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SOME INEQUALITIES FOR A CERTAIN CLASS OF MULTIVALENT FUNCTIONS USING MULTIPLIER TRANSFORMATION

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ABSTRACT. The object of the present paper is to derive several inequalities associated with differential subordinations between analytic functions and a linear operator defined for a certain family of p-valent functions, which is introduced here by means of a family of extended multiplier transformations. Some special cases and consequences of the main results are also considered.

Key words and phrases: Analytic functions, Univalent and multivalent functions, Differential subordination, Schwarz function,Linear operator, Convex functions, Starlike functions, Multiplier transformation.

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1. INTRODUCTION, DEFINITIONS AND PRELIMINARIES

Let $\mathcal{A}(p,n)$ denote the class of functions f normalized by

$$f(z) = z^p + \sum_{k=p+n}^{\infty} a_k z^k \quad (p, n \in \mathbb{N} := \{1, 2, 3, \dots\}), \tag{1.1}$$

which are analytic in the open unit disk

$$\mathbb{U} := \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

In particular, we set

$$\mathcal{A}(p,1) := \mathcal{A}_p, \mathcal{A}(1,1) := \mathcal{A} = \mathcal{A}_1 \text{ and } \mathcal{A}(1,n) := \mathcal{A}_n.$$

A function $f \in \mathcal{A}(p,n)$ is said to be in the class $\mathcal{A}(p,n;\alpha)$ if it satisfies the following inequality:

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) < \alpha \quad (z \in \mathbb{U}; \alpha > p). \tag{1.2}$$

We also denote by $\mathcal{C}(\alpha)$ and $\mathcal{S}^*(\alpha)$, respectively, the usual subclasses of $\mathcal{A}$ consisting of functions which are *convex of order* α in $\mathbb{U}$ and *starlike of order* α in $\mathbb{U}$. Thus we have (see for details, [3] and [12]),

$$\mathcal{C}(\alpha) := \left\{f : f \in \mathcal{A} \text{ and } \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \quad (z \in \mathbb{U}; 0 \leq \alpha < 1)\right\} \tag{1.3}$$

and

$$\mathcal{S}^*(\alpha) := \left\{f : f \in \mathcal{A} \text{ and } \Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad (z \in \mathbb{U}; 0 \leq \alpha < 1)\right\}. \tag{1.4}$$

In particular, we write

$$\mathcal{C}(0) =: \mathcal{C} \quad \text{and} \quad \mathcal{S}^*(0) =: \mathcal{S}^*.$$

For the above defined class $\mathcal{A}(p,n;\alpha)$ of p-valent functions, Owa et al. [6] proved the following results.

Theorem 1.1. *(Owa et al.*[6, p.8, Theorem 1]*) If*

$$f(z) \in \mathcal{A}(p,n;\alpha) \quad (p < \alpha \leq p + \frac{1}{2}n),$$

then

$$\Re\left(\frac{f(z)}{zf'(z)}\right) > \frac{2p+n}{(2\alpha+n)p} \quad (z \in \mathbb{U}). \tag{1.5}$$

Theorem 1.2. *(Owa et al.*[6, p. 10, Theorem 2]*) If*

$$f(z) \in \mathcal{A}(p,n;\alpha) \quad (p < \alpha \leq p + \frac{1}{2}n),$$

then

$$0 < \Re\left(\frac{zf'(z)}{f(z)}\right) < \frac{(2\alpha+n)p}{2p+n} \quad (z \in \mathbb{U}). \tag{1.6}$$

In fact, as already observed by Owa et al. [6, p. 10], various further special cases of (for example) Theorem 1.2 when $p = n = 1$ were considered earlier by Nunokawa [4], Saitoh et al. [8], and Singh and Singh [10].

The main object of this paper is to present an extension of each of the inequalities (1.5) and (1.6) asserted by Theorem 1.1 and Theorem 1.2, respectively, to hold true for a linear operator associated with a certain general class $\mathcal{A}(p, n; r, \lambda, \alpha)$ of p-valent functions, which we introduce here.

