ON A SUBCLASS OF UNIFORMLY CONVEX FUNCTIONS DEFINED BY THE
DZIOK-SRIVASTAVA OPERATOR
M. K. AOUF AND G. MURUGUSUNDARAMOORTHY

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MATHEMATICS DEPARTMENT, FACULTY OF SCIENCE, MANSOURA UNIVERSITY 35516, EGYPT.

SCHOOL OF SCIENCE AND HUMANITIES, VIT UNIVERSITY, VELLORE - 632014, INDIA.
mkaouf127@yahoo.com
gmsmoorthy@yahoo.com

ABSTRACT. Making use of the Dziok-Srivastava operator, we define a new subclass $T_{m}^{d}((\alpha_{1}); \alpha, \beta)$ of uniformly convex function with negative coefficients. In this paper, we obtain coefficient estimates, distortion theorems, locate extreme points and obtain radii of close-to-convexity, starlikeness and convexity for functions belonging to the class $T_{m}^{d}((\alpha_{1}); \alpha, \beta)$. We consider integral operators associated with functions belonging to the class $H_{m}^{d}((\alpha_{1}); \alpha, \beta)$ defined via the Dziok-Srivastava operator. We also obtain several results for the modified Hadamard products of functions belonging to the class $T_{m}^{d}((\alpha_{1}); \alpha, \beta)$ and we obtain properties associated with generalized fractional calculus operators.

Key words and phrases: Dziok-Srivastava operator, Analytic, Uniformly convex, Extreme points, Modified Hadamard products, Fractional calculus.

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1. Introduction

Let $S$ denote the class of functions of the form

$$(1.1) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

that are analytic and univalent in the open unit disc $U = \{z : |z| < 1\}$. Let $K(\alpha)$ and $S^*(\alpha)$ denote the subclasses of $S$ that are, respectively, convex and starlike functions of order $\alpha, 0 \leq \alpha < 1$. For convenience, we write $K(0) = K$ and $S^*(0) = S^*$ (e.g., [24]). Goodman ([6] and [7]) defined the following subclasses of $K$ and $S^*$.

**Definition 1.1.** A function $f(z)$ is uniformly convex (starlike) in $U$ if $f(z)$ is in $K$ and $S^*$ and has the property that for every circular arc $\gamma$ contained in $U$, with center $\zeta$ also in $U$, the arc $f(\gamma)$ is convex (starlike) with respect to $f(\zeta)$.

Goodman ([6] and [7]) then gave the following two-variable analytic characterizations of these classes, denoted, respectively, by UCV and UST.

**Theorem 1.1 (A).** A function $f(z)$ of the form (1.1) is in UCV if and only if

$$\text{Re} \left\{ 1 + (z - \zeta) \frac{f''(z)}{f'(z)} \right\} \geq 0, (z, \zeta) \in U \times U,$$

and is in UST if and only if

$$\text{Re} \left\{ \frac{f(z) - f(\zeta)}{(z - \zeta)f'(z)} \right\} \geq 0, (z, \zeta) \in U \times U.$$

Ma and Minda [14] and Ronning [19] independently found a more applicable one-variable characterization for UCV.

**Theorem 1.2 (B).** A function $f(z)$ of the form (1.1) is in UCV if and only if

$$\text{Re} \left\{ 1 + z\frac{f''(z)}{f'(z)} \right\} \geq \left| z\frac{f''(z)}{f'(z)} \right|, z \in U.$$

We note that [6] that the classical Alexander’s result $f(z) \in K \iff zf'(z) \in S^*$ does not hold between the classes UCV and UST. Later on, Ronning [20] introduced a new class $S_p$ of starlike functions related to UCV defined as

$$\text{(1.5)} \quad f(z) \in S_p \iff \text{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} \geq \left| \frac{zf''(z)}{f'(z)} \right| - 1, \ z \in U.$$  

Note that

$$\text{(1.6)} \quad f(z) \in UCV \iff zf'(z) \in S_p.$$  

Also in [19], Ronning generalized the classes UCV and $S_p$ by introducing a parameter $\alpha$ in the following way.

**Definition 1.2.** A function $f(z)$ of the form (1.1) is in $S_p(\alpha)$, if it satisfies the analytic characterization:

$$\text{(1.7)} \quad \text{Re} \left\{ \frac{zf'(z)}{f(z)} - \alpha \right\} \geq \left| \frac{zf'(z)}{f(z)} - 1 \right|, \alpha \in \mathbb{R}; \ z \in U.$$
and \( f(z) \in UCV(\alpha) \), the class of uniformly convex functions of order \( \alpha \), if and only if \( zf'(z) \in S_\rho(\alpha) \).

For the class \( S_\rho(\alpha) \), we get a domain whose boundary is a parabola with vertex \( w = \frac{1 + \alpha}{2} \). Also, we note that \( S_\rho(\alpha) \subset S^\ast \) for all \(-1 \leq \alpha < 1\), \( S_\rho(\alpha) \not\subset S \) for \( \alpha < -1 \) and \( UCV(\alpha) \subset K \) for \( \alpha \geq -1 \).

