ON SOME REMARKABLE PRODUCT OF THETA-FUNCTION
M. S. MAHADEVA NAIKA, M. C. MAHESHKUMAR AND K. SUSHAN BAIRY

Received 15 June, 2007; accepted 21 January, 2008; published 16 May, 2008.

DEPARTMENT OF MATHEMATICS, BANGALORE UNIVERSITY, CENTRAL COLLEGE CAMPUS,
BANGALORE-560 001, INDIA
msmnaika@rediffmail.com, softmahe@rediffmail.com, ksbairy@gmail.com

ABSTRACT. On pages 338 and 339 in his first notebook, Ramanujan records eighteen values for a certain product of theta-function. All these have been proved by B. C. Berndt, H. H. Chan and L-C. Zhang [3]. Recently M. S. Mahadeva Naika and B. N. Dharmendra [7, 8] and Mahadeva Naika and M. C. Maheshkumar [9] have obtained general theorems to establish explicit evaluations of Ramanujan’s remarkable product of theta-function. Following Ramanujan we define a new function $b_{M, N}$ as defined in (1.5). The main purpose of this paper is to establish some new general theorems for explicit evaluations of product of theta-function.

Key words and phrases: Class invariant, Modular equation, Theta-function.

2000 Mathematics Subject Classification. Primary 33C05, 11F20, 11F27.
1. Introduction

In Chapter 16 of his second notebooks [1, 2, 10], Ramanujan develops the theory of theta-function and his theta–function is defined by

\[ \varphi(q) := f(q, q) = \sum_{n=-\infty}^{\infty} q^n = (-q; q^2)_{\infty}', \]

(1.1)

\[ \psi(q) := f(q, q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \left(q^2; q^2\right)_{\infty}', \]

(1.2)

and

\[ f(-q) := f(-q, -q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(n-1)/2} = (q; q)_{\infty}, \]

(1.3)

where

\[ (a; q)_{\infty} = \prod_{n=0}^{\infty} (1 - aq^n). \]

On page 338 in his first notebook [10, p.338], Ramanujan defines

\[ a_{M, N} = \frac{Ne^{-\frac{(N-1)\pi}{4}\sqrt{MN}}}{\psi^2 \left(e^{-\frac{\pi}{\sqrt{MN}}}\right)} \left(e^{-\frac{2\pi}{\sqrt{MN}}}\right)^2 \]

(1.4)

He then, on pages 338 and 339, offers a list of eighteen particular values. All these eighteen values have been established by Berndt, Chan and Zhang [4]. Following Ramanujan we define a new function by

\[ b_{M, N} = \frac{Ne^{-\frac{(N-1)\pi}{4}\sqrt{MN}}}{\psi^2 \left(e^{-\frac{\pi}{\sqrt{MN}}}\right)} \left(e^{-\frac{2\pi}{\sqrt{MN}}}\right)^2 \]

Let \( K, K', L \) and \( L' \) denote the complete elliptic integrals of the first kind associated with the moduli \( k, k' := \sqrt{1-k^2}, l \) and \( l' := \sqrt{1-l^2} \) respectively, where \( 0 < k, l < 1 \). For a fixed positive integer \( N \), suppose that

\[ \frac{K'}{K} = \frac{L'}{L}. \]

(1.6)

Then a modular equation of degree \( N \) is a relation between \( k \) and \( l \) induced by (1.6). Following Ramanujan, set \( \alpha = k^2 \) and \( \beta = l^2 \). Then we say \( \beta \) is of degree \( N \) over \( \alpha \).

Define

\[ g_n = 2^{-\frac{1}{2}} q^{-\frac{1}{2}} \chi(-q), \]

where

\[ \chi(q) := (-q; q^2)_{\infty}. \]

Moreover, if \( q = e^{-\pi \sqrt{MN}} \) and \( \beta \) has degree \( N \) over \( \alpha \), then

\[ g_{\frac{MN}{\pi}} = (4\alpha(1-\alpha)^{-2})^{-\frac{1}{4}} \quad \text{and} \quad g_{MN} = (4\beta(1-\beta)^{-2})^{-\frac{1}{4}}. \]

(1.7)

The main purpose of this paper is to obtain some new general theorems for the explicit evaluations of remarkable product of theta-function (1.5) and also several new explicit evaluations there from.
2. MAIN THEOREMS

In this section, we establish several new general formulas for explicit evaluations of \( b_{M, N} \).

