIORI ESTIMATES

For $s \in \mathbb{R}$ and $k > 0$ set: $T_k(s) = \max(-k, \min(k,s))$ and $G_k(s) = s - T_k(s)$ and for all $n \in \mathbb{N}$, we define $A_n$ and $B_n$ as

$$A_n(u) := -\operatorname{div} a(x, T_n(u), \nabla u)$$

and

$$B_n(u) := B_n(x,u,\nabla u) = T_n(B(x,u,\nabla u)).$$

In the sequel we denote by $m^*$ either $N$ or $m$ according as we assume (3.5) or (3.6), and let $\{f_n\} \subset W^{-1}E_{\overline{M}}(\Omega)$ be a sequence of smooth functions such that

$$f_n \to f \text{ strongly in } L^{m^*}(\Omega)$$

and

$$\|f_n\|_{m^*} \le \|f\|_{m^*}.$$

Let us show that $A_n+B_n$ satisfies the conditions (i)-(iv) of Proposition 2.4 with respect to $\overline{u} = 0$ and $f_n$.

**(i).** $A_n + B_n$ is finitely continuous by [13, lemma 4.3].

**(ii).** Let $u_j \in D(A_n + B_n)$ such that:

$$(a) \qquad u_j \rightharpoonup u \in W_0^1L_M(\Omega) \text{ for } \sigma(\Pi L_M, \Pi E_{\overline{M}}),$$

$$(b) \qquad A_n(u_j) + B_n(u_j) \rightharpoonup \chi \in W^{-1}L_{\overline{M}}(\Omega) \text{ for } \sigma(\Pi L_{\overline{M}}, \Pi E_M)$$

and

$$(c) \qquad \limsup_j \langle A_n(u_j) + B_n(u_j), u_j\rangle \le \langle \chi, u\rangle.$$

We shall prove that $\{a(\cdot, T_n(u_j), \nabla u_j)\}$ is bounded in $(L_{\overline{M}}(\Omega))^N$. For that, let $\phi \in (E_M(\Omega))^N$ with $\|\phi\|_M \le 1$. From (3.3) we have

$$\int_\Omega (a(x,T_n(u_j),\nabla u_j) - a(x,T_n(u_j),\phi))\cdot(\nabla u_j - \phi)dx \ge 0.$$

Which yields

$$\int_\Omega (a(x,T_n(u_j),\nabla u_j)\cdot \phi dx \leq \int_\Omega (a(x,T_n(u_j),\nabla u_j)\cdot \nabla u_j dx - \int_\Omega a(x,T_n(u_j),\phi))\cdot(\nabla u_j-\phi)dx.$$

Using Hölder's inequality we get

$$\left|\int_\Omega B_n(x,u_j,\nabla u_j)u_j dx\right| \leq 2n\|\chi_\Omega\|_{\overline{M}}\|u_j\|_M,$$

where $\chi_\Omega$ denotes the characteristic of $\Omega$. Hence, by $(a)$ and $(c)$ we have that

$$\int_\Omega (a(x,T_n(u_j),\nabla u_j)\cdot \nabla u_j dx$$

is bounded from above. Let $r=1+k_1+k_3$. Since $\{u_j\}$ is bounded in $W_0^1L_M(\Omega)$, we can find $\lambda>0$ such that $\int_\Omega M\left(\frac{|\nabla u_j|}{\lambda}\right)dx \leq 1$. So, by using the Young's inequality we obtain

$$\begin{aligned}
&\left|\int_\Omega a(x,T_n(u_j),\phi))\cdot(\nabla u_j-\phi)dx\right| \\
&= 2r\lambda\left|\int_\Omega \frac{1}{r}a(x,T_n(u_j),\phi))\cdot\frac{1}{2\lambda}(\nabla u_j-\phi)dx\right| \\
&\leq 2r\lambda\int_\Omega \overline{M}\left(\frac{1}{r}|a(x,T_n(u_j),\phi))|\right)dx + r\lambda\int_\Omega M\left(\frac{|\nabla u_j|}{\lambda}\right)dx + r\lambda\int_\Omega M\left(\frac{|\phi|}{\lambda}\right)dx.
\end{aligned}$$

The growth condition (3.2) and the convexity of the N-function $\overline{M}$ allow us to have

$$\begin{aligned}
&\left|\int_\Omega a(x,T_n(u_j),\phi))\cdot(\nabla u_j-\phi)dx\right| \\
&\leq 2\lambda\int_\Omega \overline{M}(a_0(x))dx + 2k_1\lambda|\Omega|\overline{M}\overline{P}^{-1}M(n) + 2k_3\lambda\int_\Omega M(k_4|\phi|)dx \\
&+r\lambda\int_\Omega M\left(\frac{|\nabla u_j|}{\lambda}\right)dx + r\lambda\int_\Omega M\left(\frac{|\phi|}{\lambda}\right)dx.
\end{aligned}$$

Which gives the desired result. Therefore, there exist a subsequence, still indexed by $j$, and $l_n\in(L_{\overline{M}}(\Omega))^N$ such that

$$a(x,T_n(u_j),\nabla u_j) \rightharpoonup l_n \text{ in } (L_{\overline{M}}(\Omega))^N \text{ for } \sigma(\Pi L_{\overline{M}},\Pi E_M)$$

as $j\to\infty$. Since $\{B_n(x,u_j,\nabla u_j)\}$ is uniformly bounded in $L_{\overline{M}}(\Omega)$, we get

$$B_n(x,u_j,\nabla u_j) \rightharpoonup m_n \in L_{\overline{M}}(\Omega) \text{ for } \sigma(L_{\overline{M}},E_M)$$

as $j\to\infty$. Therefore, the linear form $\chi$ can be identified to $-\mathrm{div} l_n + m_n$. More precisely, the action of $\chi$ on an element $\phi\in W_0^1E_M(\Omega)$ is given by

$$\langle\chi,\phi\rangle = \int_\Omega l_n\cdot\nabla\phi dx + \int_\Omega m_n\phi dx.$$

We shall prove that $\nabla u_j\to\nabla u$ a.e. in $\Omega$. To do this, we argue Similarly as in [14, Theorem 5.1]. Let $\Omega^r=\{x\in\Omega : |\nabla u(x)|\leq r\}$ and denote by $\chi^r$ the characteristic function of $\Omega^r$. Fix

$r > 0$ and let $s \geq r$. By (3.3) we have

$$\begin{aligned} 0 &\leq \int_{\Omega^r} (a(x, T_n(u_j), \nabla u_j) - a(x, T_n(u_j), \nabla u)) \cdot (\nabla u_j - \nabla u)\, dx \\ &\leq \int_{\Omega^s} (a(x, T_n(u_j), \nabla u_j) - a(x, T_n(u_j), \nabla u \chi^s)) \cdot (\nabla u_j - \nabla u \chi^s)\, dx \\ &\leq \int_\Omega a(x, T_n(u_j), \nabla u_j) \cdot \nabla u_j dx - \int_\Omega a(x, T_n(u_j), \nabla u_j) \cdot \nabla u \chi^s dx \\ &\quad - \int_\Omega a(x, T_n(u_j), \nabla u \chi^s) \cdot (\nabla u_n - \nabla u \chi^s) dx. \end{aligned}$$

By [13, Proposition 4.13] and ($a$) we can suppose that $u_j \to u$ strongly in $E_M(\Omega)$ and a.e. in $\Omega$, so that we get

$$\int_\Omega B_n(x, u_j, \nabla u_j) u_j dx \to \int_\Omega m_n u dx \quad \text{as } j \to \infty.$$

Thus, from ($c$) we have

$$\limsup_j \int_\Omega a(x, T_n(u_j), \nabla u_j) \cdot \nabla u_j dx \leq \langle \chi, u \rangle - \int_\Omega m_n u dx = \int_\Omega l_n \cdot \nabla u dx.$$

For the second integral of the right-hand side

$$\limsup_j \int_\Omega a(x, T_n(u_j), \nabla u_j) \cdot \nabla u \chi^s dx = \int_\Omega l_n \cdot \nabla u \chi^s dx.$$

Since $a(x, T_n(u_j), \nabla u \chi^s)$ converges to $a(x, T_n(u), \nabla u \chi^s)$ in norm in $(E_{\overline{M}}(\Omega))^N$ by Lemma 2.1 and $\nabla u_j \rightharpoonup \nabla u$ in $(L_M(\Omega))^N$ for $\sigma(\Pi L_{\overline{M}}, \Pi E_M)$ by ($a$), we obtain for the third term of the right-hand side

$$\limsup_j \int_\Omega a(x, T_n(u_j), \nabla u \chi^s) \cdot (\nabla u_n - \nabla u \chi^s) dx = \int_{\Omega \setminus \Omega^s} a(x, T_n(u), \nabla u \chi^s) \cdot \nabla u dx.$$

Thus,

$$\begin{aligned} 0 &\leq \limsup_j \int_{\Omega^r} (a(x, T_n(u_j), \nabla u_j) - a(x, T_n(u_j), \nabla u)) \cdot (\nabla u_j - \nabla u)\, dx \\ &\leq \int_\Omega l_n \cdot \nabla u dx - \int_\Omega l_n \cdot \nabla u \chi^s dx - \int_{\Omega \setminus \Omega^s} a(x, T_n(u), \nabla u \chi^s) \cdot \nabla u dx. \end{aligned}$$