By \( \beta - UCV, 0 \leq \beta < \infty \), we denote the class of all \( \beta - \) uniformly convex functions introduced by Kanas and Wiśniowska \cite{3}. Recall that a function \( f(z) \in S \) is said to be \( \beta - \) uniformly convex in \( U \), if the image of every circular arc contained in \( U \) with center at \( \zeta \), where \( |\zeta| \leq \beta \), is convex. Note that the class \( 1 - UCV \) coincides with the class \( UCV \). Moreover, for \( \beta = 0 \) we get the class \( K \). From \cite{3} it is known that \( f(z) \in \beta - UCV \) if and only if it satisfies the following condition

\[
\text{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \beta \left| \frac{zf''(z)}{f'(z)} \right|, \quad z \in U, 0 \leq \beta < \infty .
\]  

(1.8)

We consider the class \( \beta - S^\ast, 0 \leq \beta < \infty \), of \( \beta - \) starlike functions (see \cite{9}) which are associated with \( \beta - \) uniformly convex functions by the relation

\[
f(z) \in \beta - UCV \iff zf'(z) \in \beta - S^\ast .
\]  

(1.9)

Thus, the class \( \beta - S^\ast, 0 \leq \beta < \infty \), is the subclass of \( S \), consisting of functions that satisfy the analytic condition

\[
\text{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \beta \left| \frac{zf'(z)}{f(z)} - 1 \right|, \quad z \in U .
\]  

(1.10)

Let \( (f * g)(z) \) denotes the Hadamard product (or convolution) of the functions \( f(z) \) and \( g(z) \), that is, if \( f(z) \) is given by (1.1) and \( g(z) \) is given by

\[
g(z) = z + \sum_{n=2}^{\infty} b_n z^n ,
\]  

(1.11)

then

\[
(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g * f)(z) .
\]  

(1.12)

For \( \alpha_j \in C(j = 1, 2, \ldots, l) \) and \( \beta_j \in C \setminus \{0, -1, -2, \ldots\} (j = 1, 2, \ldots, m) \), the generalized hypergeometric function \( _lF_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) \) is defined by the infinite series

\[
_lF_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) = \sum_{n=2}^{\infty} \frac{(\alpha_1)_n \ldots (\alpha_l)_n}{(\beta_1)_n \ldots (\beta_m)_n} \frac{z^n}{n!} ,
\]  

(1.13)

where \( (\lambda)_n \) is the Pochhammer symbol defined by

\[
(\lambda)_n = \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} = \left\{ \begin{array}{ll} 1 & (n = 0) ; \\ \lambda(\lambda + 1)\ldots(\lambda + n - 1) & (n \in N) . \end{array} \right.
\]  

(1.14)

Corresponding to the function

\[
h(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) = z _lF_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) ,
\]

the Dziok-Srivastava operator (\cite{4, 5, 11} and \cite{12}) \( H_m^l(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m) \) is defined by the Hadamard product
\[ h(\alpha_1, ..., \alpha_l; \beta_1, ..., \beta_m) f(z) = h(\alpha_1, ..., \alpha_l; \beta_1, ..., \beta_m; z) \ast f(z) \]

\[ = z + \sum_{n=2}^{\infty} \Gamma_n a_n z^n \]

where

\[ \Gamma_n = \frac{(\alpha_1)_{n-1} \cdots (\alpha_l)_{n-1}}{(\beta_1)_{n-1} \cdots (\beta_m)_{n-1}} \frac{1}{(n-1)!}. \]

For brevity, we write

\[ H_m^l[\alpha_1] f(z) = H_m^l(\alpha_1, ..., \alpha_l; \beta_1, ..., \beta_m) f(z). \]

It is easy to see from \((1.15)\) that

\[ z(H_m^l[\alpha_1] f(z))' = \alpha_1 H_m^l[\alpha_1 + 1] f(z) - (\alpha_1 - 1) H_m^l[\alpha_1] f(z). \]

For \(\beta \geq 0\) and \(-1 \leq \alpha < 1\), we let \(S_m^l([\alpha_1]; \alpha, \beta)\) denote the subclass of \(S\) consisting of functions \(f(z)\) of the form \((1.1)\) and satisfying the analytic criterion

\[ \Re \left\{ \frac{z(H_m^l[\alpha_1] f(z))'}{H_m^l[\alpha_1] f(z)} - \alpha \right\} > \beta \left| \frac{z(H_m^l[\alpha_1] f(z))'}{H_m^l[\alpha_1] f(z)} - 1 \right|, \quad z \in U. \]

We denote by \(T\) the subclass of \(S\) consisting of functions of the form

\[ f(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad (a_n \geq 0). \]

Further, we define the class \(T_m^l([\alpha_1]; \alpha, \beta)\) by

\[ T_m^l([\alpha_1]; \alpha, \beta) = S_m^l([\alpha_1]; \alpha, \beta) \cap T. \]

We note that

\begin{enumerate}
  \item \(T_0^1([1]; \alpha, \beta) = S_p T(\alpha, \beta)\)
  \[ = \left\{ f(z) \in T : \Re \left\{ \frac{zf'(z)}{f(z)} - \alpha \right\} > \beta \left| \frac{zf'(z)}{f(z)} - 1 \right|, \quad -1 \leq \alpha < 1, \right\} \]
  \[ \beta \geq 0, \quad z \in U \}
\end{enumerate}

The class \(S_p T(\alpha, 1) = S_p T(\alpha)\) was studied by Bharati et al. \[1\].

\begin{enumerate}
  \item \(T_1^2([a, 1]; \alpha, \beta) = S_p T(a, c; \alpha, \beta)\) (Murugusungaramoorthy and Magesh \[15\])
  \[ = \left\{ f(z) \in T : \Re \left\{ \frac{z(L(a, c)f(z))'}{L(a, c)f(z)} - \alpha \right\} > \beta \left| \frac{z(L(a, c)f(z))'}{L(a, c)f(z)} - 1 \right|, \right\} \]
  \[ -1 \leq \alpha < 1, \beta \geq 0, \ a > 0, \ c > 0, \ z \in U \}; \]
\end{enumerate}

where \(L(a, c)\) is the Carlson - Shaffer operator \[3\].