Theorem 2.1. We have

\[
b_{M, N} = Ne^{-\frac{(N-1)\pi}{4} \sqrt{\frac{M}{N}}} \psi^2 \left( e^{-\pi \sqrt{MN}} \right) \varphi^2 \left( -e^{-\pi \sqrt{MN}} \right) \psi^2 \left( e^{-\pi \sqrt{\frac{M}{N}}} \right) \varphi^2 \left( -e^{-\pi \sqrt{\frac{M}{N}}} \right),
\]

where \( M \) is any positive rational and \( N \) is a positive integer.

Proof. The identity (1.5) can be rewritten as

\[
b_{M, N} = Nq^{\frac{N-1}{2}} \psi^2 \left( -q^N \right) \varphi^2 \left( -q^{2N} \right) \psi^2 \left( -q \right) \varphi^2 \left( -q^2 \right), \quad q = e^{-\pi \sqrt{\frac{M}{N}}},
\]

If \( \beta \) is of degree \( N \) over \( \alpha \), then using Entry 10 (iii) and Entry 11 (ii) of Chapter 17 of Ramanujan’s notebooks [2, pp.122–123] in (2.2), we find that

\[
b_{M, N} = \frac{N}{m^2} \left( \frac{\beta}{\alpha} \left( \frac{1 - \beta}{1 - \alpha} \right)^2 \right)^{\frac{1}{4}}.
\]

Using Entry 10 (ii) of Chapter 17 of Ramanujan’s notebooks [2, p.122], we have

\[
\varphi^2(-q^N) \varphi^2(-q) = \frac{1}{m} \left( \frac{1 - \beta}{1 - \alpha} \right)^{\frac{1}{2}}.
\]

Using Entry 11 (i) of Chapter 17 of Ramanujan’s notebooks [2, p.123], we have

\[
q^{\frac{N-1}{2}} \psi^2(q^N) \psi^2(q) = \frac{1}{m} \left( \frac{\beta}{\alpha} \right)^{\frac{1}{4}}.
\]

Using (2.4) and (2.5) in (2.3) with \( q = e^{-\pi \sqrt{\frac{M}{N}}} \), we obtain (2.1).

Theorem 2.2. We have

\[
b_{2M, N} b_{2, N} = 1.
\]

\[
b_{2M, \frac{1}{2}} b_{2M, N} = 1.
\]

\[
b_{M, 2N} b_{\frac{M}{2}, \frac{N}{2}} = 1.
\]

Proof of (2.6). Using equation (2.1), we find that

\[
b_{2M, N} b_{2, N} = N^2 e^{-\frac{(N-1)\pi}{4} \left( \sqrt{\frac{M}{N}} + \sqrt{\frac{N}{M}} \right)} \times \psi^2 \left( e^{-\pi \sqrt{2MN}} \right) \varphi^2 \left( -e^{-\pi \sqrt{2MN}} \right) \psi^2 \left( e^{-\pi \sqrt{\frac{2}{M}}} \right) \varphi^2 \left( -e^{-\pi \sqrt{\frac{2}{M}}} \right).
\]

From Entry 27 (ii) of Chapter 16 of Ramanujan’s notebooks [2, p.43], we have

\[
e^{-\frac{\pi}{4}} \psi^2(e^{-2\mu}) \varphi^2(-e^{-\nu}) = \frac{1}{4} \sqrt{\frac{\nu}{\mu}}; \quad \mu \nu = \pi^2.
\]
Putting \( \mu = \pi \sqrt{\frac{M}{2N}} \) and \( \nu = \pi \sqrt{\frac{2N}{M}} \) in (2.10), we find that

\[
(2.11) \quad e^{-\frac{\pi}{2} \sqrt{\frac{M}{2N}}} \frac{\psi^2}{\varphi^2} \left( e^{-\pi \sqrt{\frac{2N}{M}}} \right) = \frac{1}{4} \sqrt{\frac{2N}{M}}.
\]

Putting \( \mu = \pi \sqrt{\frac{N}{2M}} \) and \( \nu = \pi \sqrt{\frac{2M}{N}} \) in (2.10), we deduce that

\[
(2.12) \quad e^{-\frac{\pi}{2} \sqrt{\frac{N}{2M}}} \frac{\psi^2}{\varphi^2} \left( e^{-\pi \sqrt{\frac{2M}{N}}} \right) = \frac{1}{4} \sqrt{\frac{2M}{N}}.
\]