Analogous to the multiplier transformation on $\mathcal{A}$, the operator $I_p(r, \lambda)$, given on $\mathcal{A}_p$ by

$$I_p(r,\lambda)f(z) := z^p + \sum_{k=p+1}^{\infty} \left(\frac{k+\lambda}{p+\lambda}\right)^r a_k z^k \quad (\lambda \geq 0; r \in \mathbb{Z}; p \in \mathbb{N}; f \in \mathcal{A}_p), \tag{1.7}$$

was studied by Sivaprasad Kumar et al. [11].

The operator $I_p(r, \lambda)$ is closely related to the Sălăgean derivative operator [9]. The operator $I_\lambda^r := I_1(r, \lambda)$ was studied by Cho and Srivastava [2] and Cho and Kim [1]. Moreover, the operator $I_r := I_1(r, 1)$ was studied earlier by Uraleggadi and Somanatha [13].

Here, in our present investigation, we define the operator $I_p(r, \lambda)$ on $\mathcal{A}(p, n)$ by

$$I_p(r,\lambda)f(z) := z^p + \sum_{k=p+n}^{\infty} \left(\frac{k+\lambda}{p+\lambda}\right)^r a_k z^k \quad (\lambda \geq 0; p \in \mathbb{N}; r \in \mathbb{Z}). \tag{1.8}$$

Making use of the linear operator $I_p(r, \lambda)$ defined by (1.8), we say that a function $f(z) \in \mathcal{A}(p, n)$ is in the aforementioned general class $\mathcal{A}(p, n; r, \lambda, \alpha)$ if it satisfies the following inequality:

$$\Re\left(\frac{I_p(r+2,\lambda)f(z)}{I_p(r+1,\lambda)f(z)}\right) < \alpha \quad (z \in \mathbb{U}; \alpha > 1; r \in \mathbb{Z}, \lambda \geq 0). \tag{1.9}$$

The Sălăgean derivative operator $D^\mu f(z)$, given on $\mathcal{A}_n$ by

$$D^\mu f(z) = D(D^{\mu-1} f(z)) = z + \sum_{k=n+1}^{\infty} (k)^\mu a_k z^k \quad (\mu \in \mathbb{N} \cup \{0\}), \tag{1.10}$$

was studied by Orhan and Kamali [5].

Also, we could observe that the Sălăgean derivative operator $D^\mu f(z)$, defined by (1.10) is a particular case of the operator $I_p(r, \lambda)f(z)$ defined by (1.8), when $p = 1$, $r = \mu$ $(\mu \in \mathbb{N} \cup \{0\})$ and $\lambda = 0$.

Thus, with this convention, a function $f(z) \in \mathcal{A}_n$ is in the class $\mathcal{A}(1, n; \mu, \alpha)$ if it satisfies the following inequality:

$$\Re\left(\frac{D^{\mu+2} f(z)}{D^{\mu+1} f(z)}\right) < \alpha, \quad (z \in \mathbb{U}; \alpha > 1; \mu \in \mathbb{N} \cup \{0\}, f \in \mathcal{A}_n). \tag{1.11}$$

Finally, for two functions f and g analytic in $\mathbb{U}$, we say that the function $f(z)$ is subordinate to $g(z)$ in $\mathbb{U}$, and write

$$f \prec g \quad \text{or} \quad f(z) \prec g(z) \quad (z \in \mathbb{U}),$$

if there exsits a Schwarz function $w(z)$, analytic in $\mathbb{U}$ with

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1 \quad (z \in \mathbb{U}),$$

such that

$$f(z) = g(w(z)) \quad (z \in \mathbb{U}). \tag{1.12}$$

In particular, if the function g is univalent in $\mathbb{U}$, the above subordination is equivalent to

$$f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

In our present investigation of the above defined general class $\mathcal{A}(p, n; r, \lambda, \alpha)$, we shall require the following lemmas.