\begin{enumerate}
  \item \(H_1^2([\lambda + 1, 1]; \alpha, \beta) = S_p T(\lambda; \alpha, \beta)\) (Shams and Kulkarni \[23\])
  \[ = \left\{ f(z) \in T : \Re \left\{ \frac{z(D_1^\lambda f(z))'}{D_1^\lambda f(z)} - \alpha \right\} > \beta \left| \frac{z(D_1^\lambda f(z))'}{D_1^\lambda f(z)} - 1 \right|, \right\} \]
  \[ 0 \leq \alpha < 1, \beta \geq 0, \lambda > -1, \ z \in U \}, \]
\end{enumerate}

where \(D_1^\lambda(\lambda > -1)\) is the Ruscheweyh derivative operator \[21\].
(IV) \[ H_1^2 ([\nu + 1; \nu + 2]; \alpha, \beta) = S_p T (\nu; \alpha, \beta) \]

\[ = \begin{cases} 
  f (z) \in T : \Re \left\{ \frac{z (J_\nu f (z))'}{J_\nu f (z)} - \alpha \right\} > \beta \left| \frac{z (J_\nu f (z))'}{J_\nu f (z)} - 1 \right|, \\
  -1 \leq \alpha < 1, \beta \geq 0, \nu > -1, z \in U \end{cases}, \]

(1.23)

where \( J_\nu (\nu > -1) \) is the generalized Bernardi-Libera-Livingston operator (2), [10] and [13].

(V) \[ H_1^2 ([2; 1 + 2]; \alpha, \beta) = S_p T (\mu; \alpha, \beta) \]

\[ = \begin{cases} 
  f (z) \in T : \Re \left\{ \frac{z (\Omega^\mu f (z))'}{\Omega^\mu f (z)} - \alpha \right\} > \beta \left| \frac{z (\Omega^\mu f (z))'}{\Omega^\mu f (z)} - 1 \right|, \\
  -1 \leq \alpha < 1, \beta \geq 0, 0 \leq \mu < 1, z \in U \end{cases}, \]

where

\[ (1.24) \]

\[ \Omega^\mu f (z) = \Gamma (2 - \mu) z^\mu D^\mu f (z) (0 \leq \mu < 1), \]

where \( \Omega^\mu \) is the Srivastava-Owa fractional derivative operator ([16] and [18]).

2. COEFFICIENT ESTIMATES

**Theorem 2.1.** A function \( f (z) \) of the form (1.1) is in the class \( S_m ([\alpha_1]; \alpha, \beta) \) if

\[ (2.1) \]

\[ \sum_{n=2}^{\infty} C_n |a_n| \leq 1 - \alpha, \]

where

\[ (2.2) \]

\[ C_n = [n (1 + \beta) - (\alpha + \beta)] \Gamma_n \]

and \( \Gamma_n \) is defined by (1.16).

**Proof.** It is suffices to show that

\[ \beta \left| \frac{z (H_m^1 [\alpha_1] f (z))'}{H_m^1 [\alpha_1] f (z)} - 1 \right| - \Re \left\{ \frac{z (H_m^1 [\alpha_1] f (z))'}{H_m^1 [\alpha_1] f (z)} - 1 \right\} \leq 1 - \alpha. \]

We have

\[ \beta \left| \frac{z (H_m^1 [\alpha_1] f (z))'}{H_m^1 [\alpha_1] f (z)} - 1 \right| - \Re \left\{ \frac{z (H_m^1 [\alpha_1] f (z))'}{H_m^1 [\alpha_1] f (z)} - 1 \right\} \leq (1 + \beta) \left| \frac{z (H_m^1 [\alpha_1] f (z))'}{H_m^1 [\alpha_1] f (z)} - 1 \right| \]

\[ \leq \frac{(1 + \beta) \sum_{n=2}^{\infty} (n - 1) \Gamma_n |a_n|}{1 - \sum_{n=2}^{\infty} \Gamma_n |a_n|} \]

This last expression is bounded above by \( (1 - \alpha) \) if

\[ \sum_{n=2}^{\infty} [n (1 + \beta) - (\alpha + \beta)] \Gamma_n |a_n| \leq 1 - \alpha, \]

and the proof is complete. \( \blacksquare \)
**Theorem 2.2.** A necessary and sufficient condition for \( f(z) \) of the form (1.19) to be in the class \( T^l_m([\alpha_1]; \alpha, \beta) \) is that

\[
\sum_{n=2}^{\infty} C_n a_n \leq 1 - \alpha ,
\]

where \( C_n \) is given by (2.2).

**Proof.** In view of Theorem 2.1, we need only to prove the necessity. If \( f(z) \in T^l_m([\alpha_1]; \alpha, \beta) \) and \( z \) is real, then (1.18) yields

\[
1 - \sum_{n=2}^{\infty} n \Gamma_n a_n z^{n-1} - \alpha \geq \beta \left| \sum_{n=2}^{\infty} (n-1) \Gamma_n a_n z^{n-1} \right|.
\]

Letting \( z \to 1^- \) along the real axis, we obtain the desired inequality

\[
\sum_{n=2}^{\infty} [n(1 + \beta) - (\alpha + \beta)] \Gamma_n a_n \leq 1 - \alpha , -1 \leq \alpha < 1, \beta \geq 0.
\]

**Corollary 2.3.** Let the function \( f(z) \) defined by (1.19) be in the class \( T^l_m([\alpha_1]; \alpha, \beta) \). Then

\[
a_n \leq \frac{(1 - \alpha)}{C_n} (n \geq 2).
\]

The result is sharp for the function \( f(z) \) given by

\[
f(z) = z - \frac{1 - \alpha}{C_n} z^n (n \geq 2),
\]

where \( C_n \) is defined by (2.2).