Then, letting $s \to \infty$, we get

$$\limsup_j \int_{\Omega^r} (a(x, T_n(u_j), \nabla u_j) - a(x, T_n(u_j), \nabla u)) \cdot (\nabla u_j - \nabla u)\, dx = 0.$$

Let us define $D_j^n$ by

$$D_j^n = (a(x, T_n(u_j), \nabla u_j) - a(x, T_n(u_j), \nabla u)) \cdot (\nabla u_j - \nabla u).$$

Since the integrand function $D_j^n$ is nonnegative by (3.3), we get

$$\limsup_j D_j^n = 0 \quad \text{in } L^1(\Omega^r).$$

Being $r > 0$ arbitrary, there exists a subsequence still indexed by $j$ such that

$$D_j^n \to 0 \quad \text{a.e. in } \Omega.$$

as $j \to \infty$. Hence, there exists a subset $U$ of $\Omega$ of zero measure such that for all $x \in \Omega \setminus U$ one has $D_j^n(x) \to 0$. Fix $n > 0$ and $x \in \Omega \setminus U$. By using (3.1) and (3.2) we arrive at

$$D_j^n(x) \geq \overline{M}^{-1} M(h(n)) M(|\nabla u_j(x)|) - c(n,x) \left(1 + \overline{M}^{-1} M(k_4 |\nabla u_j(x)|) + |\nabla u_j(x)|\right),$$

where $c(n,x)$ is a constant depending on $n$ and $x$. Thus, the sequence $\{\nabla u_j(x)\}$ is bounded in $\mathbb{R}^N$ and for a subsequence $\{u_{j'}(x)\}$, we have

$$\nabla u_{j'}(x) \to \xi \ \text{ in } \mathbb{R}^N$$

and

$$(a(x, T_n(u(x)), \xi) - a(x, T_n(u(x)), \nabla u(x))) \cdot (\xi - \nabla u(x)) = 0.$$

Since $a(x,s,\xi)$ is strictly monotone, we have $\xi = \nabla u(x)$, and then $\nabla u_j(x) \to \nabla u(x)$ for the whole sequence. It follows that

$$\nabla u_j \to \nabla u \quad \text{a.e. in } \Omega.$$

Thus, we get $m_n = B_n(x, u, \nabla u)$ and by [16, Theorem 14.6 ] we obtain $l_n = a(x, T_n(u), \nabla u) \in (L_{\overline{M}}(\Omega))^N$. Therefore, we have $u \in D(A_n + B_n)$ and $\chi = A_n(u) + B_n(u)$. From ($b$) and the equality

$$\langle A_n(u_j), u_j \rangle = \langle A_n(u_j) + B_n(u_j), u_j \rangle - \int_\Omega B_n(x, u_j, \nabla u_j) u_j dx,$$

we get

$$\begin{aligned} \limsup_j \langle A_n(u_j), u_j \rangle &\leq \langle \chi, u \rangle - \int_\Omega B_n(x, u, \nabla u) u dx \\ &= \langle A_n(u), u \rangle. \end{aligned}$$

By [14, Proposition 5] the operator $A_n$ is in particular sequentially pseudo-monotone. Therefore, we get

$$\limsup_j \langle A_n(u_j), u_j \rangle = \langle A_n(u), u \rangle.$$

Consequently, we have

$$\limsup_j \langle A_n(u_j) + B_n(u_j), u_j \rangle = \langle \chi, u \rangle.$$

**(iii).** Assume that $u \in D(A_n + B_n)$ is such that $u$ is bounded in $W_0^1 L_M(\Omega)$ and $\langle A_n(u) + B_n(u), u \rangle$ is bounded from above. We will prove that $\{a(\cdot, T_n(u), \nabla u)\}$ remains bounded in $(L_{\overline{M}}(\Omega))^N$ which implies that $A_n(u) + B_n(u)$ remains bounded in $W^{-1} L_{\overline{M}}(\Omega)$. Let $\phi \in (E_M(\Omega))^N$ with $\|\phi\|_M \leq 1$. From (3.3) we have

$$\int_\Omega (a(x, T_n(u), \nabla u) - a(x, T_n(u), \phi)) \cdot (\nabla u - \phi) dx \geq 0.$$

Thus,

$$\begin{aligned} \int_\Omega (a(x, T_n(u), \nabla u) \cdot \phi dx &\leq \int_\Omega (a(x, T_n(u), \nabla u) \cdot \nabla u dx \\ &\quad - \int_\Omega a(x, T_n(u), \phi)) \cdot (\nabla u - \phi) dx. \end{aligned}$$

Using Hölder's inequality we get

$$\left| \int_\Omega B_n(x, u, \nabla u) u dx \right| \leq 2n \|\chi_\Omega\|_{\overline{M}} \|u\|_M.$$

By writing

$$\int_\Omega (a(x, T_n(u), \nabla u) \cdot \nabla u dx = \langle A_n(u) + B_n(u), u \rangle + \int_\Omega B_n(x, u, \nabla u) u dx,$$

we obtain that $\int_\Omega (a(x,T_n(u),\nabla u)\cdot\nabla u dx$ is bounded from above. Which then implies, by similar arguments as in **(ii)**, that $\{a(\cdot,T_n(u),\nabla u)\}$ is bounded in $(L_{\overline{M}}(\Omega))^N$.
**(iv).** We prove that

$$\{u\in D(A_n+B_n)\subset W_0^1L_M(\Omega):\ \langle A_n(u)+B_n(u)-f_n,u\rangle\ \le 0\}$$

is bounded in $W_0^1L_M(\Omega)$, which yields the conclusion. If $u\in D(A_n+B_n)$ is such that: $\langle A_n(u)+B_n(u)-f_n,u\rangle\ \le 0$, then

$$\int_\Omega a(x,T_n(u),\nabla u)\cdot\nabla u dx\ +\ \int_\Omega B_n(x,u,\nabla u)u dx\le\int_\Omega f_n u dx.$$

Since $f_n$ has compact support and $B_n$ is bounded, there exists a constant $c(n)$ depending on $n$ such that by (3.1) one has

$$\overline{M}^{-1}(M(h(n)))\int_\Omega M(|\nabla u|)dx\ \le\ c(n)\int_\Omega |u|dx.$$

Let $r>0$ be a real which will be chosen later. The Young's inequality gives

$$\overline{M}^{-1}(M(h(n)))\int_\Omega M(|\nabla u|)dx\ \le\ \frac{1}{\lambda_1}\overline{M}(\lambda_1\lambda_2 rc(n))|\Omega|\ +\ \frac{1}{\lambda_1}\int_\Omega M\left(\frac{|u|}{r\lambda_2}\right)dx,$$

where $\lambda_1$ and $\lambda_2$ are the constants in inequality (2.3) of Lemma 2.2. Then, by inequality (2.3) we get

$$\overline{M}^{-1}(M(h(n)))\int_\Omega M(|\nabla u|)dx\ \le\ \frac{1}{\lambda_1}\overline{M}(\lambda_1\lambda_2 rc(n))|\Omega|\ +\ \int_\Omega M\left(\frac{1}{r}|\nabla u|\right)dx.$$

The choice $r\ge\max\left(1,\dfrac{2}{\overline{M}^{-1}M(h(n))}\right)$ guarantees that $\int_\Omega M(|\nabla u|)dx$ is bounded and so is $\|u\|_{W_0^1L_M(\Omega)}$. Hence, condition **(iv)** is filled.
Therefore, by applying Proposition 2.4 there exists at least one solution $u_n\in D(A_n+B_n)\subset W_0^1L_M(\Omega)$ to the approximate equation

$$-\mathrm{div}\, a(x,T_n(u_n),\nabla u_n)\ +\ B_n(x,u_n,\nabla u_n)\ =\ f_n$$

in the sense that

$$\int_\Omega a(x,T_n(u_n),\nabla u_n)\cdot\nabla v dx\ +\ \int_\Omega B_n(x,u_n,\nabla_n)v dx\ =\ \int_\Omega f_n v dx \tag{4.1}$$

for all $v\ \in\ W_0^1L_M(\Omega)$.