Lemma 1.3. *(cf. Miller and Mocanu* [3, p. 35, Theorem 2.3i(i)]*) Let Ω be a set in the complex plane $\mathbb{C}$ and suppose that $\Phi(u,v;z)$ is a complex-valued mapping:*

$$\Phi : \mathbb{C}^2 \times \mathbb{U} \to \mathbb{C},$$

where

$$u = u_1 + iu_2 \quad \text{and} \quad v = v_1 + iv_2.$$

Also, let $\Phi(iu_2, v_1; z) \notin \Omega$ for all $z \in \mathbb{U}$ and for all real u_2 and v_1 such that

$$v_1 \leq -\frac{1}{2}n(1+u_2^2). \tag{1.13}$$

If

$$q(z) = 1 + c_n z^n + c_{n+1} z^{n+1} + \cdots$$

is analytic in $\mathbb{U}$ and

$$\Phi(q(z), zq'(z); z) \in \Omega \quad (z \in \mathbb{U}),$$

then

$$\Re\{q(z)\} > 0 \quad (z \in \mathbb{U}).$$

Lemma 1.4. *(cf. Miller and Mocanu* [3, p. 132, Theorem 3.4h]*) Let $\psi(z)$ be univalent in $\mathbb{U}$ and suppose that the functions ϑ and φ are analytic in a domain $\mathbb{D} \supset \psi(\mathbb{U})$ with $\varphi(\zeta) \neq 0$ when $\zeta \in \psi(\mathbb{U})$. Define the functions $Q(z)$ and $h(z)$ by*

$$Q(z) := z\psi'(z)\varphi(\psi(z)) \quad \text{and} \quad h(z) := \vartheta(\psi(z)) + Q(z), \tag{1.14}$$

and assume that (i) $Q(z)$ is starlike univalent in $\mathbb{U}$ and (ii) $\Re\left(\frac{zh'(z)}{Q(z)}\right) > 0 \quad (z \in \mathbb{U})$. If

$$\vartheta(q(z)) + zq'(z)\varphi(q(z)) \prec h(z) \quad (z \in \mathbb{U}), \tag{1.15}$$

then

$$q(z) \prec \psi(z) \quad (z \in \mathbb{U})$$

and $\psi(z)$ is the best dominant.

Lemma 1.5. *(Ravichandran et al.* [7, pp. 8, Lemma 3]*) Let the functions $q(z)$ and $\psi(z)$ be analytic in $\mathbb{U}$ and suppose that*

$$\psi(z) \neq 0 \quad (z \in \mathbb{U})$$

is also univalent in $\mathbb{U}$ and that $z\psi'(z)/\psi(z)$ is starlike univalent in $\mathbb{U}$. If

$$\Re\left(\frac{\alpha}{\beta}\frac{1}{\psi(z)} + \left[1 + \frac{z\psi''(z)}{\psi'(z)} - \frac{z\psi'(z)}{\psi(z)}\right]\right) > 0 \quad (z \in \mathbb{U}; \alpha, \beta \in \mathbb{C}; \beta \neq 0) \tag{1.16}$$

and

$$\frac{\alpha}{q(z)} - \beta\frac{zq'(z)}{q(z)} \prec \frac{\alpha}{\psi(z)} - \beta\frac{z\psi'(z)}{\psi(z)} \quad (z \in \mathbb{U}; \alpha, \beta \in \mathbb{C}; \beta \neq 0), \tag{1.17}$$

then

$$q(z) \prec \psi(z) \quad (z \in \mathbb{U})$$

and $q(z)$ is the best dominant.

2. INEQUALITIES ASSOCIATED WITH THE LINEAR OPERATOR $I_p(r,\lambda)$

In view of Lemma 1.3 of the preceeding section, we first prove Theorem 2.1 below.

Theorem 2.1. *Let the parameter α satisfy the following inequality:*

$$1 < \alpha \leq 1 + \frac{n}{2(p+\lambda)}, \quad \textit{where} \quad p+\lambda > 0. \tag{2.1}$$

If $f(z) \in \mathcal{A}(p,n;r,\lambda,\alpha)$, then

$$\Re\left(\frac{I_p(r,\lambda)f(z)}{I_p(r+1,\lambda)f(z)}\right) > \frac{2(p+\lambda)+n}{n+2\alpha(p+\lambda)} \quad (z \in \mathbb{U}) \tag{2.2}$$

and

$$\Re\left(\frac{I_p(r+1,\lambda)f(z)}{I_p(r,\lambda)f(z)}\right) < \frac{n+2\alpha(p+\lambda)}{2(p+\lambda)+n} \quad (z \in \mathbb{U}). \tag{2.3}$$