### 3. Growth and Distortion Theorem

**Theorem 3.1.** Let the function \( f(z) \) defined by (1.19) be in the class \( T^l_m([\alpha_1]; \alpha, \beta) \). If the sequence \( \{C_n\} \) is nondecreasing, then

\[
|z| - \frac{1 - \alpha}{(2 - \alpha + \beta) \Gamma_2} |z|^2 \leq |f(z)| \leq |z| + \frac{1 - \alpha}{(2 - \alpha + \beta) \Gamma_2} |z|^2 (z \in U).
\]

If the sequence \( \{C_n\} \) is nondecreasing, then

\[
1 - \frac{2(1 - \alpha)}{(2 - \alpha + \beta) \Gamma_2} |z| \leq |f'(z)| \leq 1 + \frac{2(1 - \alpha)}{(2 - \alpha + \beta) \Gamma_2} |z| (z \in U),
\]

where \( C_n \) is defined by (2.2). The result is sharp, with the extremal function \( f(z) \) defined by

\[
f(z) = z - \frac{1 - \alpha}{(2 - \alpha + \beta) \Gamma_2} z^2.
\]

**Proof.** Let a function \( f(z) \) of the form (1.19) belong to the class \( T^l_m([\alpha_1]; \alpha, \beta) \). If the sequence \( \{C_n\} \) is nondecreasing and positive, by Theorem 2.2, we have

\[
\sum_{n=2}^{\infty} a_n \leq \frac{(1 - \alpha)}{(2 - \alpha + \beta) \Gamma_2} ,
\]
and if the sequence \( \{ \frac{C_n}{a_n} \} \) is nondecreasing and positive, by Theorem 2.2 we have

\[
\sum_{n=2}^{\infty} n a_n \leq \frac{2(1 - \alpha)}{(2 - \alpha + \beta)\Gamma_2}.
\]

Making use of the conditions (3.4) and (3.5), in conjunction with the definition (1.19), we readily obtain the assertion (3.1) and (3.2) of 3.1.

4. Extreme Points

In view of the necessary and sufficient conditions of Theorem 2.2, the family \( T_m([\alpha_1]; \alpha, \beta) \) is closed under convex linear combinations. This leads to the determination of the extreme points for the family.

**Theorem 4.1.** Let \( C_n \) be defined by (2.2) and let us put

\[
f_1(z) = z
\]

and

\[
f_n(z) = z - \frac{1 - \alpha}{C_n} z^n \quad (n \geq 2)
\]

for \(-1 \leq \alpha < 1\) and \(\beta \geq 0\). Then \( f(z) \) is in the class \( T_m([\alpha_1]; \alpha, \beta) \) if and only if it can be expressed in the form:

\[
f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z),
\]

where \( \lambda_n \geq 0 (n \geq 1) \) and \( \sum_{n=1}^{\infty} \lambda_n = 1 \).

**Proof.** Assume that

\[
f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z) = z - \sum_{n=2}^{\infty} \frac{1 - \alpha}{C_n} \lambda_n z^n.
\]

Then it follows that

\[
\sum_{n=2}^{\infty} \frac{C_n}{1 - \alpha} \cdot \frac{1 - \alpha}{C_n} \lambda_n = \sum_{n=2}^{\infty} \lambda_n = 1 - \lambda_1 \leq 1.
\]

So by Theorem 2.2 \( f(z) \in T_m([\alpha_1]; \alpha, \beta) \).

Conversely, assume that the function \( f(z) \) defined by (1.19) belongs to the class \( T_m([\alpha_1]; \alpha, \beta) \). Then

\[
a_n \leq \frac{1 - \alpha}{C_n} \quad (n \geq 2).
\]

Setting

\[
\lambda_n = \frac{C_n}{1 - \alpha} a_n \quad (n \geq 2)
\]

and

\[
\lambda_1 = 1 - \sum_{n=2}^{\infty} \lambda_n,
\]

we can see that \( f(z) \) can be expressed in the form (4.3). This completes the proof of Theorem 4.1.
Corollary 4.2. The extreme points of the class $T^d_m([\alpha_1]; \alpha, \beta)$ are the functions $f_n(z)(n \geq 1)$ given by Theorem 4.1.

5. Radii of Close-to-Convexity, Starlikeness and Convexity

Theorem 5.1. Let the function $f(z)$ defined by (1.19) be in the class $T^d_m([\alpha_1]; \alpha, \beta)$. Then $f(z)$ is close-to-convex of order $\rho$ ($0 \leq \rho < 1$) in $|z| < r_1$, where

$$r_1 = \inf_n \left\{ \frac{(1 - \rho)C_n}{n(1 - \alpha)} \right\} \frac{1}{n-1} (n \geq 2),$$

(5.1)

where $C_n$ is defined by (2.2). The result is sharp, with the extremal function $f(z)$ given by (2.5).

Proof. We must show that

$$|f'(z) - 1| \leq 1 - \rho$$

for $|z| < r_1$, where $r_1$ is given by (5.1). Indeed we find from the definition (1.19) that

$$|f'(z) - 1| \leq \sum_{n=2}^{\infty} na_n |z|^{n-1}.$$ 

Thus

$$|f'(z) - 1| \leq 1 - \rho$$

if

$$\sum_{n=2}^{\infty} \left( \frac{n}{1 - \rho} \right) a_n |z|^{n-1} \leq 1.$$ 

(5.2)

But, by Theorem 2.2, (5.2) will be true if

$$\left( \frac{n}{1 - \rho} \right) |z|^{n-1} \leq \frac{C_n}{1 - \alpha},$$

that is, if

$$|z| \leq \left\{ \frac{(1 - \rho)C_n}{n(1 - \alpha)} \right\} \frac{1}{n-1} (n \geq 2).$$

(5.3)

Theorem 5.1 follows easily from (5.3).

Theorem 5.2. Let the function $f(z)$ defined by (1.19) be in the class $T^d_m([\alpha_1]; \alpha, \beta)$. Then the function $f(z)$ is starlike of order $\rho$ ($0 \leq \rho < 1$) in $|z| < r_2$, where

$$r_2 = \inf_n \left\{ \frac{(1 - \rho)C_n}{(n - \rho)(1 - \alpha)} \right\} \frac{1}{n-1} (n \geq 2),$$

(5.4)

where $C_n$ is defined by (2.2). The result is sharp, with the extremal function $f(z)$ given by (2.5).