Putting \( \mu = \pi \sqrt{\frac{MN}{2}} \) and \( \nu = \pi \sqrt{\frac{2}{MN}} \) in (2.10), we find that

\[
(2.13) \quad e^{-\frac{\pi}{2} \sqrt{\frac{MN}{2}}} \frac{\psi^2}{\varphi^2} \left( e^{-\pi \sqrt{\frac{2}{MN}}} \right) = \frac{1}{4} \sqrt{\frac{2}{MN}}.
\]

Putting \( \mu = \pi \sqrt{\frac{1}{2MN}} \) and \( \nu = \pi \sqrt{\frac{2}{MN}} \) in (2.10), we deduce that

\[
(2.14) \quad e^{-\frac{\pi}{2} \sqrt{\frac{1}{2MN}}} \frac{\psi^2}{\varphi^2} \left( e^{-\pi \sqrt{\frac{2}{MN}}} \right) = \frac{1}{4} \sqrt{\frac{2}{MN}}.
\]

Using (2.11), (2.12), (2.13) and (2.14) in (2.9), we obtain the required result (2.6).

Proofs of (2.7) and (2.8) are similar to the proof of (2.6). So we omit the proof.

**Corollary 2.1.** We have

\[
(2.15) \quad b_{2, N} = 1.
\]

**Proof.** Putting \( M = 1 \) in (2.6), we obtain the result (2.15).

**Theorem 2.3.** We have

\[
(2.16) \quad b_{2M, N} = b_{2N, M} = b_{2, \frac{1}{N}} = b_{\frac{1}{2M}, N}.
\]

**Proof.** Replacing \( M \) by \( 2M \) in (2.1), we deduce that

\[
(2.17) \quad b_{2M, N} = Ne^{-\frac{(N-1)}{4} \sqrt{\frac{2N}{M}}} \frac{\psi^2}{\varphi^2} \left( e^{-\pi \sqrt{2MN}} \right) \frac{\varphi^2}{\psi^2} \left( e^{-\pi \sqrt{2MN}} \right).
\]

Putting \( \mu = \pi \sqrt{\frac{M}{2N}} \) and \( \nu = \pi \sqrt{\frac{2N}{M}} \) in (2.10), we find that

\[
(2.18) \quad \psi^2 \left( e^{-\pi \sqrt{\frac{2N}{M}}} \right) = \frac{1}{4} \sqrt{\frac{2N}{M}} e^\frac{\pi}{2} \sqrt{\frac{2N}{M}} \varphi^2 \left( e^{-\pi \sqrt{\frac{2N}{M}}} \right).
\]

Putting \( \mu = \pi \sqrt{\frac{N}{2M}} \) and \( \nu = \pi \sqrt{\frac{2M}{N}} \) in (2.10), we deduce that

\[
(2.19) \quad \varphi^2 \left( e^{-\pi \sqrt{\frac{2M}{N}}} \right) = 4 \sqrt{\frac{N}{2M}} e^{-\frac{\pi}{2} \sqrt{\frac{2M}{N}}} \psi^2 \left( e^{-\pi \sqrt{\frac{2M}{N}}} \right).
\]

Using (2.18) and (2.19) in (2.17), we obtain the first equality of (2.16).

The proofs of the other equalities are similar to the first equality. So we omit the details.
Theorem 2.4. We have
\[ b_{M, 2N} = b_{2N, M} b_{\frac{2N}{M}}. \]

Proof. Replacing \( N \) by \( 2N \) in (2.1), we obtain
\[ b_{M, 2N} = 2N e^{\frac{(2N-1)\pi}{4} \sqrt{\frac{2N}{M}}} \psi^2 \left( e^{-\pi \sqrt{2M/N}} \right) \varphi^2 \left( e^{-\pi \sqrt{2M/N}} \right). \]

Replacing \( M \) by \( 2N \) and \( N \) by \( M \) in (2.1), we deduce that
\[ b_{2N, M} = M e^{\frac{(M-1)\pi}{4} \sqrt{\frac{2N}{M}}} \psi^2 \left( e^{-\pi \sqrt{2M/N}} \right) \varphi^2 \left( e^{-\pi \sqrt{2M/N}} \right). \]

Using (2.21), (2.22) and (2.1), we obtain the required result.