**Lemma 4.1.** *Let $u_n$ be a solution of (4.1). For all $t,\epsilon$ in $\mathbb{R}_+^*$, one has the following inequalities:*

$$\begin{aligned}&\int_{\{t<u_n\le t+\epsilon\}} a(x,T_n(u_n),\nabla u_n)\cdot\nabla u_n e^{\gamma(u_n^+)}dx\\ &\le\int_{\{u_n>t\}} f_n^+ e^{\gamma(u_n^+)}T_\epsilon(G_t(u_n^+))dx.\end{aligned}\tag{4.2}$$

$$\begin{aligned}&\int_{\{-t-\epsilon<u_n\le -t\}} a(x,T_n(u_n),\nabla u_n)\cdot\nabla u_n e^{\gamma(u_n^-)}dx\\ &\le\int_{\{u_n<-t\}} f_n^- e^{\gamma(u_n^-)}T_\epsilon(G_t(u_n^-))dx.\end{aligned}\tag{4.3}$$

*Proof.* **(4.2)-** Observe that by [11, lemma 2] the function

$$e^{\gamma(T_k(u_n^+))}T_\epsilon(G_t(T_k(u_n^+)))$$

belongs to $W_0^1L_M(\Omega)\cap L^\infty(\Omega)$ for all $k>0$. Thus, testing by this function in (4.1), we obtain

$$\begin{aligned}
&\int_\Omega a(x,T_n(u_n),\nabla u_n)\cdot\nabla T_k(u_n^+)\frac{\beta(T_k(u_n^+))}{\overline{M}^{-1}M(h(|T_k(u_n^+)|))}\\
&\quad\times e^{\gamma(T_k(u_n^+))}T_\epsilon(G_t(T_k(u_n^+)))dx\\
&+\int_{\{t<T_k(u_n^+)\le t+\epsilon\}} a(x,T_n(u_n),\nabla u_n)\cdot\nabla T_k(u_n^+)e^{\gamma(T_k(u_n^+))}dx \qquad (4.4)\\
&+\int_\Omega B_n(x,u_n,\nabla u_n)e^{\gamma(T_k(u_n^+))}T_\epsilon(G_t(T_k(u_n^+)))dx\\
&=\int_\Omega f_n e^{\gamma(T_k(u_n^+))}T_\epsilon(G_t(T_k(u_n^+)))dx.
\end{aligned}$$

Now, we will pass to the limit as $k$ tends to $+\infty$ in (4.4). Note that

$$\begin{aligned}
&\int_\Omega a(x,T_n(u_n),\nabla u_n)\cdot\nabla T_k(u_n^+)\frac{\beta(T_k(u_n^+))}{\overline{M}^{-1}(M(h(|T_k(u_n^+)|)))}e^{\gamma(T_k(u_n^+))}T_\epsilon(G_t(T_k(u_n^+)))dx\\
&=\int_{\{0\le u_n<k\}} a(x,T_n(u_n),\nabla u_n)\cdot\nabla u_n\frac{\beta(u_n^+)}{\overline{M}^{-1}(M(h(u_n^+)))}e^{\gamma(u_n^+)}T_\epsilon(G_t(u_n^+))dx.
\end{aligned}$$

By (3.1) and the positivity of $h$ and $\beta$, the integrand function is nonnegative, it follows by applying the monotone convergence theorem, that

$$\begin{aligned}
&\int_\Omega a(x,T_n(u_n),\nabla u_n)\cdot\nabla T_k(u_n^+)\frac{\beta(T_k(u_n^+))}{\overline{M}^{-1}(M(h(|T_k(u_n^+)|)))}e^{\gamma(T_k(u_n^+))}T_\epsilon(G_t(T_k(u_n^+)))dx\\
&\to\int_\Omega a(x,T_n(u_n),\nabla u_n)\cdot\nabla u_n\frac{\beta(u_n^+)}{\overline{M}^{-1}(M(h(u_n^+)))}e^{\gamma(u_n^+)}T_\epsilon(G_t(u_n^+))dx
\end{aligned}$$

as $k\to\infty$. For the second integral in the left-hand side of (4.4), we write

$$\begin{aligned}
&\int_{\{t<T_k(u_n^+)\le t+\epsilon\}} a(x,T_n(u_n),\nabla u_n)\cdot\nabla T_k(u_n^+)e^{\gamma(T_k(u_n^+))}dx\\
&=\int_{\{t<u_n\le t+\epsilon\}\cap\{0\le u_n<k\}} a(x,T_n(u_n),\nabla u_n)\cdot\nabla u_n e^{\gamma(u_n^+)}dx.
\end{aligned}$$

By similar arguments as above, we have

$$\begin{aligned}
&\int_{\{t<T_k(u_n^+)\le t+\epsilon\}} a(x,T_n(u_n),\nabla u_n)\cdot\nabla T_k(u_n^+)e^{\gamma(T_k(u_n^+))}dx\\
&\to\int_{\{t<u_n\le t+\epsilon\}} a(x,T_n(u_n),\nabla u_n)\cdot\nabla u_n e^{\gamma(u_n^+)}dx
\end{aligned}$$

as $k\to\infty$. Since the functions $B_n$, $f_n$ and $\gamma$ are bounded, we apply the Lebesgue's dominated convergence theorem for the remaining integrals in (4.4). Consequently, letting $k$ tend to $\infty$ in

(4.4) we obtain

$$\begin{aligned}&\int_\Omega a(x,T_n(u_n),\nabla u_n)\cdot\nabla u_n\frac{\beta(u_n^+)}{\overline{M}^{-1}(M(h(u_n^+)))}e^{\gamma(u_n^+)}T_\epsilon(G_t(u_n^+))dx\\&+\int_{\{t<u_n\le t+\epsilon\}}a(x,T_n(u_n),\nabla u_n)\cdot\nabla u_n e^{\gamma(u_n^+)}dx\\&+\int_\Omega B_n(x,u_n,\nabla u_n)e^{\gamma(u_n^+)}T_\epsilon(G_t(u_n^+))dx\\&=\int_\Omega f_n e^{\gamma(u_n^+)}T_\epsilon(G_t(u_n^+))dx.\end{aligned}$$

Note that $u_n^+=|u_n|$ on the set $\{x\in\Omega: u_n(x)\ge 0\}$. Thus (3.1) and (3.4) imply that

$$\begin{aligned}&\int_\Omega a(x,T_n(u_n),\nabla u_n)\cdot\nabla u_n\frac{\beta(u_n^+)}{\overline{M}^{-1}(M(h(u_n^+)))}e^{\gamma(u_n^+)}T_\epsilon(G_t(u_n^+))dx\\&+\int_\Omega B_n(x,u_n,\nabla u_n)e^{\gamma(u_n^+)}T_\epsilon(G_t(u_n^+))dx\ \ge\ 0,\end{aligned}$$

and so

$$\int_{\{t<u_n\le t+\epsilon\}}a(x,T_n(u_n),\nabla u_n)\cdot\nabla u_n e^{\gamma(u_n^+)}dx\le\int_\Omega f_n e^{\gamma(u_n^+)}T_\epsilon(G_t(u_n^+))dx.$$

Since $T_\epsilon(G_t(u_n^+))$ is different from zero only on $\{u_n>t\}$ and $f_n\le f_n^+$ we have

$$\int_{\{t<u_n\le t+\epsilon\}}a(x,T_n(u_n),\nabla u_n)\cdot\nabla u_n e^{\gamma(u_n^+)}dx\le\int_{\{u_n>t\}}f_n^+e^{\gamma(u_n^+)}T_\epsilon(G_t(u_n^+))dx,$$

and (4.2) is proved.

**(4.3)-** For all $k>0$, the function

$$-e^{\gamma(T_k(u_n^-))}T_\epsilon(G_t(T_k(u_n^-)))$$

belongs to $W_0^1L_M(\Omega)\cap L^\infty(\Omega)$, (see [11, lemma 2]), so that one can take it as test function in (4.1) and obtain

$$\begin{aligned}&-\int_\Omega a(x,T_n(u_n),\nabla u_n)\cdot\nabla T_k(u_n^-)\frac{\beta(T_k(u_n^-))}{\overline{M}^{-1}(M(h(|T_k(u_n^-)|)))}\\&\quad\times e^{\gamma(T_k(u_n^-))}T_\epsilon(G_t(T_k(u_n^-)))dx\\&-\int_{\{t<T_k(u_n^-)\le t+\epsilon\}}a(x,T_n(u_n),\nabla u_n)\cdot\nabla T_k(u_n^-)e^{\gamma(T_k(u_n^-))}dx\\&-\int_\Omega B_n(x,u_n,\nabla u_n)e^{\gamma(T_k(u_n^-))}T_\epsilon(G_t(T_k(u_n^-)))dx\\&=\ -\int_\Omega f_n e^{\gamma(T_k(u_n^-))}T_\epsilon(G_t(T_k(u_n^-)))dx.\end{aligned}\tag{4.5}$$

The first integral in the left-hand side of (4.5) is written

$$\begin{aligned}&-\int_\Omega a(x,T_n(u_n),\nabla u_n)\cdot\nabla T_k(u_n^-)\frac{\beta(T_k(u_n^-))}{\overline{M}^{-1}(M(h(|T_k(u_n^-)|)))}e^{\gamma(T_k(u_n^-))}T_\epsilon(G_t(T_k(u_n^-)))dx\\&=\int_{\{-k<u_n\le 0\}}a(x,T_n(u_n),\nabla u_n)\cdot\nabla u_n\frac{\beta(u_n^-)}{\overline{M}^{-1}(M(h(u_n^-)))}e^{\gamma(u_n^-)}T_\epsilon(G_t(u_n^-))dx.\end{aligned}$$