Proof. It is sufficient to show that

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq 1 - \rho$$

for $|z| < r_2$, where
where \( r_2 \) is given by (5.4). Indeed we find, again from the definition (1.19) that
\[
\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \frac{\sum_{n=2}^{\infty} (n-1)a_n |z|^{n-1}}{1 - \sum_{n=2}^{\infty} a_n |z|^{n-1}}.
\]
Thus
\[
\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq 1 - \rho
\]
if
(5.5)
\[
\sum_{n=2}^{\infty} \left( \frac{n-\rho}{1-\rho} \right) a_n |z|^{n-1} \leq 1.
\]
But, by Theorem 2.2, (5.5) will be true if
\[
\left( \frac{n-\rho}{1-\rho} \right) |z|^{n-1} \leq \frac{C_n}{1-\alpha},
\]
that is, if
(5.6)
\[
|z| \leq \left\{ \frac{(1-\rho)C_n}{(n-\rho)(1-\alpha)} \right\} \frac{1}{n-1} (n \geq 2).
\]
Theorem 5.2 now follows easily from (5.6).

**Corollary 5.3.** Let the function \( f(z) \) defined by (1.19) be in the class \( T_{m}^{l}([\alpha_1]; \alpha, \beta) \). Then \( f(z) \) is convex of order \( \rho \) \((0 \leq \rho < 1)\) in \(|z| < r_3\), where
(5.7)
\[
r_3 = \inf_n \left\{ \frac{(1-\rho)C_n}{(n-\rho)(1-\alpha)} \right\} \frac{1}{n-1} (n \geq 2),
\]
where \( C_n \) is defined by (2.2). The result is sharp, with the extremal function \( f(z) \) given by (2.5).

### 6. A Family of Integral Operators

**Theorem 6.1.** Let the function \( f(z) \) defined by (1.19) be in the class \( T_{m}^{l}([\alpha_1]; \alpha, \beta) \) and let \( c \) be a real number such that \( c > -1 \). Then the function \( F(z) \) defined by
(6.1)
\[
F(z) = \frac{c+1}{z^c} \int_{0}^{z} t^{c-1} f(t) dt \ (c > -1)
\]
also belongs to the class \( T_{m}^{l}([\alpha_1]; \alpha, \beta) \).

**Proof.** From the representation (6.1) of \( F(z) \), it follows that
\[
F(z) = z - \sum_{n=2}^{\infty} b_n z^n,
\]
where
\[
b_n = \left( \frac{c+1}{c+n} \right) a_n.
\]
Therefore, we have
\[
\sum_{n=2}^{\infty} C_n b_n = \sum_{n=2}^{\infty} C_n \frac{c + 1}{c + n} a_n \\
\leq \sum_{n=2}^{\infty} C_n a_n \leq 1 - \alpha ,
\]
since \( f(z) \in T_m^l([\alpha_1]; \alpha, \beta) \). Hence, by Theorem 2.2 \( F(z) \in T_m^l([\alpha_1]; \alpha, \beta) \). On the other hand, the converse is not true. This leads to a radius of univalence result. 

**Theorem 6.2.** Let the function \( F(z) = z - \sum_{n=2}^{\infty} a_n z^n (a_n \geq 0) \) be in the class \( T_m^l([\alpha_1]; \alpha, \beta) \), and let \( c \) be a real number such that \( c > -1 \). Then the function \( f(z) \) given by (6.1) is univalent in \( |z| < R^* \), where

\[
R^* = \inf_n \left\{ \frac{(c + 1)C_n}{n(c + n(1 - \alpha))} \right\}^{1 \over n - 1} (n \geq 2),
\]

The result is sharp.

**Proof.** Form (6.1), we have
\[
f(z) = z^{1-c}(z^cF(z))' = z - \sum_{n=2}^{\infty} \left( \frac{c + n}{c + 1} \right) a_n z^n .
\]
In order to obtain the required result, it suffices to show that
\[
|f'(z) - 1| < 1 \quad \text{whenever} \quad |z| < R^* ,
\]
where \( R^* \) is given by (6.2). Now
\[
|f'(z) - 1| \leq \sum_{n=2}^{\infty} \frac{n(c + n)}{(c + 1)} a_n |z|^{n-1} .
\]
Thus \( |f'(z) - 1| < 1 \) if
\[
\sum_{n=2}^{\infty} \frac{n(c + n)}{(c + 1)} a_n |z|^{n-1} < 1 .
\]
But Theorem 2.2 confirms that
\[
\sum_{n=2}^{\infty} \frac{C_n}{1 - \alpha} a_n \leq 1 .
\]
Hence (6.4) will be satisfied if
\[
\frac{n(c + n)}{(c + 1)} |z|^{n-1} \leq \frac{C_n}{1 - \alpha} ,
\]
that is, if
\[
|z| < \left\{ \frac{(c + 1)C_n}{n(c + n)(1 - \alpha)} \right\}^{1 \over n - 1} (n \geq 2) .
\]
Therefore, the function \( f(z) \) given by (6.1) is univalent in \( |z| < R^* \). Sharpness of the result follows if we take
\[
f(z) = z - \frac{(c + n)(1 - \alpha)}{(c + 1)C_n} z^n (n \geq 2) .
\]
7. Modified Hadamard Products

Let the functions $f_\nu(z) (\nu = 1, 2)$ be defined by

$$f_\nu(z) = z - \sum_{n=2}^{\infty} a_{n,\nu} z^n (a_{n,\nu} \geq 0; \nu = 1, 2).$$

The modified Hadamard product of $f_1(z)$ and $f_2(z)$ is defined by

$$(f_1 \ast f_2)(z) = z - \sum_{n=2}^{\infty} a_{n,1}a_{n,2} z^n.$$  

**Theorem 7.1.** Let each of the functions $f_\nu(z) (\nu = 1, 2)$ defined by (7.1) be in the class $T_m^\ell([\alpha_1]; \alpha, \beta)$. If the sequence $\{C_n\}$ is nondecreasing, then $(f_1 \ast f_2)(z) \in T_m^\ell([\alpha_1]; \delta([\alpha_1], \alpha, \beta), \beta)$, where

$$\delta([\alpha_1], \alpha, \beta) = 1 - \frac{(1 + \beta)(1 - \alpha)^2}{(2 - \alpha + \beta)^2 \Gamma_2 - (1 - \alpha)^2}.$$  

The result is sharp.