Theorem 2.5. We have
\[ b_{6, 3} = \frac{1}{3}. \]

Proof. If \( \beta \) is of degree 3 over \( \alpha \), then using Entry 5 (vii) of Chapter 19 of Ramanujan’s notebooks [2, p.230], we find that
\[ m^2 \left( \frac{\alpha(1-\alpha)^2}{\beta(1-\beta)^2} \right)^{\frac{1}{4}} + \frac{9}{m^2} \left( \frac{\alpha(1-\alpha)^2}{\beta(1-\beta)^2} \right)^{-\frac{1}{4}} = \left( \frac{\beta(1-\alpha)^2}{\alpha(1-\beta)^2} \right)^{\frac{1}{4}} + \left( \frac{\beta(1-\alpha)^2}{\alpha(1-\beta)^2} \right)^{-\frac{1}{4}}. \]

Using (1.7) and (2.3) with \( N = 3 \) in the above identity (2.24), we obtain (2.23).

Corollary 2.2. We have
\[ b_{6, 3} = \frac{1}{3}. \]

Proof. From the table in Chapter 34 of Ramanujan’s notebooks [3, p.200], we have
\[ g_{18} = \left( \sqrt{3} + \sqrt{2} \right)^{\frac{1}{3}} \]
and
\[ g_{\frac{2}{3}} = g_2 = 1. \]

Using (2.26) and (2.27) in (2.23) with \( M = 6 \), we obtain (2.25).

Theorem 2.6. We have
\[ \frac{1}{\sqrt{b_{M, 5}}} + \sqrt{b_{M, 5}} = \frac{1}{\sqrt{5}} \left( \frac{g_{3M}^6}{g_{M}^3} + \frac{g_{M}^3}{g_{3M}^6} \right). \]

Proof. If \( \beta \) is of degree 5 over \( \alpha \), then using Entry 13 (xii) of Chapter 19 of Ramanujan’s notebooks [2, p.281], we find that
\[ m \left( \frac{\alpha(1-\alpha)^2}{\beta(1-\beta)^2} \right)^{\frac{1}{5}} + \frac{5}{m} \left( \frac{\alpha(1-\alpha)^2}{\beta(1-\beta)^2} \right)^{-\frac{1}{5}} = \left( \frac{\beta(1-\alpha)^2}{\alpha(1-\beta)^2} \right)^{\frac{1}{5}} + \left( \frac{\beta(1-\alpha)^2}{\alpha(1-\beta)^2} \right)^{-\frac{1}{5}}. \]

Using (1.7) and (2.3) with \( N = 5 \) in the above identity (2.29), we obtain (2.28).
Theorem 2.7. We have

\[ \frac{1}{\sqrt{b_{M, \gamma}}} + \sqrt{b_{M, \gamma}} = \frac{1}{\gamma} \left( \frac{g_{27M}}{g_{44M}} + \frac{g_{47M}}{g_{87M}} - 8 \left( \frac{g_{27M}}{g_{47M}} + \frac{g_{47M}}{g_{87M}} \right) \right). \]

Proof. If \( \beta \) is of degree 7 over \( \alpha \), then using Entry 19 (v) of Chapter 19 of Ramanujan’s notebooks \[2, p.314\], we find that

\[ m^2 \left( \frac{\alpha(1-\alpha)^2}{(\beta(1-\beta)^2)} \right)^{\frac{1}{4}} + \frac{49}{m^2} \left( \frac{\alpha(1-\alpha)^2}{\beta(1-\beta)^2} \right)^{-\frac{1}{4}} = \left( \frac{\beta(1-\alpha)^2}{\alpha(1-\beta)^2} \right)^{\frac{1}{4}} \]

\[ + \left( \frac{\beta(1-\alpha)^2}{\alpha(1-\beta)^2} \right)^{-\frac{1}{4}} - 8 \left( \frac{\beta(1-\alpha)^2}{\alpha(1-\beta)^2} \right)^{\frac{1}{17}} + \left( \frac{\beta(1-\alpha)^2}{\alpha(1-\beta)^2} \right)^{-\frac{1}{17}}. \]

Using (1.7) and (2.3) with \( N = 7 \) in the above identity (2.31), we obtain (2.30).