Thus, applying the monotone convergence theorem we get

$$
\begin{aligned}
&-\int_\Omega a(x,T_n(u_n),\nabla u_n)\cdot\nabla T_k(u_n^-)\frac{\beta(T_k(u_n^-))}{\overline{M}^{-1}(M(h(|T_k(u_n^-)|)))}e^{\gamma(T_k(u_n^-))}T_\epsilon(G_t(T_k(u_n^-)))dx\\
&\to \int_\Omega a(x,T_n(u_n),\nabla u_n)\cdot\nabla u_n\frac{\beta(u_n^-)}{\overline{M}^{-1}(M(h(u_n^-)))}e^{\gamma(u_n^-)}T_\epsilon(G_t(u_n^-))dx.
\end{aligned}
$$

as $k\to\infty$. For the second integral in the left-hand side of (4.5), we write

$$
\begin{aligned}
&-\int_{\{t<T_k(u_n^-)\le t+\epsilon\}} a(x,T_n(u_n),\nabla u_n)\cdot\nabla T_k(u_n^-)e^{\gamma(T_k(u_n^-))}dx\\
&=\int_{\{t<T_k(u_n^-)\le t+\epsilon\}\cap\{-k<u_n\le 0\}} a(x,T_n(u_n),\nabla u_n)\cdot\nabla u_n e^{\gamma(u_n^-)}dx\\
&=\int_{\{-t-\epsilon\le u_n<-t\}\cap\{-k<u_n\le 0\}} a(x,T_n(u_n),\nabla u_n)\cdot\nabla u_n e^{\gamma(u_n^-)}dx.
\end{aligned}
$$

Applying again the monotone convergence theorem, we obtain

$$
\begin{aligned}
&-\int_{\{t<T_k(u_n^-)\le t+\epsilon\}} a(x,T_n(u_n),\nabla u_n)\cdot\nabla T_k(u_n^-)e^{\gamma(T_k(u_n^-))}dx\\
&\to\int_{\{-t-\epsilon\le u_n<-t\}} a(x,T_n(u_n),\nabla u_n)\cdot\nabla u_n e^{\gamma(u_n^-)}dx,
\end{aligned}
$$

as $k\to\infty$. For the remaining integrals in (4.5), Lebesgue's dominated convergence theorem may be applied since $B_n$, $f_n$ and $\gamma$ are bounded. Hence, letting $k$ tend to $+\infty$ in (4.5), we get

$$
\begin{aligned}
&\int_\Omega a(x,T_n(u_n),\nabla u_n)\cdot\nabla u_n\frac{\beta(u_n^-)}{\overline{M}^{-1}(M(h(u_n^-)))}e^{\gamma(u_n^-)}T_\epsilon(G_t(u_n^-))dx\\
&+\int_{\{-t-\epsilon\le u_n<-t\}} a(x,T_n(u_n),\nabla u_n)\cdot\nabla u_n e^{\gamma(u_n^-)}dx\\
&-\int_\Omega B_n(x,u_n,\nabla u_n)e^{\gamma(u_n^-)}T_\epsilon(G_t(u_n^-))dx\\
&=\ -\int_\Omega f_n e^{\gamma(u_n^-)}T_\epsilon(G_t(u_n^-))dx.
\end{aligned}
$$

Since $u_n^-=|u_n|$ on the set$\{x\in\Omega: u_n(x)\le 0\}$, by using (3.1) and (3.4) we have

$$
\begin{aligned}
&\int_\Omega a(x,T_n(u_n),\nabla u_n)\cdot\nabla u_n\frac{\beta(u_n^-)}{\overline{M}^{-1}(M(h(u_n^-)))}e^{\gamma(u_n^-)}T_\epsilon(G_t(u_n^-))dx\\
&-\int_\Omega B_n(x,u_n,\nabla u_n)e^{\gamma(u_n^-)}T_\epsilon(G_t(u_n^-))dx\ \ge\ 0.
\end{aligned}
$$

Observe that $-f_n\le f_n^-$ and $\{u_n^->t\}\cap\{u_n\le 0\}=\{u_n<-t\}$, and so we obtain

$$
\int_{\{-t-\epsilon\le u_n<-t\}} a(x,T_n(u_n),\nabla u_n)\cdot\nabla u_n e^{\gamma(u_n^-)}dx\ \le\ \int_{\{u_n<-t\}} f_n^- e^{\gamma(u_n^-)}T_\epsilon(G_t(u_n^-))dx.
$$

Which proves (4.3). ∎

Now, we are able to prove the following auxiliary result:

**Lemma 4.2.** *There exists a constant $c_0$, not depending on $n$, such that for almost every $t>0$*

$$
-\frac{d}{dt}\int_{\{|u_n|>t\}}\overline{M}^{-1}(M(h(|u_n|)))M(|\nabla u_n|)dx\le c_0\int_{\{|u_n|>t\}}|f_n|dx. \tag{4.6}
$$

*Proof.* Since the two functions $e^{\gamma(u_n^+)}$ and $e^{\gamma(u_n^-)}$ are bounded in $L^\infty(\Omega)$, we sum up both inequalities (4.2) and (4.3) obtaining a constant $c_0 > 0$, not depending on $n$, such that

$$\int_{\{t<|u_n|\le t+\epsilon\}} a(x, T_n(u_n), \nabla u_n)\cdot \nabla u_n dx \le \epsilon c_0 \int_{\{|u_n|>t\}} |f_n| dx,$$

and by (3.1) we get

$$\int_{\{t<|u_n|\le t+\epsilon\}} \overline{M}^{-1}(M(h(|u_n|)))M(|\nabla u_n|)dx \le \epsilon c_0 \int_{\{|u_n|>t\}} |f_n| dx.$$

Then, dividing by $\epsilon$ and letting $\epsilon$ tend to $0^+$ we obtain (4.6). ∎

Inequality (4.6) was proved in [17] when h is a constant function and $B \equiv 0$. The following comparison lemma plays a fundamental role to get uniform estimation for solutions of approximate equations (4.1) in $L^\infty(\Omega)$, it is quite similar to that proved in [17] when h is a constant function. The proof we give here is based on (4.6) and on techniques inspired from those in [17].

**Lemma 4.3.** *Let $K(t) = \frac{M(t)}{t}$ and $\mu_n(t) = |\{x \in \Omega : |u_n(x)| > t\}|$, for all $t > 0$. We have for almost every $t > 0$:*

$$h(t) \le \frac{2M(1)(-\mu_n'(t))}{\overline{M}^{-1}(M(1))NC_N^{\frac{1}{N}}\mu_n(t)^{1-\frac{1}{N}}} K^{-1}\left(\frac{c_0\int_{\{|u_n|>t\}}|f_n|dx}{\overline{M}^{-1}(M(1))NC_N^{\frac{1}{N}}\mu_n(t)^{1-\frac{1}{N}}}\right). \tag{4.7}$$

*where $C_N$ stands for the measure of the unit ball in $\mathbb{R}^N$ and $c_0$ is the constant which appears in (4.6).*

*Proof.* The hypotheses made on the N-function $M$, allow to affirm that the function $C(t) = \dfrac{1}{K^{-1}(t)}$ is decreasing and convex (see [17]). Hence, Jensen's inequality yields

$$C\left(\frac{\int_{\{t<|u_n|\le t+k\}} \overline{M}^{-1}(M(h(|u_n|)))M(|\nabla u_n|)dx}{\int_{\{t<|u_n|\le t+k\}} \overline{M}^{-1}(M(h(|u_n|)))|\nabla u_n|dx}\right)$$

$$= C\left(\frac{\int_{\{t<|u_n|\le t+k\}} K(|\nabla u_n|)\overline{M}^{-1}(M(h(|u_n|)))|\nabla u_n|dx}{\int_{\{t<|u_n|\le t+k\}} \overline{M}^{-1}(M(h(|u_n|)))|\nabla u_n|dx}\right)$$

$$\le \frac{\int_{\{t<|u_n|\le t+k\}} \overline{M}^{-1}(M(h(|u_n|)))dx}{\int_{\{t<|u_n|\le t+k\}} \overline{M}^{-1}(M(h(|u_n|)))|\nabla u_n|dx}$$

$$\le \frac{\overline{M}^{-1}(M(h(t)))(-\mu_n(t+k)+\mu_n(t))}{\overline{M}^{-1}(M(h(t+k)))\int_{\{t<|u_n|\le t+k\}} |\nabla u_n|dx}.$$

Taking into account that $\overline{M}^{-1}(M(h(t))) \leq \overline{M}^{-1}(M(1))$, using the convexity of $C$ and then letting $k \to 0^+$, we obtain for almost every $t > 0$

$$\frac{\overline{M}^{-1}(M(1))}{\overline{M}^{-1}(M(h(t)))}\, C\left(\frac{-\dfrac{d}{dt}\displaystyle\int_{\{|u_n|>t\}} \overline{M}^{-1}(M(h(|u_n|)))M(|\nabla u_n|)dx}{\overline{M}^{-1}(M(1))(-\dfrac{d}{dt}\displaystyle\int_{\{|u_n|>t\}} |\nabla u_n|dx)}\right)$$
$$\leq \frac{-\mu_n'(t)}{-\dfrac{d}{dt}\displaystyle\int_{\{|u_n|>t\}} |\nabla u_n|dx}.$$

Recall the following inequality, (see [17]):

$$-\frac{d}{dt}\int_{\{|u_n|>t\}} |\nabla u_n|dx \geq NC_N^{\frac{1}{N}}\mu_n(t)^{1-\frac{1}{N}} \quad \text{for almost every } t > 0. \tag{4.8}$$

The monotonicity of the function $C$, $(4.6)$ and $(4.8)$ yield

$$\frac{1}{\overline{M}^{-1}(M(h(t)))}$$
$$\leq \frac{-\mu_n'(t)}{\overline{M}^{-1}(M(1))NC_N^{\frac{1}{N}}\mu_n(t)^{1-\frac{1}{N}}}\; K^{-1}\left(\frac{c_0\displaystyle\int_{\{|u_n|>t\}} |f_n|dx}{\overline{M}^{-1}(M(1))NC_N^{\frac{1}{N}}\mu_n(t)^{1-\frac{1}{N}}}\right).$$

Using (2.1) and the fact that $0 < h(t) \leq 1$, we obtain (4.7). ∎

## 5. PROOF OF THEOREM 3.1

Using Lemma 4.3, we prove Theorem 3.1 in six steps.