**Proof.** Employing the technique used earlier by Schild and Silverman [22], we need to find the largest $\delta = \delta([\alpha_1], \alpha, \beta)$ such that

$$\sum_{n=2}^{\infty} \frac{[n(1 + \beta) - (\delta + \beta)]\Gamma_n}{1 - \delta} a_{n,1}a_{n,2} \leq 1.$$  

Since

$$\sum_{n=2}^{\infty} \frac{[n(1 + \beta) - (\alpha + \beta)]\Gamma_n}{1 - \alpha} a_{n,1} \leq 1$$  

and

$$\sum_{n=2}^{\infty} \frac{[n(1 + \beta) - (\alpha + \beta)]\Gamma_n}{1 - \alpha} a_{n,2} \leq 1,$$

by the Cauchy-Schwarz inequality, we have

$$\sum_{n=2}^{\infty} \frac{[n(1 + \beta) - (\alpha + \beta)]\Gamma_n}{1 - \alpha} \sqrt{a_{n,1}a_{n,2}} \leq 1.$$  

Thus it is sufficient to show that

$$\frac{[n(1 + \beta) - (\delta + \beta)]\Gamma_n}{1 - \delta} a_{n,1}a_{n,2} \leq \frac{[n(1 + \beta) - (\alpha + \beta)]\Gamma_n}{1 - \alpha} \sqrt{a_{n,1}a_{n,2}}(n \geq 2),$$  

that is, that

$$\sqrt{a_{n,1}a_{n,2}} \leq \frac{[n(1 + \beta) - (\alpha + \beta)](1 - \delta)}{[n(1 + \beta) - (\delta + \beta)](1 - \alpha)}(n \geq 2).$$  

Note that

$$\sqrt{a_{n,1}a_{n,2}} \leq \frac{(1 - \alpha)}{[n(1 + \beta) - (\alpha + \beta)]\Gamma_n}(n \geq 2).$$
Consequently, we need only to prove that
\[
\frac{1 - \alpha}{[n(1 + \beta) - (\alpha + \beta)] \Gamma_n} \leq \frac{[n(1 + \beta) - (\alpha + \beta)](1 - \delta)}{[n(1 + \beta) - (\delta + \beta)](1 - \alpha)} \quad (n \geq 2),
\]
or, equivalently, that
\[
\delta \leq 1 - \frac{(n - 1)(1 + \beta)(1 - \alpha)^2}{(n + 1 + \beta)(1 - \alpha)^2} \quad (n \geq 2).
\]
Since
\[
\Phi(n) = 1 - \frac{(n - 1)(1 + \beta)(1 - \alpha)^2}{(n + 1 + \beta)(1 - \alpha)^2} \Gamma_n - (1 - \alpha)^2
\]
is an increasing function of \(n(n \geq 2)\), letting \(n = 2\) in (7.13), we obtain
\[
\delta \leq \Phi(2) = 1 - \frac{(1 + \beta)(1 - \alpha)^2}{(2 - \alpha + \beta)(2 - \gamma + \beta) \Gamma_2 - (1 - \alpha)^2},
\]
which proves the main assertion of Theorem 7.1.

Finally, by taking the functions \(f_\nu(z)(\nu = 1, 2)\) given by
\[
f_\nu(z) = z - \frac{1 - \alpha}{(2 - \alpha + \beta) \Gamma_2} z^2 (\nu = 1, 2),
\]
we can see that the result is sharp.

Proceeding as in the proof of Theorem 7.1, we get

**Theorem 7.2.** Let the function \(f_1(z)\) defined by (7.1) be in the class \(T_m^d([\alpha_1]; \alpha, \beta)\) and the function \(f_2(z)\) defined by (7.1) be in the class \(T_m^d([\alpha_1]; \gamma, \beta)\). If the sequence \(\{C_n\}\) is nondecreasing, then \((f_1 * f_2)(z) \in T_m^d([\alpha_1]; [\alpha_1, \gamma, \beta, \beta])\), where
\[
\zeta([\alpha_1], \alpha, \gamma, \beta) = 1 - \frac{(1 + \beta)(1 - \alpha)(1 - \gamma)}{(2 - \alpha + \beta)(2 - \gamma + \beta) \Gamma_2 - (1 - \alpha)(1 - \gamma)}.
\]
The result is the best possible for the functions
\[
f_1(z) = z - \frac{1 - \alpha}{(2 - \alpha + \beta) \Gamma_2} z^2
\]
and
\[
f_2(z) = z - \frac{1 - \gamma}{(2 - \gamma + \beta) \Gamma_2} z^2.
\]