Theorem 2.8. We have

\[ \frac{1}{\sqrt{b_{M, \gamma}}} + \sqrt{b_{M, \gamma}} = \frac{1}{3} \left( \frac{g_{39M}}{g_{49M}} + \frac{g_{49M}}{g_{99M}} - 4 \right). \]

Proof. If \( \gamma \) is of degree 9 over \( \alpha \), then using Entries 3 (x), (xi) of Chapter 20 of Ramanujan’s notebooks \[2, p.352\], we find that

\[ \sqrt{mm^l} \left( \frac{\alpha(1-\alpha)^2}{\gamma(1-\gamma)^2} \right)^{\frac{1}{17}} + \frac{3}{\sqrt{mm^l}} \left( \frac{\alpha(1-\alpha)^2}{\gamma(1-\gamma)^2} \right)^{-\frac{1}{17}} = \left( \frac{\gamma(1-\alpha)^2}{\alpha(1-\gamma)^2} \right)^{\frac{1}{17}} + \left( \frac{\gamma(1-\alpha)^2}{\alpha(1-\gamma)^2} \right)^{-\frac{1}{17}}. \]

Using (1.7) and (2.3) with \( N = 9 \) in the above identity (2.33), we obtain (2.32).

Theorem 2.9. We have

\[ \frac{1}{\sqrt{b_{M, 13}}} + \sqrt{b_{M, 13}} = \frac{1}{13} \left( \frac{g_{139M}}{g_{4913M}} + \frac{g_{4913M}}{g_{1939M}} - 14 \left( \frac{g_{139M}}{g_{4913M}} + \frac{g_{4913M}}{g_{1939M}} \right) \right). \]

Proof. If \( \beta \) is of degree 13 over \( \alpha \), then using Entries 8 (iii), (iv) of Chapter 20 of Ramanujan’s Notebooks \[2, p.376\], we find that

\[ m \left( \frac{\alpha(1-\alpha)^2}{\beta(1-\beta)^2} \right)^{\frac{1}{13}} + \frac{13}{m} \left( \frac{\alpha(1-\alpha)^2}{\beta(1-\beta)^2} \right)^{-\frac{1}{13}} = \left( \frac{\beta(1-\alpha)^2}{\alpha(1-\beta)^2} \right)^{\frac{1}{13}} + \left( \frac{\beta(1-\alpha)^2}{\alpha(1-\beta)^2} \right)^{-\frac{1}{13}} \]

\[ - 4 \left[ \left( \frac{\beta(1-\alpha)^2}{\alpha(1-\beta)^2} \right)^{\frac{1}{13}} + \left( \frac{\beta(1-\alpha)^2}{\alpha(1-\beta)^2} \right)^{-\frac{1}{13}} \right]. \]

Using (1.7) and (2.3) with \( N = 13 \) in the above identity (2.35), we obtain (2.34).

Theorem 2.10. We have

\[ \frac{1}{\sqrt{b_{M, 25}}} + \sqrt{b_{M, 25}} = \frac{1}{5} \left[ \left( \frac{g_{25M}}{g_{425M}} + \frac{g_{425M}}{g_{825M}} \right)^3 - 4 \left( \frac{g_{25M}}{g_{425M}} + \frac{g_{425M}}{g_{825M}} \right)^2 - 3 \left( \frac{g_{25M}}{g_{425M}} + \frac{g_{425M}}{g_{825M}} \right) + 8 \right]. \]
Proof. If $\gamma$ is of degree 25 over $\alpha$, then using Entries 15 (i), (ii) of Chapter 19 of Ramanujan’s notebooks [2, p.384], we find that

\begin{equation}
(3.1) \quad \frac{1}{\sqrt{A_M}} + \sqrt{A_M} = \frac{1}{3} \left[ (V_M + V_{M^{-1}})^3 - 7(V_M + V_{M^{-1}}) \right],
\end{equation}

where

\begin{equation}
A_M = b_{M,3}b_{25M,3} \quad \text{and} \quad V_M = \frac{g_{M,3}g_{25M,3}}{g_{3M,975M}}.
\end{equation}

Proof. If $\beta$, $\gamma$, and $\delta$ are third, fifth and fifteenths degree over $\alpha$ respectively, then using Entries 11 (x) and (xi) of Chapter 20 of Ramanujan’s notebooks [2, p.384], we find that