Proof. Define the function $q(z)$ by

$$(1-\beta)q(z)+\beta = \frac{I_p(r,\lambda)f(z)}{I_p(r+1,\lambda)f(z)} \quad (z \in \mathbb{U}), \tag{2.4}$$

where

$$\beta := \frac{2(p+\lambda)+n}{n+2\alpha(p+\lambda)}. \tag{2.5}$$

Then, clearly, $q(z)$ is analytic in $\mathbb{U}$ and

$$q(z) = 1 + c_n z^n + c_{n+1}z^{n+1} + \ldots \quad (z \in \mathbb{U}).$$

By means of a simple computation, we observe from (2.4) that

$$\frac{(1-\beta)zq'(z)}{(1-\beta)q(z)+\beta} = \frac{z[I_p(r,\lambda)f(z)]'}{I_p(r,\lambda)f(z)} - \frac{z[I_p(r+1,\lambda)f(z)]'}{I_p(r+1,\lambda)f(z)}. \tag{2.6}$$

Making use of the familiar identity:

$$(p+\lambda)I_p(r+1,\lambda)f(z) = z[I_p(r,\lambda)f(z)]' + \lambda I_p(r,\lambda)f(z), \tag{2.7}$$

we find from (2.6) that

$$\frac{(1-\beta)zq'(z)}{(1-\beta)q(z)+\beta} = (p+\lambda)\frac{I_p(r+1,\lambda)f(z)}{I_p(r,\lambda)f(z)} - (p+\lambda)\frac{I_p(r+2,\lambda)f(z)}{I_p(r+1,\lambda)f(z)},$$

which, in view of (2.4), yields

$$\frac{I_p(r+2,\lambda)f(z)}{I_p(r+1,\lambda)f(z)} = \frac{1}{(1-\beta)q(z)+\beta} - \left(\frac{1}{p+\lambda}\right)\left[\frac{(1-\beta)zq'(z)}{(1-\beta)q(z)+\beta}\right]. \tag{2.8}$$

If we define $\Phi(u,v;z)$ by

$$\Phi(u,v;z) := \frac{1}{(1-\beta)u+\beta} - \left(\frac{1}{p+\lambda}\right)\left[\frac{(1-\beta)v}{(1-\beta)u+\beta}\right], \tag{2.9}$$

then, by the hypothesis of Theorem 2.1 that $f \in \mathcal{A}(p,n;r,\lambda,\alpha)$, we have

$$\Re\{\Phi(q(z),zq'(z);z)\} = \Re\left(\frac{I_p(r+2,\lambda)f(z)}{I_p(r+1,\lambda)f(z)}\right) < \alpha \quad (z \in \mathbb{U}; \alpha > 1).$$

We will now show that

$$\Re\{\Phi(iu_2,v_1;z)\} \geq \alpha$$

for all $z \in \mathbb{U}$ and for all real u_2 and v_1 constrained by the inequality (1.13). Indeed we find from (2.9) that

$$\begin{aligned}\Re\{\Phi(iu_2, v_1; z)\} &= \Re\left\{\frac{1}{(1-\beta)iu_2+\beta} - \left(\frac{1}{p+\lambda}\right)\frac{(1-\beta)v_1}{(1-\beta)iu_2+\beta}\right\} \\ &= \Re\left\{\frac{\beta-(1-\beta)iu_2}{(1-\beta)^2u_2^2+\beta^2} - \left(\frac{1}{p+\lambda}\right)\frac{(\beta-(1-\beta)iu_2)(1-\beta)v_1}{(1-\beta)u_2^2+\beta^2}\right\} \\ &= \frac{\beta}{(1-\beta)^2u_2^2+\beta^2} - \left(\frac{1}{p+\lambda}\right)\frac{\beta(1-\beta)v_1}{(1-\beta)^2u_2^2+\beta^2},\end{aligned}$$

so that by using (1.13), we have

$$\Re\{\Phi(iu_2, v_1; z)\} \geq \frac{\beta}{(1-\beta)^2u_2^2+\beta^2} + \frac{1}{p+\lambda}\left(\frac{\beta(1-\beta)\frac{n}{2}(1+u_2^2)}{(1-\beta)^2u_2^2+\beta^2}\right)$$

or equivalently,

$$\Re\{\Phi(iu_2, v_1; z)\} \geq \frac{\beta}{p+\lambda}\left[\frac{(p+\lambda)+\frac{n}{2}(1-\beta)(1+u_2^2)}{(1-\beta)^2u_2^2+\beta^2}\right] \quad (z \in \mathbb{U}). \tag{2.10}$$