**step 1:** $L^\infty$-bound.

If we assume (3.5), using the Hölder's inequality

$$\int_{\{|u_n|>t\}} |f_n|dx \leq \|f\|_N\mu_n(t)^{1-\frac{1}{N}},$$

(4.7) becomes

$$h(t) \leq \frac{2M(1)(-\mu_n'(t))}{\overline{M}^{-1}(M(1))NC_N^{\frac{1}{N}}\mu_n(t)^{1-\frac{1}{N}}}K^{-1}\left(\frac{c_0\|f\|_N}{\overline{M}^{-1}(M(1))NC_N^{\frac{1}{N}}}\right).$$

Then, integrating between $0$ and $s$, we get

$$H(s) \leq \frac{2M(1)}{\overline{M}^{-1}(M(1))NC_N^{\frac{1}{N}}}K^{-1}\left(\frac{c_0\|f\|_N}{\overline{M}^{-1}(M(1))NC_N^{\frac{1}{N}}}\right)\int_0^s \frac{-\mu_n'(t)}{\mu_n(t)^{1-\frac{1}{N}}}dt.$$

Hence, a change of variables yields

$$H(s) \leq \frac{2M(1)}{\overline{M}^{-1}(M(1))NC_N^{\frac{1}{N}}}K^{-1}\left(\frac{c_0\|f\|_N}{\overline{M}^{-1}(M(1))NC_N^{\frac{1}{N}}}\right)\int_{\mu_n(s)}^{|\Omega|} \frac{dt}{t^{1-\frac{1}{N}}}.$$

By (2.4) we get

$$H(u_n^*(\sigma)) \leq \frac{2M(1)}{\overline{M}^{-1}(M(1))NC_N^{\frac{1}{N}}}K^{-1}\left(\frac{c_0\|f\|_N}{\overline{M}^{-1}(M(1))NC_N^{\frac{1}{N}}}\right)\int_{\sigma}^{|\Omega|} \frac{dt}{t^{1-\frac{1}{N}}}.$$

So that

$$H(u_n^*(0)) \leq \frac{2M(1)}{\overline{M}^{-1}(M(1))NC_N^{\frac{1}{N}}} K^{-1}\left(\frac{c_0\|f\|_N}{\overline{M}^{-1}(M(1))NC_N^{\frac{1}{N}}}\right) N|\Omega|^{\frac{1}{N}}.$$

Since $u_n^*(0) = \|u_n\|_\infty$, the assumption made on $H$ (i.e $\lim_{s\to+\infty} H(s) = +\infty$) shows that the sequence $\{u_n\}$ is uniformly bounded in $L^\infty(\Omega)$. Moreover, if we denote by $H^{-1}$ the inverse function of $H$, one has:

$$\|u_n\|_\infty \leq H^{-1}\left(\frac{2M(1)}{\overline{M}^{-1}(M(1))NC_N^{\frac{1}{N}}} K^{-1}\left(\frac{c_0\|f\|_N}{\overline{M}^{-1}(M(1))NC_N^{\frac{1}{N}}}\right) N|\Omega|^{\frac{1}{N}}\right). \tag{5.1}$$

Now, we assume that (3.6) is filled. Then, using the Hölder's inequality

$$\int_{\{|u_n|>t\}} |f_n|dx \leq \|f\|_m \mu_n(t)^{1-\frac{1}{m}}$$

in (4.7), we obtain

$$H(s) \;\leq\; \frac{2M(1)}{\overline{M}^{-1}(M(1))NC_N^{\frac{1}{N}}}\int_0^s \frac{-\mu_n'(t)}{\mu_n(t)^{1-\frac{1}{N}}} K^{-1}\left(\frac{c_0\|f\|_m}{\overline{M}^{-1}(M(1))NC_N^{\frac{1}{N}}\mu_n(t)^{\frac{1}{m}-\frac{1}{N}}}\right)dt.$$

A change of variables gives

$$H(s) \;\leq\; \frac{2M(1)}{\overline{M}^{-1}(M(1))NC_N^{\frac{1}{N}}}\int_{\mu_n(s)}^{|\Omega|} K^{-1}\left(\frac{c_0\|f\|_m}{\overline{M}^{-1}(M(1))NC_N^{\frac{1}{N}}\sigma^{\frac{1}{m}-\frac{1}{N}}}\right)\frac{d\sigma}{\sigma^{1-\frac{1}{N}}}.$$

As above, (2.4) gives

$$H(u_n^*(\tau)) \;\leq\; \frac{2M(1)}{\overline{M}^{-1}(M(1))NC_N^{\frac{1}{N}}}\int_{\tau}^{|\Omega|} K^{-1}\left(\frac{c_0\|f\|_m}{\overline{M}^{-1}(M(1))NC_N^{\frac{1}{N}}\sigma^{\frac{1}{m}-\frac{1}{N}}}\right)\frac{d\sigma}{\sigma^{1-\frac{1}{N}}}.$$

Then, we have

$$H(\|u_n\|_\infty) \;\leq\; \frac{2M(1)}{\overline{M}^{-1}(M(1))NC_N^{\frac{1}{N}}}\int_{0}^{|\Omega|} K^{-1}\left(\frac{c_0\|f\|_m}{\overline{M}^{-1}(M(1))NC_N^{\frac{1}{N}}\sigma^{\frac{1}{m}-\frac{1}{N}}}\right)\frac{d\sigma}{\sigma^{1-\frac{1}{N}}}.$$

A change of variables gives

$$H(\|u_n\|_\infty) \;\leq\; \frac{2M(1)c_0^r\|f\|_m^r}{(\overline{M}^{-1}(M(1)))^{r+1}N^rC_N^{\frac{r+1}{N}}}\int_\lambda^{+\infty} rt^{-r-1}K^{-1}(t)dt,$$

where $\lambda = \dfrac{c_0\|f\|_m}{\overline{M}^{-1}(M(1))NC_N^{\frac{1}{N}}|\Omega|^{\frac{1}{rN}}}$. Then, by an integration by parts we obtain that

$$H(\|u_n\|_\infty) \leq \frac{2M(1)c_0^r\|f\|_m^r}{(\overline{M}^{-1}(M(1)))^{r+1}N^rC_N^{\frac{r+1}{N}}}\left(\frac{K^{-1}(\lambda)}{\lambda^r} + \int_{K^{-1}(\lambda)}^{+\infty}\left(\frac{s}{M(s)}\right)^r ds\right).$$

The assumption made on $H$ guarantees that the sequence $\{u_n\}$ is uniformly bounded in $L^\infty(\Omega)$. Indeed, denoting by $H^{-1}$ the inverse function of $H$, one has

$$\|u_n\|_\infty \leq$$

$$H^{-1}\left(\frac{2M(1)c_0^r\|f\|_m^r}{(\overline{M}^{-1}(M(1)))^{r+1}N^rC_N^{\frac{r+1}{N}}}\left(\frac{K^{-1}(\lambda)}{\lambda^r} + \int_{K^{-1}(\lambda)}^{+\infty}\left(\frac{s}{M(s)}\right)^r ds\right)\right). \tag{5.2}$$

Consequently, in both cases the sequence $\{u_n\}$ is uniformly bounded in $L^\infty(\Omega)$, so that in the sequel, we will denote by $c$ the constant appearing either in (5.1) or in (5.2), that is :

$$\|u_n\|_\infty \leq c. \tag{5.3}$$

**Step 2:** Estimation in $W_0^1L_M(\Omega)$.
In order to obtain an estimation in $W_0^1L_M(\Omega)$, we need to prove the following

**Lemma 5.1.**
(a) $\displaystyle\int_{\{0\leq u_n\}} a(x,T_n(u_n),\nabla u_n)\cdot\nabla u_n e^{\gamma(u_n^+)}dx \;\leq\; \int_\Omega f_n^+ e^{\gamma(u_n^+)}u_n^+dx.$

(b) $\displaystyle\int_{\{u_n\leq 0\}} a(x,T_n(u_n),\nabla u_n)\cdot\nabla u_n e^{\gamma(u_n^-)}dx \;\leq\; \int_\Omega f_n^- e^{\gamma(u_n^-)}u_n^-dx.$

