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ERROR INEQUALITIES FOR WEIGHTED INTEGRATION FORMULAE AND APPLICATIONS

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ABSTRACT. Weighted integration formulae are derived. Error inequalities for the weighted integration formulae are obtained. Applications to some special functions are also given.

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1. INTRODUCTION

In recent years a number of authors have considered weighted integral inequalities. These inequalities, very often, give error bounds for weighted quadrature formulae. The authors considered both, 1-dimensional and n-dimensional cases. For example, this topic is considered in [2]–[7] and [9]-[12]. In many cases obtained error inequalities are generalizations of the well-known Ostrowski integral inequality.

In this paper we establish a general weighted integration formula and emphasize a particular case of this formula. We also give few error inequalities for the weighted integration formula which can be considered (in some sense) as generalizations of the Ostrowski inequality but here we call more attention to applications of these inequalities. We use Appell-like sequences of functions to obtain the general integration formula. This further leads to use of the Beta and Gamma functions. We only briefly sketch possible applications of Bernoulli polynomials in such formulae. Finally, as illustrative examples of applications we give applications to some special functions. We consider the Fresnel and Dawson integrals and Error function.

2. MAIN RESULTS

We recall some properties of the Beta function

$$B(\alpha,\beta)=\int_0^1(1-t)^{\alpha-1}t^{\beta-1}dt,\ \alpha,\beta>0, \tag{2.1}$$

and the Gamma function

$$\Gamma(z)=\int_0^\infty t^{z-1}e^{-t}dt,\ \ z>0. \tag{2.2}$$

We have

$$B(\alpha,\beta)=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \tag{2.3}$$

$$\Gamma(n)=(n-1)!,\ n\in N, \tag{2.4}$$

$$\Gamma(n+\frac{1}{2})=\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n}\Gamma(\frac{1}{2})=\frac{(2n-1)!!}{2^n}\Gamma(\frac{1}{2}), \tag{2.5}$$

$$\Gamma(\frac{1}{2})=\sqrt{\pi}. \tag{2.6}$$

We also define

$$\|f\|_\infty=\sup_{t\in[a,b]}|f(t)|,$$

$$\|f\|_1=\int_a^b|f(t)|\,dt.$$

Definition 2.1. We say that the sequence $\{Q_k(t,s)\}_0^\infty$ forms an Appell-like (or harmonic) sequence of functions with respect to the first variable t if

$$\frac{\partial Q_k(t,s)}{\partial t}=Q_{k-1}(t,s),\ Q_0(t,s)=1. \tag{2.7}$$

Further, we define the functions

$$P_k(t)=\int_a^t Q_{k-1}(t,s)w(s)ds,\ \ t\in[a,b]\,,\ k=1,2,..., \tag{2.8}$$

where $w(s)$ is an integrable function. Then we have

$$\begin{aligned} P_k'(t) &= \int_a^t \frac{\partial Q_{k-1}(t,s)}{\partial t}w(s)ds+Q_{k-1}(t,t)w(t) \\ &= \int_a^t Q_{k-2}(t,s)w(s)ds+Q_{k-1}(t,t)w(t) \\ &= P_{k-1}(t)+Q_{k-1}(t,t)w(t). \end{aligned} \tag{2.9}$$

If $Q_k(t,t)=0$, $k\geq 1$, then $P_k'(t)=P_{k-1}(t)$, $P_0(t)=w(t)$ and $\{P_k(t)\}_1^\infty$ is an Appell-like (or harmonic) sequence of functions.