\begin{equation}
(3.2) \quad mm' \left( \frac{\alpha \gamma (1 - \alpha)^2 (1 - \gamma)^2}{\beta (1 - \beta)^2 (1 - \delta)^2} \right)^{\frac{1}{8}} + \frac{9}{mm'} \left( \frac{\alpha \gamma (1 - \alpha)^2 (1 - \gamma)^2}{\beta (1 - \beta)^2 (1 - \delta)^2} \right)^{-\frac{1}{8}} \left( \frac{\beta \delta (1 - \alpha)^2 (1 - \gamma)^2}{\alpha (1 - \beta)^2 (1 - \delta)^2} \right)^{\frac{1}{8}} + \left( \frac{\beta \delta (1 - \alpha)^2 (1 - \gamma)^2}{\alpha (1 - \beta)^2 (1 - \delta)^2} \right)^{-\frac{1}{8}} - 4 \left( \frac{\beta \delta (1 - \alpha)^2 (1 - \gamma)^2}{\alpha (1 - \beta)^2 (1 - \delta)^2} \right)^{\frac{1}{8}} \left( \frac{\beta \delta (1 - \alpha)^2 (1 - \gamma)^2}{\alpha (1 - \beta)^2 (1 - \delta)^2} \right)^{-\frac{1}{8}}.
\end{equation}

Using (1.7) and (2.3) with $N = 3$ in the above identity (3.2), we obtain (3.1).

Theorem 3.1. We have

\begin{equation}
(3.3) \quad \frac{1}{\sqrt{A_M}} + \sqrt{A_M} = V_M^3 + V_{M^{-1}}^3 + 4,
\end{equation}

where

\begin{equation}
A_M = b_{25M,3}b_{M,3} \quad \text{and} \quad V_M = \frac{g_{M,3}g_{25M,3}}{g_{3M,975M}}.
\end{equation}

Proof. If $\beta$, $\gamma$, and $\delta$ are of third, fifth and fifteenths degree over $\alpha$ respectively, then by using Entries 11 (viii) and (ix) of Chapter 20 of Ramanujan’s notebooks [2, p.384], we find that

\begin{equation}
(3.4) \quad \sqrt{\frac{m'}{m}} \left( \frac{\beta \gamma (1 - \beta)^2 (1 - \gamma)^2}{\alpha \delta (1 - \alpha)^2 (1 - \delta)^2} \right)^{\frac{1}{16}} - \sqrt{\frac{m'}{m'}} \left( \frac{\beta \gamma (1 - \beta)^2 (1 - \gamma)^2}{\alpha \delta (1 - \alpha)^2 (1 - \delta)^2} \right)^{-\frac{1}{16}} \left( \frac{\beta \gamma (1 - \beta)^2 (1 - \gamma)^2}{\alpha \delta (1 - \alpha)^2 (1 - \delta)^2} \right)^{\frac{1}{16}} + \left( \frac{\beta \gamma (1 - \beta)^2 (1 - \gamma)^2}{\alpha \delta (1 - \alpha)^2 (1 - \delta)^2} \right)^{-\frac{1}{16}}.
\end{equation}

Using (1.7) and (2.3) with $N = 3$ in the above identity (3.4), we obtain (3.3).
Theorem 3.3. We have
\[
\frac{1}{\sqrt{A_M}} + \sqrt{A_M} = V^3_M + V^{-3}_M + 4,
\]
where
\[
A_M = \frac{b_{19M, 5}}{b_{M, 5}} \text{ and } V_M = \frac{g_M g_{49M}}{g_{25M} g_{92M}}.
\]

Proof. Using (1.7) and (2.3) with \( N = 5 \) in the above identity (3.4), we obtain (3.5). 

Theorem 3.4. We have
\[
\frac{1}{\sqrt{A_M}} + \sqrt{A_M} = (V_M + V^{-1}_M)^3 + (V_M + V^{-1}_M),
\]
where
\[
A_M = \frac{b_{19M, 3}}{b_{M, 3}} \text{ and } V_M = \frac{g_M g_{147M}}{g_{7M} g_{92M}}.
\]

Proof. If \( \beta, \gamma \) and \( \delta \) are of third, seventh and twenty-first degree over \( \alpha \) respectively, then by using Entries 13 (i) and (ii) of Chapter 20 of Ramanujan’s notebooks [2, p.401], we find that
\[
\frac{m'}{m} \left( \frac{\beta \gamma (1 - \beta)^2 (1 - \gamma)^2}{\alpha \delta (1 - \alpha)^2 (1 - \delta)^2} \right)^{\frac{1}{2}} + \frac{m'}{m} \left( \frac{\beta \gamma (1 - \beta)^2 (1 - \gamma)^2}{\alpha \delta (1 - \alpha)^2 (1 - \delta)^2} \right)^{-\frac{k}{2}}
\]
\[
= \left( \frac{\alpha \delta (1 - \beta)^2 (1 - \gamma)^2}{\beta \gamma (1 - \alpha)^2 (1 - \delta)^2} \right)^{\frac{1}{2}} + \left( \frac{\alpha \delta (1 - \beta)^2 (1 - \gamma)^2}{\beta \gamma (1 - \alpha)^2 (1 - \delta)^2} \right)^{-\frac{k}{2}}
\]
\[
+ 4 \left( \frac{\alpha \delta (1 - \beta)^2 (1 - \gamma)^2}{\beta \gamma (1 - \alpha)^2 (1 - \delta)^2} \right)^{\frac{1}{2}} + 4 \left( \frac{\alpha \delta (1 - \beta)^2 (1 - \gamma)^2}{\beta \gamma (1 - \alpha)^2 (1 - \delta)^2} \right)^{-\frac{k}{2}}.
\]