*Proof.* Note that by (5.3), $u_n^+\in W_0^1L_M(\Omega)\cap L^\infty(\Omega)$ and $e^{\gamma(u_n^+)}\in W^1L_M(\Omega)\cap L^\infty(\Omega)$. Hence, $e^{\gamma(u_n^+)}u_n^+$ belongs to $W_0^1L_M(\Omega)\cap L^\infty(\Omega)$ and so it is an admissible test function in (4.1). Taking it so, it yields

$$\begin{aligned}
&\int_{\{0\leq u_n\}} a(x,T_n(u_n),\nabla u_n)\cdot\nabla u_n\frac{\beta(u_n^+)}{\overline{M}^{-1}M(h(|u_n^+|))}e^{\gamma(u_n^+)}u_n^+dx\\
&+\int_{\{0\leq u_n\}} a(x,T_n(u_n),\nabla u_n)\cdot\nabla u_n e^{\gamma(u_n^+)}dx\\
&+\int_{\{0\leq u_n\}} B_n(x,u_n,\nabla u_n)e^{\gamma(u_n^+)}u_n^+dx\\
&=\int_\Omega f_n e^{\gamma(u_n^+)}u_n^+dx,
\end{aligned}$$

and by (3.1) and (3.4), one gets

$$\begin{aligned}
&\int_{\{0\leq u_n\}} a(x,T_n(u_n),\nabla u_n)\cdot\nabla u_n\frac{\beta(u_n^+)}{\overline{M}^{-1}M(h(u_n^+))}e^{\gamma(u_n^+)}u_n^+dx\\
&+\int_{\{0\leq u_n\}} B_n(x,u_n,\nabla u_n)e^{\gamma(u_n^+)}u_n^+dx \;\geq\; 0.
\end{aligned}$$

It follows that

$$\int_{\{0\leq u_n\}} a(x,T_n(u_n),\nabla u_n)\cdot\nabla u_n e^{\gamma(u_n^+)}dx \;\leq\; \int_\Omega f_n^+ e^{\gamma(u_n^+)}u_n^+dx,$$

and (a) is proved. To prove (b), we choose $v=-e^{\gamma(u_n^-)}u_n^-$ in (4.1) to obtain

$$\begin{aligned}
&\int_{\{u_n\leq 0\}} a(x,T_n(u_n),\nabla u_n)\cdot\nabla u_n\frac{\beta(u_n^-)}{\overline{M}^{-1}M(h(|u_n^-|))}e^{\gamma(u_n^-)}u_n^-dx\\
&+\int_{\{u_n\leq 0\}} a(x,T_n(u_n),\nabla u_n)\cdot\nabla u_n e^{\gamma(u_n^-)}dx\\
&-\int_{\{u_n\leq 0\}} B_n(x,u_n,\nabla u_n)e^{\gamma(u_n^-)}u_n^-dx\\
&=-\int_\Omega f_n e^{\gamma(u_n^-)}u_n^-dx,
\end{aligned}$$

using again (3.1) and (3.4), one has

$$\begin{aligned}&\int_{\{u_n\le 0\}} a(x,T_n(u_n),\nabla u_n)\cdot\nabla u_n \frac{\beta(u_n^-)}{\overline{M}^{-1}M(h(|u_n^-|))}e^{\gamma(u_n^-)}u_n^-dx\\ &-\int_{\{u_n\le 0\}} B_n(x,u_n,\nabla u_n)e^{\gamma(u_n^-)}u_n^-dx \ \ge\ 0,\end{aligned}$$

which implies that

$$\int_{\{u_n\le 0\}} a(x,T_n(u_n),\nabla u_n)\cdot\nabla u_n e^{\gamma(u_n^-)}dx \ \le\ \int_\Omega f_n^- e^{\gamma(u_n^-)}u_n^-dx,$$

and the lemma is proved. ∎

Now, summing up both inequalities (a) and (b) of the above lemma and taking into account that $\gamma$ is bounded, we deduce that there exists a constant $c_1$ not depending on $n$ such that

$$\int_\Omega a(x,T_n(u_n),\nabla u_n)\cdot\nabla u_n dx \ \le\ c_1\int_\Omega |f_n u_n|dx. \tag{5.4}$$

By (3.1) and (5.3), we obtain

$$\int_\Omega M(|\nabla u_n|)dx \ \le\ \frac{cc_1\|f\|_{m^*}|\Omega|^{1-\frac{1}{m^*}}}{\overline{M}^{-1}M(h(c))}. \tag{5.5}$$

Hence, the sequence $\{u_n\}$ is bounded in $W_0^1L_M(\Omega)$. Consequently, there exist a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, and a function $u\in W_0^1L_M(\Omega)$ such that

$$u_n \rightharpoonup u \ \text{ weakly in } \ W_0^1L_M(\Omega) \ \text{ for } \ \sigma(\Pi L_M,\Pi E_{\overline{M}}), \tag{5.6}$$

and

$$u_n\to u \ \text{ in } \ E_M(\Omega) \ \text{ strongly and a.e. in } \ \Omega. \tag{5.7}$$

**Step 3:** Almost everywhere convergence of the gradients.
Let us begin by the following lemma which will be used in the sequel:

**Lemma 5.2.** *The sequence* $\{a(x,T_n(u_n),\nabla u_n)\}$ *is uniformly bounded in* $(L_{\overline{M}}(\Omega))^N$.

*Proof.* We will use the dual norm of $(L_{\overline{M}}(\Omega))^N$. Let $\varphi\in(E_M(\Omega))^N$ such that $\|\varphi\|_M\le 1$. By (3.3) we have

$$\left(a(x,T_n(u_n),\nabla u_n)-a(x,T_n(u_n),\frac{\varphi}{k_4})\right)\cdot\left(\nabla u_n-\frac{\varphi}{k_4}\right)\ge 0.$$

Then

$$\begin{aligned}\int_\Omega a(x,T_n(u_n),\nabla u_n)\cdot\varphi dx \ &\le\ k_4\int_\Omega a(x,T_n(u_n),\nabla u_n)\cdot\nabla u_n dx\\ &\quad -k_4\int_\Omega a(x,T_n(u_n),\frac{\varphi}{k_4})\cdot\nabla u_n dx\\ &\quad +\int_\Omega a(x,T_n(u_n),\frac{\varphi}{k_4})\cdot\varphi dx.\end{aligned}$$

Let $\lambda = 1 + k_1 + k_3$. Using (3.2), (5.4), (5.5) and Young's inequality, we obtain

$$\begin{aligned}\int_\Omega a(x, T_n(u_n), \nabla u_n)\cdot \varphi dx \quad \leq \quad & k_4 c c_1 \|f\|_{m^*}|\Omega|^{1-\frac{1}{m^*}} + k_4\lambda \frac{c c_1\|f\|_{m^*}|\Omega|^{1-\frac{1}{m^*}}}{\overline{M}^{-1}M(h(c))} \\ & + (1+k_4)\int_\Omega \overline{M}(|a_0(x)|)dx \\ & + k_1(1+k_4)\overline{MP}^{-1}M(k_2 c)|\Omega| \\ & + k_3(1+k_4) + \lambda,\end{aligned}$$

which gives the desired result. ∎

From (5.3) and (5.6) one deduces that $u \in W_0^1L_M(\Omega)\cap L^\infty(\Omega)$, so that by [12, Theorem 4], there exists a sequence $\{v_j\}$ in $\mathcal{D}(\Omega)$ such that

$$v_j \to u \quad \text{in} \quad W_0^1 L_M(\Omega)$$

as $j\to\infty$, for the modular convergence and almost everywhere in $\Omega$. Moreover, we have

$$\|v_j\|_\infty \leq (N+1)\|u\|_\infty.$$

For $s > 0$, we denote by $\chi_j^s$ the characteristic function of the set

$$\Omega_j^s = \{x \in \Omega : |\nabla v_j(x)| \leq s\},$$

and by $\chi^s$ the characteristic function of the set

$$\Omega^s = \{x \in \Omega : |\nabla u(x)| \leq s\}.$$

Being $\beta$ continuous, thanks to (5.3) the sequence $\{\beta(u_n)\}$ is bounded, so that, there exists a constant $\beta_0$ such that

$$\|\beta(u_n)\|_\infty \leq \beta_0. \tag{5.8}$$

Consider the function $\varphi(t) = te^{\sigma t^2}$, $\sigma > 0$, and the real $\sigma_0 = \dfrac{\beta_0}{\overline{M}^{-1}M(h(c))}$ where $c$ is the constant in (5.3). It is well known that if $\sigma > (\frac{\sigma_0}{2})^2$, one has for all $t \in \mathbb{R}$

$$\varphi'(t) - \sigma_0|\varphi(t)| \geq \frac{1}{2}.$$

The choice of $\varphi(u_n - v_j)$ as test function in (4.1), yields for $n > c$

$$\begin{aligned}& \int_\Omega a(x, u_n, \nabla u_n)\cdot\nabla(u_n - v_j)\varphi'(u_n - v_j)dx \\ & + \int_\Omega B_n(x, u_n, \nabla u_n)\varphi(u_n - v_j)dx \\ & = \int_\Omega f_n\varphi(u_n - v_j)dx.\end{aligned} \tag{5.9}$$

In what follows, $\epsilon_i(n, j)$ $(i = 0, 1, 2, ...)$ denote various sequences of real numbers which converge to $0$ when $n$ and $j \to \infty$, i.e.