Theorem 2.1. *Let the sequence $\{Q_k(t,s)\}_0^\infty$ forms an Appell-like sequence of functions with respect to the first variable t and let $P_k(t)$ be defined by (2.8). If $f\in C^n(a,b)$ then*

$$\begin{aligned} \int_a^b w(t)f(t)dt &= \sum_{k=1}^n(-1)^{k+1}P_k(b)f^{(k-1)}(b) \\ &\quad +\sum_{k=2}^n(-1)^{k+1}\int_a^b Q_{k-1}(t,t)w(t)f^{(k-1)}(t)dt+R(f), \end{aligned} \tag{2.10}$$

where

$$|R(f)|\leq \left\|f^{(n)}\right\|_\infty \|P_n\|_1\,. \tag{2.11}$$

Proof. Integrating by parts, we obtain

$$\begin{aligned} (-1)^n\int_a^b P_n(t)f^{(n)}(t)dt &= (-1)^n\left[P_n(b)f^{(n-1)}(b)-P_n(a)f^{(n-1)}(a)\right] \\ &\quad +(-1)^{n-1}\int_a^b P_n'(t)f^{(n-1)}(t)dt. \end{aligned} \tag{2.12}$$

From (2.12) and (2.9) it follows that

$$\begin{aligned} (-1)^n\int_a^b P_n(t)f^{(n)}(t)dt &= (-1)^nP_n(b)f^{(n-1)}(b) \\ &\quad +(-1)^{n-1}\int_a^b Q_{n-1}(t,t)w(t)f^{(n-1)}(t)dt \\ &\quad +(-1)^{n-1}\int_a^b P_{n-1}(t)f^{(n-1)}(t)dt, \end{aligned}$$

since $P_n(a)=0$. In a similar way we get

$$\begin{aligned} (-1)^{n-1}\int_a^b P_{n-1}(t)f^{(n-1)}(t)dt &= (-1)^{n-1}P_{n-1}(b)f^{(n-2)}(b) \\ &\quad +(-1)^{n-2}\int_a^b Q_{n-2}(t,t)w(t)f^{(n-2)}(t)dt \\ &\quad +(-1)^{n-2}\int_a^b P_{n-2}(t)f^{(n-2)}(t)dt. \end{aligned}$$

Continuing in this way we get

$$(2.13)\quad (-1)^n \int_a^b P_n(t) f^{(n)}(t)dt = \sum_{k=1}^{n} (-1)^k P_k(b) f^{(k-1)}(b) + \sum_{k=2}^{n} (-1)^{k-1} \int_a^b Q_{k-1}(t,t) w(t) f^{(k-1)}(t)dt + \int_a^b w(t) f(t)dt.$$

We see that (2.13) is equivalent to (2.10). We also have

$$R(f) = (-1)^n \int_a^b P_n(t) f^{(n)}(t)dt$$

such that it is not difficult to see that (2.11) holds, too. ∎

Corollary 2.2. *Let the assumptions of Theorem 2.1 hold. If* $Q_k(t,t) = 0$*,* $k \geq 1$*, then*

$$(2.14)\quad \int_a^b w(t) f(t)dt = \sum_{k=1}^{n} (-1)^{k+1} P_k(b) f^{(k-1)}(b) + R(f),$$

where

$$(2.15)\quad |R(f)| \leq \left\| f^{(n)} \right\|_\infty \|P_n\|_1 .$$

Remark 2.1. Here we do not consider all possibilities of applications of the above theorem. We only mention some facts about that. We can choose

$$Q_n(t,s) = \sum_{k=0}^{n} \frac{c_k}{(n-k)!} (t-s)^{n-k},$$

where c_k are arbitrary coefficients. Further, in [8] we can find some Appell (or harmonic) sequences of polynomials. They can be used for construction of the functions $Q_k(t,s)$. For example,

$$Q_k(t,s) = \frac{(b-a)^k}{k!} B_k \left(\frac{t-s}{b-a} \right),$$

where $B_k(t)$ are Bernoulli polynomials. We have $B_k'(t) = kB_{k-1}(t)$ and $Q_k(t,t) = \frac{(b-a)^k}{k!} B_k$, where $B_k = B_k(0)$ are Bernoulli numbers ($B_{2k+1} = 0$, $k \geq 1$). More about this polynomials and numbers can be found in [1].

Here we emphasize only one application (to special functions) of the above theorem. For that purpose we need the following variant of Theorem 2.1.