**Theorem 7.3.** Let the functions \(f_\nu(z)(\nu = 1, 2)\) defined by (7.1) be in the class \(T_m^d([\alpha_1]; \alpha, \beta)\). If the sequence \(\{C_n\}\) is nondecreasing. Then the function
\[
h(z) = z - \sum_{n=2}^{\infty} (a_{n,1}^2 + a_{n,2}^2) z^n
\]
belongs to the class \(T_m^d([\alpha_1]; \Theta([\alpha_1], \alpha, \beta, \beta))\), where
\[
\Theta([\alpha_1], \alpha, \beta) = 1 - \frac{2(1 + \beta)(1 - \alpha)^2}{(2 + \beta - \alpha)^2 \Gamma_2 - 2(1 - \alpha)^2}.
\]
The result is sharp for the functions \(f_\nu(z)\) defined by (7.15).
Proof. By virtue of Theorem 2.2, we obtain
\[
\sum_{n=2}^{\infty} \left\{ \frac{[n(1+\beta) - (\alpha + \beta)] \Gamma_n}{1 - \alpha} \right\}^2 a_{n,1}^2 \leq \left\{ \sum_{n=2}^{\infty} \frac{[n(1+\beta) - (\alpha + \beta)] \Gamma_n}{1 - \alpha} a_{n,1} \right\}^2 \leq 1
\]
and
\[
\sum_{n=2}^{\infty} \left\{ \frac{[n(1+\beta) - (\alpha + \beta)] \Gamma_n}{1 - \alpha} \right\}^2 a_{n,2}^2 \leq \left\{ \sum_{n=2}^{\infty} \frac{[n(1+\beta) - (\alpha + \beta)] \Gamma_n}{1 - \alpha} a_{n,2} \right\}^2 \leq 1.
\]
It follows from (7.21) and (7.22) that
\[
\sum_{n=2}^{\infty} \frac{[n(1+\beta) - (\alpha + \beta)] \Gamma_n}{1 - \alpha} \right\}^2 (a_{n,1}^2 + a_{n,2}^2) \leq 1.
\]
Therefore, we need to find the largest \( \Im([\alpha_1], \alpha, \beta) \) such that
\[
\frac{[n(1+\beta) - (\Im + \beta)] \Gamma_n}{1 - \Im} \leq \frac{1}{2} \left\{ \frac{[n(1+\beta) - (\alpha + \beta)] \Gamma_n}{1 - \alpha} \right\}^2 (n \geq 2),
\]
that is,
\[
\Im \leq 1 - \frac{2(n-1)(1+\beta)(1-\alpha)^2}{[n(1+\beta) - (\alpha + \beta)]^2 \Gamma_n - 2(1-\alpha)^2} (n \geq 2).
\]
Since
\[
D(n) = 1 - \frac{2(n-1)(1+\beta)(1-\alpha)^2}{[n(1+\beta) - (\alpha + \beta)]^2 \Gamma_n - 2(1-\alpha)^2}
\]
is an increasing function of \( n (n \geq 2) \), we readily have
\[
\Im \leq D(2) = 1 - \frac{2(1+\beta)(1-\alpha)^2}{(2 + \beta - \alpha)^2 \Gamma_2 - 2(1-\alpha)^2},
\]
and Theorem 7.3 follows at once. 

8. Properties associated with generalized fractional calculus operators

In terms of the Gauss hypergeometric function
\[
\sum_{n=0}^{\infty} \frac{\partial n (\mu) \n!}{(\nu) n!} z^n
\]
\((z \in U; \delta, \mu, \nu \in C; \nu \neq 0, -1, -2, ...)
\),

where (again) \((\lambda)\) \(n\) denotes the Pochhammer symbol defined in (1.14), the generalized fractional calculus operators \( I^\alpha_{\delta, \mu, \nu} \) and \( J^\alpha_{\delta, \mu, \nu} \) are defined below (cf., e.g., [17] and [25]).

Definition 8.1. (Generalized Fractional Integral operator). The generalized fractional integral of order \( \mu \) is defined, for a function \( f(z) \), by
\[
I^\mu_{\delta, \mu, \nu} f(z) = \frac{z^{-\mu-\nu}}{\Gamma(\mu)} \int_{0}^{z} (z - \zeta)^{\mu-1} 2F_1 (\mu + \nu; -\eta; \mu; 1 - \zeta / z) : f(\zeta) d\zeta
\]
\((\mu > 0; \epsilon > \max \{0, \nu - \eta\} - 1)\)
where \( f(z) \) is an analytic function in a simply-connected region of the \( z \)-plane containing the origin, and the multiplicity of \( (z - \zeta)^{\mu-1} \) is removed by requiring \( \log(z - \zeta) \) to be real when \( (z - \zeta) > 0 \), provided further that

\[
(\text{8.3}) \quad f(z) = O(|z|^\epsilon) \quad (z \to 0).
\]

**Definition 8.2.** (Generalized Fractional Derivative Operator). The generalized fractional derivative of order \( \mu \) is defined, for a function \( f(z) \), by

\[
(\text{8.4}) \quad J_{0,z}^{\mu,\nu,\eta} f(z) = \begin{cases}
\frac{1}{\Gamma(1-\mu)} \frac{d}{dz} \left\{ z^{\mu-\nu} \int_0^z (z - \zeta)^{-\mu} \, \text{}_2F_1(\nu - \mu, 1 - \eta; 1 - \mu; f(\zeta)) \right\} (0 \leq \mu < 1), \\
1 - \frac{\zeta}{z} f(\zeta) d\zeta \right\} (0 \leq \mu < 1), \\
\frac{d^n}{dz^n} J_{0,z}^{\mu-n,\nu,\eta} f(z) (n \leq \mu < n + 1; n \in \mathbb{N}) \quad (\epsilon > \max \{0, \nu - \eta\} - 1),
\end{cases}
\]

where \( f(z) \) is constrained, and the multiplicity of \((z - \zeta)^{\mu-1}\) is removed, as in Definition 8.1, and \( \epsilon \) is given by the order estimate (8.3).

\[
(\text{8.5}) \quad f(z) = O(|z|^\epsilon) \quad (z \to 0).
\]

It follows from Definition 8.1 and Definition 8.2 that

\[
(\text{8.6}) \quad I_{0,z}^{\mu,\nu,\eta} f(z) = D_z^{-\mu} f(z) \quad (\mu > 0),
\]

and

\[
(\text{8.7}) \quad J_{0,z}^{\mu,\nu,\eta} f(z) = D_z^{\mu} f(z) \quad (0 \leq \mu < 1),
\]

where \( D_z^{\mu} (\mu \in \mathbb{R}) \) is the fractional operator considered by Owa \([16]\) and (subsequently) by Owa and Srivastava \([18]\) and Srivastava and Owa \([24]\). Furthermore, in terms of Gamma functions, Definitions 8.1 and 8.2 readily yield.