Using (1.7) and (2.3) with \( N = 3 \) in the above identity (3.7), we obtain (3.6). 

Theorem 3.5. We have
\[
\frac{1}{\sqrt{A_M}} + \sqrt{A_M} = (V_M + V^{-1}_M)^3 + (V_M + V^{-1}_M),
\]
where
\[
A_M = \frac{b_{19M, 7}}{b_{M, 7}} \text{ and } V_M = \frac{g_M g_{63M}}{g_{7M} g_{92M}}.
\]

Proof. Using (1.7) and (2.3) with \( N = 7 \) in the above identity (3.7), we obtain (3.8). 

Theorem 3.6. We have
\[
\frac{1}{\sqrt{A_M}} + \sqrt{A_M} = (V_M + V^{-1}_M)^3 + 4 (V_M + V^{-1}_M)^2 + 5 (V_M + V^{-1}_M),
\]
where
\[
A_M = \frac{b_{169M, 3}}{b_{M, 3}} \text{ and } V_M = \frac{g_M g_{507M}}{g_{25M} g_{169M}}.
\]

Proof. If \( \beta, \gamma \) and \( \delta \) are of third, thirteenth and thirty-ninth degree over \( \alpha \) respectively, then by using Entry 19 (iv) of Chapter 20 of Ramanujan’s notebooks [2, p.426], we find that
\[
\sqrt{m'} / m \left( \frac{\beta \gamma (1 - \beta)^2 (1 - \gamma)^2}{\alpha \delta (1 - \alpha)^2 (1 - \delta)^2} \right)^{\frac{1}{2}} + \sqrt{m'} / m \left( \frac{\beta \gamma (1 - \beta)^2 (1 - \gamma)^2}{\alpha \delta (1 - \alpha)^2 (1 - \delta)^2} \right)^{-\frac{k}{2}}
\]
\[
= \left( \frac{\alpha \delta (1 - \beta)^2 (1 - \gamma)^2}{\beta \gamma (1 - \alpha)^2 (1 - \delta)^2} \right)^{\frac{1}{2}} + \left( \frac{\alpha \delta (1 - \beta)^2 (1 - \gamma)^2}{\beta \gamma (1 - \alpha)^2 (1 - \delta)^2} \right)^{-\frac{k}{2}}
\]
Using Entries 18 (vi), (vii) of Chapter 20 of Ramanujan’s notebooks \[2, \text{p.426}\], we find that

\[\frac{1}{\sqrt{A_M}} + \sqrt{A_M} = (V_M + V_M^{-1})^3 + 4 (V_M + V_M^{-1})^2 + 5 (V_M + V_M^{-1}) + 4,\]

where

\[A_M = \frac{b_{25M, 7}}{b_{M, 7}} \quad \text{and} \quad V_M = \frac{g_M g_{175M}}{g_{7M} g_{25M}}.\]

Proof. Using (1.7) and (2.3) with \(N = 5\) in the above identity (3.13), we obtain (3.12). \[\Box\]

Theorem 3.9. We have

\[\frac{1}{\sqrt{A_M}} + \sqrt{A_M} = \left(V_M + V_M^{-1}\right)^3 + 4 \left(V_M + V_M^{-1}\right)^2 + 5 \left(V_M + V_M^{-1}\right) + 4,\]

where

\[A_M = \frac{b_{25M, 7}}{b_{M, 7}} \quad \text{and} \quad V_M = \frac{g_M g_{175M}}{g_{7M} g_{25M}}.\]

Proof. Using (1.7) and (2.3) with \(N = 7\) in the above identity (3.13), we obtain (3.14). \[\Box\]