Theorem 2.3. *Let* $f \in C^n(a,b)$ *and* $\beta > -1$*. Then*

$$(2.16)\quad \int_a^b (t-a)^\beta f(t)dt = \sum_{k=1}^{n} (-1)^{k+1} \frac{\Gamma(\beta+1)}{\Gamma(k+\beta+1)} (b-a)^{k+\beta} f^{(k-1)}(b) + R(f),$$

where

$$(2.17)\quad |R(f)| \leq \frac{\Gamma(\beta+1)}{\Gamma(n+\beta+1)} (b-a)^{n+\beta+1} \left\| f^{(n)} \right\|_\infty .$$

Proof. We define

$$Q_k(t,s) = \frac{(t-s)^k}{k!} \text{ and } w(s) = (s-a)^\beta. \tag{2.18}$$

From (2.8) and (2.18) we have

$$P_k(b) = \int_a^b Q_{k-1}(b,s)(s-a)^\beta ds = \int_a^b \frac{(b-s)^{k-1}}{(k-1)!}(s-a)^\beta ds.$$

If we substitute $u = s - a$ then we get

$$P_k(b) = \int_a^b \frac{(b-a-u)^{k-1}}{(k-1)!} u^\beta du = \frac{(b-a)^{k-1}}{(k-1)!} \int_a^b \left(1 - \frac{u}{b-a}\right)^{k-1} u^\beta du.$$

We now substitute $v = u/(b-a)$. Then we have

$$P_k(b) = \frac{(b-a)^{k+\beta}}{(k-1)!} \int_0^1 (1-v)^{k-1} v^\beta dv. \tag{2.19}$$

From (2.19) and (2.1)-(2.4) it follows that

$$\begin{aligned} P_k(b) &= \frac{(b-a)^{k+\beta}}{(k-1)!} B(k, \beta+1) = \frac{(b-a)^{k+\beta}}{(k-1)!} \frac{\Gamma(k)\Gamma(\beta+1)}{\Gamma(k+\beta+1)} \\ &= \frac{\Gamma(\beta+1)}{\Gamma(k+\beta+1)} (b-a)^{k+\beta}. \end{aligned} \tag{2.20}$$

From (2.14) and (2.20) we see that (2.16) holds. From (2.15) and (2.20) we find that

$$\begin{aligned} |R(f)| &\leq \left\|f^{(n)}\right\|_\infty \int_a^b \left| \int_a^b \frac{(b-s)^{n-1}}{(n-1)!} (s-a)^\beta ds \right| dt \\ &= \left\|f^{(n)}\right\|_\infty \int_a^b |P_n(b)|\, dt = \frac{\Gamma(\beta+1)}{\Gamma(n+\beta+1)} (b-a)^{n+\beta+1} \left\|f^{(n)}\right\|_\infty \end{aligned}$$

such that (2.17) holds, too. ∎

Corollary 2.4. *Let $f \in C^n(a,b)$. Then*

$$\int_a^b \frac{f(t)}{\sqrt{t-a}} dt = \sum_{k=1}^n (-1)^{k+1} \frac{2^k (b-a)^{k-\frac{1}{2}}}{(2k-1)!!} f^{(k-1)}(b) + R(f), \tag{2.21}$$

where

$$|R(f)| \leq \frac{2^n}{(2n-1)!!} (b-a)^{n+\frac{1}{2}} \left\|f^{(n)}\right\|_\infty. \tag{2.22}$$