$$\lim_{j\to\infty}\lim_{n\to\infty}\epsilon_i(n, j) = 0.$$

In view of (5.3) and (5.7), we have $\varphi(u_n - v_j) \to \varphi(u - v_j)$ weakly* in $L^\infty(\Omega)$ as $n \to \infty$, then as $n \to \infty$

$$\int_\Omega f_n\varphi(u_n - v_j)dx \;\to\; \int_\Omega f\varphi(u - v_j)dx,$$

and since $u - v_j \to 0$ weakly* in $L^\infty(\Omega)$ as $j \to \infty$, we get

$$\int_\Omega f\varphi(u - v_j)dx \to 0 \quad \text{as } j \to \infty.$$

So that

$$\int_\Omega f_n\varphi(u_n - v_j)dx = \epsilon_0(n, j).$$

For the first term on the left-hand side of (5.9), we write

$$\begin{aligned}
&\int_\Omega a(x, u_n, \nabla u_n) \cdot \nabla(u_n - v_j)\varphi'(u_n - v_j)dx \\
&= \int_\Omega \left(a(x, u_n, \nabla u_n) - a(x, u_n, \nabla v_j\chi_j^s)\right) \cdot \left(\nabla u_n - \nabla v_j\chi_j^s\right)\varphi'(u_n - v_j)dx \\
&+ \int_\Omega a(x, u_n, \nabla v_j\chi_j^s) \cdot (\nabla u_n - \nabla v_j\chi_j^s)\varphi'(u_n - v_j)dx \\
&- \int_{\Omega\setminus\Omega_j^s} a(x, u_n, \nabla u_n) \cdot \nabla v_j\varphi'(u_n - v_j)dx.
\end{aligned}$$

As a consequence of Lemma 5.2, there exists $l \in (L_{\overline{M}}(\Omega))^N$ such that

$$a(x, u_n, \nabla u_n) \rightharpoonup l \text{ weakly in } (L_{\overline{M}}(\Omega))^N \text{ for } \sigma(\Pi L_{\overline{M}}, \Pi E_M).$$

Since $\nabla v_j\chi_{\Omega\setminus\Omega_j^s} \in (E_M(\Omega))^N$, we have

$$\int_{\Omega\setminus\Omega_j^s} a(x, u_n, \nabla u_n) \cdot \nabla v_j\varphi'(u_n - v_j)dx \to \int_{\Omega\setminus\Omega_j^s} l \cdot \nabla v_j\varphi'(u - v_j)dx$$

as $n \to \infty$, and the modular convergence of $\{v_j\}$, gives

$$\int_{\Omega\setminus\Omega_j^s} l \cdot \nabla v_j\varphi'(u - v_j)dx \to \int_{\Omega\setminus\Omega^s} l \cdot \nabla u dx$$

as $j \to \infty$. So that

$$\int_{\Omega\setminus\Omega_j^s} a(x, u_n, \nabla u_n) \cdot \nabla v_j\varphi'(u_n - v_j)dx = \int_{\Omega\setminus\Omega^s} l \cdot \nabla u dx + \epsilon_1(n, j).$$

Since $a(x, u_n, \nabla v_j\chi_j^s)\varphi'(u_n - v_j) \to a(x, u, \nabla v_j\chi_j^s)\varphi'(u - v_j)$ strongly in $(E_{\overline{M}}(\Omega))^N$ as $n \to \infty$ by lemma 2.1 and $\nabla u_n \rightharpoonup \nabla u$ weakly in $(L_M(\Omega))^N$ by (5.6), we obtain

$$\begin{aligned}
&\int_\Omega a(x, u_n, \nabla v_j\chi_j^s) \cdot (\nabla u_n - \nabla v_j\chi_j^s)\varphi'(u_n - v_j)dx \\
&\to \int_\Omega a(x, u, \nabla v_j\chi_j^s) \cdot (\nabla u - \nabla v_j\chi_j^s)\varphi'(u - v_j)dx
\end{aligned}$$

as $n \to \infty$, and since $\nabla v_j\chi_j^s \to \nabla u\chi^s$ strongly in $(E_M(\Omega))^N$ as $j \to \infty$, we get

$$\int_\Omega a(x, u, \nabla v_j\chi_j^s) \cdot (\nabla u - \nabla v_j\chi_j^s)\varphi'(u - v_j)dx \to 0$$

as $j \to \infty$. Thus,

$$\int_\Omega a(x, u_n, \nabla v_j\chi_j^s) \cdot (\nabla u_n - \nabla v_j\chi_j^s)\varphi'(u_n - v_j)dx = \epsilon_2(n, j).$$

Hence, (5.9) becomes

$$\tag{5.10}\begin{aligned}&\int_\Omega \big(a(x,u_n,\nabla u_n)-a(x,u_n,\nabla v_j\chi_j^s)\big)\cdot\big(\nabla u_n-\nabla v_j\chi_j^s\big)\,\varphi'(u_n-v_j)dx\\ &+\int_\Omega B_n(x,u_n,\nabla u_n)\varphi(u_n-v_j)dx \;=\; \int_{\Omega\setminus\Omega^s} l\cdot\nabla u dx \;+\; \epsilon_3(n,j).\end{aligned}$$

Now, we evaluate the second term on the left-hand side of (5.10). Using (3.4) and then (3.1) and (5.3), we obtain

$$\begin{aligned}&\left|\int_\Omega B_n(x,u_n,\nabla u_n)\varphi(u_n-v_j)dx\right|\\ &\le \int_\Omega \beta(u_n)M(|\nabla u_n|)|\varphi(u_n-v_j)|dx\\ &\le \int_\Omega \frac{\beta(u_n)}{\overline{M}^{-1}M(h(|u_n|))}a(x,u_n,\nabla u_n)\cdot\nabla u_n|\varphi(u_n-v_j)|dx\\ &\le \sigma_0\int_\Omega \big(a(x,u_n,\nabla u_n)-a(x,u_n,\nabla v_j\chi_j^s)\big)\cdot\big(\nabla u_n-\nabla v_j\chi_j^s\big)\,|\varphi(u_n-v_j)|dx\\ &+\sigma_0\int_\Omega a(x,u_n,\nabla v_j\chi_j^s)\cdot(\nabla u_n-\nabla v_j\chi_j^s)|\varphi(u_n-v_j)|dx\\ &+\sigma_0\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla v_j\chi_j^s|\varphi(u_n-v_j)|dx.\end{aligned}$$

As above, we have

$$\sigma_0\int_\Omega a(x,u_n,\nabla v_j\chi_j^s)\cdot(\nabla u_n-\nabla v_j\chi_j^s)|\varphi(u_n-v_j)|dx \;=\; \epsilon_4(n,j),$$

$$\sigma_0\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla v_j\chi_j^s|\varphi(u_n-v_j)|dx \;=\; \epsilon_5(n,j).$$

Then

$$\begin{aligned}&\left|\int_\Omega B_n(x,u_n,\nabla u_n)\varphi(u_n-v_j)dx\right|\\ &\le \sigma_0\int_\Omega \big(a(x,u_n,\nabla u_n)-a(x,u_n,\nabla v_j\chi_j^s)\big)\cdot\big(\nabla u_n-\nabla v_j\chi_j^s\big)\,|\varphi(u_n-v_j)|dx\\ &+\epsilon_6(n,j).\end{aligned}$$

This inequality and (5.10) allow to have

$$\begin{aligned}&\int_\Omega \big(a(x,u_n,\nabla u_n)-a(x,u_n,\nabla v_j\chi_j^s)\big)\cdot\big(\nabla u_n-\nabla v_j\chi_j^s\big)\\ &\times\left(\varphi'(u_n-v_j)-\sigma_0|\varphi(u_n-v_j)|\right)dx\\ &\le \int_{\Omega\setminus\Omega^s} l\cdot\nabla u dx \;+\; \epsilon_7(n,j).\end{aligned}$$

and then

$$\tag{5.11}\begin{aligned}&\int_\Omega \big(a(x,u_n,\nabla u_n)-a(x,u_n,\nabla v_j\chi_j^s)\big)\cdot\big(\nabla u_n-\nabla v_j\chi_j^s\big)\,dx\\ &\le\; 2\int_{\Omega\setminus\Omega^s} l\cdot\nabla u dx \;+\; 2\epsilon_7(n,j).\end{aligned}$$

On the other hand

$$\begin{aligned}
&\int_\Omega (a(x,u_n,\nabla u_n) - a(x,u_n,\nabla u\chi^s)) \cdot (\nabla u_n - \nabla u\chi^s)\, dx \\
&= \int_\Omega \left(a(x,u_n,\nabla u_n) - a(x,u_n,\nabla v_j\chi_j^s)\right) \cdot \left(\nabla u_n - \nabla v_j\chi_j^s\right) dx \\
&+ \int_\Omega a(x,u_n,\nabla u_n) \cdot (\nabla v_j\chi_j^s - \nabla u\chi^s) dx \\
&- \int_\Omega a(x,u_n,\nabla u\chi^s) \cdot (\nabla u_n - \nabla u\chi^s) dx \\
&+ \int_\Omega a(x,u_n,\nabla v_j\chi_j^s) \cdot (\nabla u_n - \nabla v_j\chi_j^s) dx.
\end{aligned}$$

Similar arguments as above show that

$$\int_\Omega a(x,u_n,\nabla u_n) \cdot (\nabla v_j\chi_j^s - \nabla u\chi^s) dx = \epsilon_8(n,j),$$

$$\int_\Omega a(x,u_n,\nabla u\chi^s) \cdot (\nabla u_n - \nabla u\chi^s) dx = \epsilon_9(n,j),$$

$$\int_\Omega a(x,u_n,\nabla v_j\chi_j^s) \cdot (\nabla u_n - \nabla v_j\chi_j^s) dx = \epsilon_{10}(n,j). \tag{5.12}$$