From the inequalities in (2.1), we get

$$\frac{n}{2}\beta^2 \geq \left((p+\lambda)+\frac{n}{2}(1-\beta)\right)(1-\beta),$$

and hence the function

$$\frac{(p+\lambda)+\frac{1}{2}n(1-\beta)(1+x^2)}{(1-\beta)^2x^2+\beta^2}$$

is an increasing function for $x \geq 0$. Thus we find from (2.10) that

$$\Re\{\Phi(iu_2, v_1; z)\} \geq \frac{1}{p+\lambda}\left(\frac{(p+\lambda)+\frac{n}{2}(1-\beta)}{\beta}\right) = \alpha \quad (z \in \mathbb{U}).$$

The first assertion (2.2) of Theorem 2.1 follows by applying Lemma 1.3.

Next, we define the function $\psi(z)$ by

$$\psi(z) := \frac{I_p(r,\lambda)f(z)}{I_p(r+1,\lambda)f(z)} \quad (z \in \mathbb{U}),$$

where β is given by (2.5). Then, in view of the already proven assertion (2.2) of Theorem 2.1, we have

$$\Re\{\psi(z)\} > \beta > 0 \quad (z \in \mathbb{U}) \tag{2.11}$$

so that,

$$\Re\left(\frac{1}{\psi(z)}\right) > 0 \quad (z \in \mathbb{U}). \tag{2.12}$$

Since (2.12) holds true, we have

$$\Re\{\psi(z)\}\Re\left(\frac{1}{\psi(z)}\right) \leq |\psi(z)|.\frac{1}{|\psi(z)|} = 1,$$

or

$$\Re\left(\frac{1}{\psi(z)}\right) \leq \frac{1}{\Re\{\psi(z)\}} \quad (z \in \mathbb{U}),$$

which, in view of (2.11), yields

$$0 < \Re\left(\frac{1}{\psi(z)}\right) < \frac{1}{\beta} \quad (z \in \mathbb{U})$$

which is the second assertion (2.3) of Theorem 2.1.

The following result is a special case of Theorem 2.1 obtained by taking $f(z) \in \mathcal{A}(1,n)$ with $p = 1, r = \mu(\mu \in \mathbb{N} \cup \{0\})$ and $\lambda = 0$.

Corollary 2.2. *If $f(z) \in \mathcal{A}(1,n;\mu,\alpha)$ $(1 < \alpha \leq 1 + \frac{n}{2})$, then*

$$\Re\left(\frac{D^{\mu}f(z)}{D^{\mu+1}f(z)}\right) > \frac{2+n}{n+2\alpha} \quad (z \in \mathbb{U}),$$

and

$$\Re\left(\frac{D^{\mu+1}f(z)}{D^{\mu}f(z)}\right) < \frac{n+2\alpha}{2+n} \quad (z \in \mathbb{U}).$$

3. FURTHER RESULTS INVOLVING DIFFERENTIAL SUBORDINATION BETWEEN ANALYTIC FUNCTIONS

In this section, we prove the following result involving differential subordination between analytic functions.