Proof. We use (2.5) and apply Theorem 2.3 with $\beta = -1/2$. ∎

3. Applications to special functions

We consider the Fresnel integrals

$$FS(x) = \int_0^x \sin t^2 dt \text{ and } FC(x) = \int_0^x \cos t^2 dt, \tag{3.1}$$

the Dawson integral

$$D(x) = \int_0^x e^{t^2} dt \tag{3.2}$$

and the Error function

$$Erf(x) = \frac{1}{2\sqrt{\pi}} \int_0^x e^{-t^2} dt. \tag{3.3}$$

Proposition 3.1. *Let $FS(x)$ be defined by (3.1). Then*

$$FS(x) = x \sin x^2 S_n(x) + x \cos x^2 C_n(x) + R_n, \tag{3.4}$$

where

$$S_n(x) = \sum_{j=0}^{\left[\frac{n-1}{2}\right]} (-1)^j \frac{2^{2j} x^{4j}}{(4j+1)!!}, \tag{3.5}$$

$$C_n(x) = \sum_{j=0}^{\left[\frac{n-2}{2}\right]} (-1)^{j+1} \frac{2^{2j+1} x^{4j+2}}{(4j+3)!!} \tag{3.6}$$

and

$$|R_n| \le \frac{2^n x^{2n+1}}{(2n-1)!!}. \tag{3.7}$$

Proof. We substitute $a = 0$ and $f(t) = \sin t$ in (2.21). We have

$$f^{(2j)}(b) = (-1)^j \sin b, \tag{3.8}$$

$$f^{(2j+1)}(b) = (-1)^j \cos b,\ j = 0, 1, 2, ... \tag{3.9}$$

If $k-1$ is even, $k = 2j+1$, then (3.8) holds and

$$(-1)^k \frac{2^k (b-a)^{k-\frac{1}{2}}}{(2k-1)!!} f^{(k-1)}(b) = -\frac{2^{2j+1} b^{2j+\frac{1}{2}}}{(4j+1)!!} (-1)^j \sin b. \tag{3.10}$$

If $k-1$ is odd, $k = 2j+2$, then (3.9) holds and

$$(-1)^k \frac{2^k (b-a)^{k-\frac{1}{2}}}{(2k-1)!!} f^{(k-1)}(b) = -\frac{2^{2j+2} b^{2j+\frac{3}{2}}}{(4j+3)!!} (-1)^j \cos b. \tag{3.11}$$

From (2.21), (3.10) and (3.11) we get

$$\begin{aligned} \int_0^b \frac{\sin t}{\sqrt{t}} dt &= -\sum_{j=0}^{\left[\frac{n-1}{2}\right]} \frac{2^{2j+1} b^{2j+\frac{1}{2}}}{(4j+1)!!} (-1)^{j+1} \sin b \\ &\quad - \sum_{j=0}^{\left[\frac{n-2}{2}\right]} \frac{2^{2j+2} b^{2j+\frac{3}{2}}}{(4j+3)!!} (-1)^j \cos b. \end{aligned} \tag{3.12}$$

We also have

$$\int_0^x \sin u^2 du = \frac{1}{2} \int_0^{x^2} \frac{\sin t}{\sqrt{t}} dt. \tag{3.13}$$

If we now substitute $b = x^2$ in (3.12) then from (3.13) we see that (3.4) holds. We easily get the estimate (3.7) from (2.22). ∎

Proposition 3.2. *Let $FC(x)$ be defined by (3.1). Then*

$$FC(x) = x\cos x^2 S_n(x) - x\sin x^2 C_n(x) + R_n, \tag{3.14}$$

where $S_n(x)$ and $C_n(x)$ are defined by (3.5) and (3.6) and

$$|R_n| \le \frac{2^n x^{2n+1}}{(2n-1)!!}. \tag{3.15}$$

Proof. The proof is similar to the proof of Proposition 3.1.

We substitute $a = 0$ and $f(t) = \cos t$ in (2.21). We have

$$f^{(2j)}(b) = (-1)^j \cos b, \tag{3.16}$$

$$f^{(2j+1)}(b) = (-1)^{j+1} \sin b,\ j = 0, 1, 2, ... \tag{3.17}$$

If $k-1$ is even, $k = 2j+1$, then (3.16) holds and

$$(-1)^k \frac{2^k (b-a)^{k-\frac{1}{2}}}{(2k-1)!!} f^{(k-1)}(b) = -\frac{2^{2j+1} b^{2j+\frac{1}{2}}}{(4j+1)!!}(-1)^j \cos b. \tag{3.18}$$