It follows, by using (5.11), that

$$\begin{aligned}
&\int_\Omega (a(x,u_n,\nabla u_n) - a(x,u_n,\nabla u\chi^s)) \cdot (\nabla u_n - \nabla u\chi^s)\, dx \\
&\leq 2\int_{\Omega\setminus\Omega^s} l\cdot \nabla u dx + \epsilon_{11}(n,j).
\end{aligned}$$

Let now $r \leq s$, we write

$$\begin{aligned}
0 &\leq \int_{\Omega^r} (a(x,u_n,\nabla u_n) - a(x,u_n,\nabla u)) \cdot (\nabla u_n - \nabla u)\, dx \\
&\leq \int_{\Omega^s} (a(x,u_n,\nabla u_n) - a(x,u_n,\nabla u)) \cdot (\nabla u_n - \nabla u)\, dx \\
&= \int_{\Omega^s} (a(x,u_n,\nabla u_n) - a(x,u_n,\nabla u\chi^s)) \cdot (\nabla u_n - \nabla u\chi^s)\, dx \\
&\leq \int_{\Omega} (a(x,u_n,\nabla u_n) - a(x,u_n,\nabla u\chi^s)) \cdot (\nabla u_n - \nabla u\chi^s)\, dx \\
&\leq 2\int_{\Omega\setminus\Omega^s} l\cdot \nabla u dx + \epsilon_{11}(n,j).
\end{aligned}$$

Since $l\cdot\nabla u \in L^1(\Omega)$, letting $s \to \infty$, we get

$$\int_{\Omega^r} (a(x,u_n,\nabla u_n) - a(x,u_n,\nabla u)) \cdot (\nabla u_n - \nabla u)\, dx \to 0 \tag{5.13}$$

as $n \to \infty$. Let $D_n$ be defined by

$$D_n = (a(x,u_n,\nabla u_n) - a(x,u_n,\nabla u)) \cdot (\nabla u_n - \nabla u)\, .$$

As a consequence of (5.13), one has $D_n \to 0$ strongly in $L^1(\Omega^r)$, extracting a subsequence, still denoted by $\{u_n\}$, we get

$$D_n \to 0 \quad \text{a.e in } \Omega^r.$$

Then, there exists a subset $Z$ of $\Omega^r$ of zero measure such that: $D_n(x) \to 0$ for all $x \in \Omega^r \setminus Z$. Fix $x \in \Omega^r \setminus Z$, by using (3.1) and (3.2) we obtain

$$D_n(x) \geq \overline{M}^{-1} M(h(c)) M(|\nabla u_n(x)|) - c_1(x) \left(1 + \overline{M}^{-1} M(k_4 |\nabla u_n(x)|) + |\nabla u_n(x)|\right),$$

where $c$ is the constant which appears in (5.3) and $c_1(x)$ is a constant which does not depend on $n$. Thus, the sequence $\{\nabla u_n(x)\}$ is bounded in $\mathbb{R}^N$ and then for a subsequence $\{u_{n'}(x)\}$ we have

$$\nabla u_{n'}(x) \to \xi \ \text{ in } \mathbb{R}^N$$

and

$$(a(x, u(x), \xi) - a(x, u(x), \nabla u(x))) \cdot (\xi - \nabla u(x)) = 0.$$

Since $a(x, s, \xi)$ is strictly monotone, we have $\xi = \nabla u(x)$, and then $\nabla u_n(x) \to \nabla u(x)$ for the whole sequence. It follows that

$$\nabla u_n \to \nabla u \quad \text{a.e. in } \Omega^r.$$

Consequently, as $r$ is arbitrary, one can deduce that

(5.14) $$\nabla u_n \to \nabla u \quad \text{a.e. in } \Omega.$$

Then, by Lemma 5.2 and [16, Theorem 14.6] we have

(5.15) $$a(x, T_n(u_n), \nabla u_n) \rightharpoonup a(x, u, \nabla u) \text{ weakly in } (L_{\overline{M}}(\Omega))^N \quad \text{for } \sigma(\Pi L_{\overline{M}}, \Pi E_M).$$

**Step 4:** Modular convergence of the gradients.
Going back to (5.11), we can write for $n > c$

$$\begin{aligned}\int_\Omega a(x, u_n, \nabla u_n) \nabla u_n dx \quad \leq \quad & \int_\Omega a(x, u_n, \nabla u_n) \cdot \nabla v_j \chi_j^s dx \\ & + \int_\Omega a(x, u_n, \nabla v_j \chi_j^s) \cdot (\nabla u_n - \nabla v_j \chi_j^s) dx \\ & + 2 \int_{\Omega \setminus \Omega^s} l \cdot \nabla u dx \ + \ 2\epsilon_7(n, j),\end{aligned}$$

and by (5.12), we get

$$\begin{aligned}\int_\Omega a(x, u_n, \nabla u_n) \cdot \nabla u_n dx \quad \leq \quad & \int_\Omega a(x, u_n, \nabla u_n) \cdot \nabla v_j \chi_j^s dx \\ & + 2 \int_{\Omega \setminus \Omega^s} l \cdot \nabla u dx \ + \ \epsilon_{12}(n, j).\end{aligned}$$

Passing to the limit superior over $n$ and then to the limit over $j$ in both sides of this inequality, we obtain

$$\limsup_{n \to \infty} \int_\Omega a(x, u_n, \nabla u_n) \cdot \nabla u_n dx \ \leq \ \int_\Omega a(x, u, \nabla u) \cdot \nabla u \chi^s dx \ + \ \int_{\Omega \setminus \Omega^s} l \cdot \nabla u dx,$$

and by letting $s \to \infty$, one has

$$\limsup_{n \to \infty} \int_\Omega a(x, u_n, \nabla u_n) \cdot \nabla u_n dx \ \leq \ \int_\Omega a(x, u, \nabla u) \cdot \nabla u dx.$$

Fatou's lemma allows us to have

$$\lim_{n \to \infty} \int_\Omega a(x, u_n, \nabla u_n) \cdot \nabla u_n dx \ = \ \int_\Omega a(x, u, \nabla u) \cdot \nabla u dx.$$

Hence, by Lemma 2.3, we conclude that

$$a(x,u_n,\nabla u_n)\cdot\nabla u_n \;\to\; a(x,u,\nabla u)\cdot\nabla u \quad \text{in } L^1(\Omega). \tag{5.16}$$

Then, by (3.1), (5.3) and the convexity of the N-function $M$ we obtain for $n > c$

$$\begin{aligned} M\left(\frac{|\nabla u_n-\nabla u|}{2}\right) &\le \frac{1}{2\overline{M}^{-1}(M(h(c)))}\overline{M}^{-1}(M(h(|u_n|)))M(|\nabla u_n|) \\ &\quad + \frac{1}{2\overline{M}^{-1}(M(h(c)))}\overline{M}^{-1}(M(h(|u|)))M(|\nabla u|) \\ &\le \frac{1}{2\overline{M}^{-1}(M(h(c)))}\,a(x,u_n,\nabla u_n)\cdot\nabla u_n \\ &\quad + \frac{1}{2\overline{M}^{-1}(M(h(c)))}\,a(x,u,\nabla u)\cdot\nabla u. \end{aligned}$$

Therefore, by (5.16) and Vitali's theorem we conclude that

$$u_n \;\to\; u \quad \text{in } W_0^1L_M(\Omega)$$

for the modular convergence.
**Step5:** Equi-integrability of the non-linearities.
We will now prove that

$$B_n(x,u_n,\nabla u_n) \;\to\; B(x,u,\nabla u) \quad \text{strongly in } L^1(\Omega), \tag{5.17}$$

by using Vitali's theorem. Thanks to (5.7) and (5.14), one has

$$B_n(x,u_n,\nabla u_n) \;\to\; B(x,u,\nabla u) \quad \text{a.e in } \Omega.$$

It remains to show the uniform equi-integrability of the sequence $\{B_n(x,u_n,\nabla u_n)\}$. By (3.4) and (5.8), we have

$$|B_n(x,u_n,\nabla u_n)| \le \beta_0 M(|\nabla u_n|).$$

Let $E$ be a measurable subset of $\Omega$. Thanks to (3.1), (5.3) and (5.8), We have

$$\int_E |B(x,u_n,\nabla u_n)|dx \;\le\; \frac{\beta_0}{\overline{M}^{-1}M(h(c))}\int_E a(x,u_n,\nabla u_n)\cdot\nabla u_n dx.$$

Using (5.16) and Vitali's theorem we obtain the equi-integrability of the sequence $\{B_n(x,u_n,\nabla u_n)\}$. Which proves (5.17).
**Step 6:** Passage to the limit.
Let $v \in W_0^1L_M(\Omega)\cap L^\infty(\Omega)$, by virtue of (5.15) and (5.17) it is easy to pass to the limit in (4.1) and obtain

$$\int_\Omega a(x,u,\nabla u)\cdot\nabla v dx + \int_\Omega B(x,u,\nabla u)v dx = \int_\Omega f v dx.$$

Moreover we have $u \in W_0^1L_M(\Omega)\cap L^\infty(\Omega)$. Thus, $u$ is a weak solution of (3.7).

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