Theorem 3.1. *Let the function $\psi(z) \neq 0$ $(z \in \mathbb{U})$ be analytic and univalent in $\mathbb{U}$ and suppose that $z\psi'(z)/\psi(z)$ is starlike univalent in $\mathbb{U}$ and*

$$\Re\left(\frac{p+\lambda}{\psi(z)} + \left[1 + \frac{z\psi''(z)}{\psi'(z)} - \frac{z\psi'(z)}{\psi(z)}\right]\right) > 0 \quad (z \in \mathbb{U}; (p+\lambda) \in \mathbb{C}\backslash\{0\}). \tag{3.1}$$

If $f \in \mathcal{A}_p$ satisfies the following subordination:

$$\frac{I_p(r+2,\lambda)f(z)}{I_p(r+1,\lambda)f(z)} \prec \frac{p+\lambda}{\psi(z)} - \frac{z\psi'(z)}{\psi(z)} \quad (z \in \mathbb{U}), \tag{3.2}$$

then

$$\frac{I_p(r,\lambda)f(z)}{I_p(r+1,\lambda)f(z)} \prec \psi(z) \quad (z \in \mathbb{U}) \tag{3.3}$$

and $\psi(z)$ is the best dominant.

Proof. Let the function $q(z)$ be defined by

$$q(z) := \frac{I_p(r,\lambda)f(z)}{I_p(r+1,\lambda)f(z)} \quad (z \in \mathbb{U}; f \in \mathcal{A}_p),$$

so that, by a simple computation, we have

$$\frac{zq'(z)}{q(z)} = \frac{z[I_p(r,\lambda)f(z)]'}{I_p(r,\lambda)f(z)} - \frac{z[I_p(r+1,\lambda)f(z)]'}{I_p(r+1,\lambda)f(z)}, \tag{3.4}$$

which follows also from (2.6) in the special case when $\beta = 0$.

Making use of the familiar identity (2.7) once again, we find that

$$\begin{aligned}
\frac{I_p(r+2,\lambda)f(z)}{I_p(r+1,\lambda)f(z)} &= \frac{I_p(r+1,\lambda)f(z)}{I_p(r,\lambda)f(z)} - \left(\frac{1}{p+\lambda}\right)\left[\frac{z[I_p(r,\lambda)f(z)]'}{I_p(r,\lambda)f(z)} - \frac{z[I_p(r+1,\lambda)f(z)]'}{I_p(r+1,\lambda)f(z)}\right] \\
&= \frac{1}{q(z)} - \left(\frac{1}{p+\lambda}\right)\frac{zq'(z)}{q(z)} \\
&= \frac{1}{p+\lambda}\left[\frac{p+\lambda}{q(z)} - \frac{zq'(z)}{q(z)}\right],
\end{aligned}$$

which, in light of the hypothesis (3.2) of Theorem 2, yields the following subordination:

$$\frac{p+\lambda}{q(z)} - \frac{zq'(z)}{q(z)} \prec \frac{p+\lambda}{\psi(z)} - \frac{z\psi'(z)}{\psi(z)} \quad (z \in \mathbb{U}).$$

The assertion (3.3) of Theorem 3.1 now follows from Lemma 1.5.

Remark 3.1. If the function $\psi(z)$ is such that

$$\Re\{\psi(z)\} > 0 \quad (z \in \mathbb{U})$$

and if $z\psi'(z)/\psi(z)$ is starlike in $\mathbb{U}$, then the condition (3.1) is satisfied for $p+\lambda > 0$.

As a special case, when $p = 1$, $r = \mu$ $(\mu \in \mathbb{N} \cup \{0\})$ and $\lambda = 0$, Theorem 3.1 yields the following result.

Corollary 3.2. *Let the function* $\psi(z) \neq 0$ $(z \in \mathbb{U})$ *be analytic and univalent in* $\mathbb{U}$ *and suppose that* $z\psi'(z)/\psi(z)$ *is starlike univalent in* $\mathbb{U}$ *and*

$$\Re\left(\frac{1}{\psi(z)} + \left[1 + \frac{z\psi''(z)}{\psi'(z)} - \frac{z\psi'(z)}{\psi(z)}\right]\right) > 0 \quad (z \in \mathbb{U}).$$

If $f(z) \in \mathcal{A}(1,1;\mu,\alpha)$ *satisfies the following subordination:*

$$\frac{D^{\mu+2}f(z)}{D^{\mu+1}f(z)} \prec \frac{1}{\psi(z)} - \frac{z\psi'(z)}{\psi(z)} \quad (z \in \mathbb{U}),$$

then

$$\frac{D^{\mu}f(z)}{D^{\mu+1}f(z)} \prec \psi(z) \quad (z \in \mathbb{U}).$$

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