If $k-1$ is odd, $k = 2j+2$, then (3.17) holds and

$$(-1)^k \frac{2^k (b-a)^{k-\frac{1}{2}}}{(2k-1)!!} f^{(k-1)}(b) = -\frac{2^{2j+2} b^{2j+\frac{3}{2}}}{(4j+3)!!}(-1)^j \sin b. \tag{3.19}$$

From (2.21), (3.18) and (3.19) we get

$$\begin{aligned} \int_0^b \frac{\cos t}{\sqrt{t}} dt \quad &= \quad -\sum_{j=0}^{\left[\frac{n-1}{2}\right]} \frac{2^{2j+1} b^{2j+\frac{1}{2}}}{(4j+1)!!}(-1)^j \cos b \\ &\quad -\sum_{j=0}^{\left[\frac{n-2}{2}\right]} \frac{2^{2j+2} b^{2j+\frac{3}{2}}}{(4j+3)!!}(-1)^j \sin b. \end{aligned} \tag{3.20}$$

We also have

$$\int_0^x \cos u^2 du = \frac{1}{2}\int_0^{x^2} \frac{\cos t}{\sqrt{t}} dt. \tag{3.21}$$

If we now substitute $b = x^2$ in (3.20) then from (3.21) we see that (3.14) holds. We get the estimate (3.15) from (2.22). ∎

Proposition 3.3. *Let $D(x)$ be defined by (3.2). Then*

$$D(x) = \sum_{k=1}^{n} (-1)^{k+1} \frac{2^k x^{2k-1}}{(2k-1)!!} e^{x^2} + R_n, \tag{3.22}$$

where

$$|R_n| \le \frac{2^n x^{2n+1}}{(2n-1)!!} e^{x^2}. \tag{3.23}$$

Proof. We substitute $a = 0$ and $f(t) = e^t$ in (2.21). We have

$$f^{(k)}(t) = e^t,\ k = 0, 1, 2, ...$$

We also have

$$\int_0^x e^{u^2} du = \frac{1}{2}\int_0^{x^2} \frac{e^t}{\sqrt{t}} dt \tag{3.24}$$

and

$$(-1)^k \frac{2^k (b-a)^{k-\frac{1}{2}}}{(2k-1)!!} f^{(k-1)}(b) = (-1)^k \frac{2^k b^{k-\frac{1}{2}}}{(2k-1)!!} e^b. \tag{3.25}$$

From (2.21), (3.24) and (3.25) with $b = x^2$ we easily find that (3.22) holds. The estimate (3.23) follows immediately from (2.22). ∎

Proposition 3.4. *Let $Erf(x)$ be defined by (3.3). Then*

$$Erf(x) = \frac{1}{2\sqrt{\pi}} \sum_{k=1}^{n} \frac{2^{k-1} x^{2k-1}}{(2k-1)!!} e^{-x^2} + R_n \tag{3.26}$$

where

$$|R_n| \le \frac{1}{2\sqrt{\pi}} \frac{2^n x^{2n+1}}{(2n-1)!!} e^{-x^2}. \tag{3.27}$$

Proof. The proof is similar to the proof of Proposition 3.3.

We substitute $a = 0$ and $f(t) = e^{-t}$ in (2.21). We have

$$f^{(k)}(t) = (-1)^k e^{-t}, k = 0, 1, 2, ...$$

We also have

$$\int_0^x e^{-u^2} du = \frac{1}{2} \int_0^{x^2} \frac{e^{-t}}{\sqrt{t}} dt \tag{3.28}$$

and

$$(-1)^k \frac{2^k (b-a)^{k-\frac{1}{2}}}{(2k-1)!!} f^{(k-1)}(b) = \frac{2^k b^{k-\frac{1}{2}}}{(2k-1)!!} e^b. \tag{3.29}$$

From (2.21), (3.28) and (3.29) with $b = x^2$ we easily find that (3.26) holds. The estimate (3.27) follows immediately from (2.22). ∎

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