**Lemma 8.1.** \([25]\). The generalized fractional integral and the generalized fractional derivative of a power function are given by

\[
(\text{8.8}) \quad I_{0,z}^{\mu,\nu,\eta} z^\rho = \frac{\Gamma(\rho + 1) \Gamma(\rho - \nu + 1 + \eta)}{\Gamma(\rho - \nu + 1) \Gamma(\rho + \mu + \eta + 1)} z^{\rho - \nu} \quad (\mu > 0; \rho > \max \{0, \nu - \eta\} - 1),
\]

and

\[
(\text{8.9}) \quad J_{0,}^{\mu,\nu,\eta} z^\rho = \frac{\Gamma(\rho + 1) \Gamma(\rho - \nu + 1 + \eta)}{\Gamma(\rho - \nu + 1) \Gamma(\rho - \mu + \eta + 1)} z^{\rho - \nu} \quad (0 \leq \mu < 1; \rho > \max \{0, \nu - \eta\} - 1).
\]

**Theorem 8.2.** Let the function \( f(z) \) defined by (1.19) be in the class \( T_m^\alpha ([\alpha_1]; \alpha, \beta) \). If the sequence \( \{C_n\} \) is nondecreasing, then

\[
\frac{\Gamma(2 - \nu + \eta)}{\Gamma(2 - \nu) \Gamma(2 + \mu + \eta)} |z|^{1-\nu} \left\{ 1 - \frac{2(1 - \alpha)(2 - \nu + \eta)}{(2 - \nu)(2 + \mu + \eta)(2 - \alpha + \beta) \Gamma_2} |z| \right\} \leq |I_{0,z}^{\mu,\nu,\eta} f(z)| \leq
\]

\[
AJMAA, \text{Vol. 5, No. 1, Art. 3, pp. 1-17, 2008}
Each of these results is sharp for the function \( f(z) \) defined by (3.3).

**Proof.** Making use of the assertion (8.8) of Lemma 8.1, we find from (1.19) that

\[
F(z) = \frac{\Gamma(2 - \nu + \eta)}{\Gamma(2 - \nu)\Gamma(2 + \mu + \eta)} z^\nu J_{0,z}^{\mu,\nu,\eta} f(z)
\]

where

\[
\Phi(n) = \frac{(1)_n(2 - \nu + \eta)_{n-1}}{(2 - \nu)_{n-1}(2 + \mu + \eta)_{n-1}} \quad (n \in \mathbb{N}\{1\})
\]

The function \( \Phi(n) \) defined by (8.14) can easily be seen to be nonincreasing under the parametric constraints stated already with (8.10), and we thus have

\[
0 < \Phi(n) \leq \Phi(2) = \frac{2(2 - \nu + \eta)}{(2 - \nu)(2 + \mu + \eta)} \quad (n \in \mathbb{N}\{1\})
\]

Now the assertion (8.10) of Theorem 8.2 would follow readily from (3.4), (8.13) and (8.15).

The assertion (8.11) of Theorem 8.2 can be proven similarly by noting from (8.9) that

\[
G(z) = \frac{\Gamma(2 - \nu + \eta)}{\Gamma(2 - \nu)\Gamma(2 + \mu + \eta)} z^\nu J_{0,z}^{\mu,\nu,\eta} f(z)
\]

where

\[
0 < \Psi(n) = \frac{(1)_n(2 - \nu + \eta)_{n-1}}{(2 - \nu)_{n-1}(2 + \mu + \eta)_{n-1}} \quad (n \in \mathbb{N}\{1\})
\]

under the parametric constraints stated already with (8.11).

Finally, by observing that the equalities in each of the assertions (8.10) and (8.11) are attained by the function \( f(z) \) given by (3.3), we complete the proof of Theorem 8.2.
In view of the relationships (8.6) and (8.7), by setting \( \nu = -\mu \) and \( \nu = \mu \) in our assertions (8.10) and (8.11), respectively, we obtain

**Corollary 8.3.** Let the function \( f(z) \) defined by (1.19) be in the class \( T_m([\alpha_1]; \alpha, \beta) \). If the sequence \( \{C_n\} \) is nondecreasing, then

\[
\frac{|z|^{1+\mu}}{\Gamma(2+\mu)} \left\{ 1 - \frac{2(1-\alpha)}{(2+\mu)(2-\alpha+\beta)\Gamma^2_2} |z| \right\} \leq |D_z^\mu f(z)| \leq \frac{|z|^{1+\mu}}{\Gamma(2+\mu)} \left\{ 1 + \frac{2(1-\alpha)}{(2+\mu)(2-\alpha+\beta)\Gamma^2_2} |z| \right\} \quad (z \in U; \mu > 0).
\]

and

\[
\frac{|z|^{1-\mu}}{\Gamma(2-\mu)} \left\{ 1 - \frac{2(1-\alpha)}{(2-\mu)(2-\alpha+\beta)\Gamma^2_2} |z| \right\} \leq |D_z^{-\mu} f(z)| \leq \frac{|z|^{1-\mu}}{\Gamma(2-\mu)} \left\{ 1 + \frac{2(1-\alpha)}{(2-\mu)(2-\alpha+\beta)\Gamma^2_2} |z| \right\} \quad (z \in U; 0 \leq \mu < 1).
\]

Each of these results is sharp for the function \( f(z) \) given by (3.3).

**REFERENCES**