Theorem 3.10. We have

\[\frac{1}{\sqrt{A_M}} + \sqrt{A_M} = \frac{1}{3} \left(\left(V_M + V_M^{-1}\right)^3 - 4 \left(V_M + V_M^{-1}\right)^2 - 3 \left(V_M + V_M^{-1}\right) + 12\right),\]

where

\[A_M = b_{M, 3} b_{121M, 3} \quad \text{and} \quad V_M = \frac{g_M g_{121M}}{g_{3M} g_{363M}}.\]
Proof. If \( \beta, \gamma \) and \( \delta \) are third, eleventh and thirty-third degree over \( \alpha \) respectively, then using Entries 14 (i) and (ii) of Chapter 20 of Ramanujan’s notebooks [2], p.408, we find that

\[
(3.16) \quad \sqrt{\frac{m^\prime}{m}} \left( \frac{\alpha \gamma (1 - \alpha)^2 (1 - \gamma)^2}{\beta \delta (1 - \beta)^2 (1 - \delta)^2} \right)^{\frac{1}{m^\prime}} + \frac{3}{\sqrt{m^\prime}} \left( \frac{\alpha \gamma (1 - \alpha)^2 (1 - \gamma)^2}{\beta \delta (1 - \beta)^2 (1 - \delta)^2} \right)^{-\frac{1}{m^\prime}}
\]

\[
= \left( \frac{\beta \delta (1 - \alpha)^2 (1 - \gamma)^2}{\alpha \gamma (1 - \beta)^2 (1 - \delta)^2} \right)^{\frac{1}{m^\prime}} + \left( \frac{\beta \delta (1 - \alpha)^2 (1 - \gamma)^2}{\alpha \gamma (1 - \beta)^2 (1 - \delta)^2} \right)^{-\frac{1}{m^\prime}}
\]

Using (1.7) and (2.3) with \( N = 3 \) in the above identity (3.16), we obtain (3.15). 

Theorem 3.11. We have

\[
(3.17) \quad \frac{1}{\sqrt{A_M}} + \sqrt{A_M} = V^3_M + V^{-3}_M,
\]

where

\[
A_M = \frac{b_{81M, 3}}{b_{M, 3}} \quad \text{and} \quad V_M = \frac{g_{64}g_{243M}}{g_{3M}g_{27M}}.
\]

Proof. If \( \beta, \gamma \) and \( \delta \) are of third, ninth and twenty-seventh degree over \( \alpha \) respectively, then by using Entry 5 (i) of Chapter 20 of Ramanujan’s notebooks [2], p.360 and its reciprocal equation, we find that

\[
(3.18) \quad \sqrt{\frac{m^\prime}{m}} \left( \frac{\alpha \gamma (1 - \beta)^2 (1 - \gamma)^2}{\alpha \delta (1 - \alpha)^2 (1 - \delta)^2} \right)^{\frac{1}{m^\prime}} - \sqrt{\frac{m^\prime}{m}} \left( \frac{\beta \gamma (1 - \beta)^2 (1 - \gamma)^2}{\beta \delta (1 - \alpha)^2 (1 - \delta)^2} \right)^{\frac{1}{m^\prime}}
\]

\[
= \left( \frac{\alpha \delta (1 - \beta)^2 (1 - \gamma)^2}{\beta \gamma (1 - \alpha)^2 (1 - \delta)^2} \right)^{\frac{1}{m^\prime}} - \left( \frac{\alpha \delta (1 - \beta)^2 (1 - \gamma)^2}{\beta \gamma (1 - \alpha)^2 (1 - \delta)^2} \right)^{-\frac{1}{m^\prime}}.
\]

Using (1.7) and (2.3) with \( N = 3 \) in the above identity (3.18), we obtain (3.17).

Theorem 3.12. We have

\[
(3.19) \quad \frac{1}{\sqrt{A_M}} + \sqrt{A_M} = V^3_M + V^{-3}_M,
\]

where

\[
A_M = \frac{b_{9M, 3}}{b_{M, 9}} \quad \text{and} \quad V_M = \frac{g_{64}g_{M}}{g_{9M}g_{81M}}.
\]

Proof. Using (1.7) and (2.3) with \( N = 9 \) in the above identity (3.18), we obtain (3.19).

4. Explicit Evaluations of \( b_{M, N} \)

In this section, we establish several explicit evaluations of \( b_{M, N} \).

Theorem 4.1. We have

\[
(4.1) \quad b_{5, 3} = \sqrt{190 - 105\sqrt{3} - \sqrt{186 - 105\sqrt{3}}} \quad \frac{2}{2},
\]

\[
(4.2) \quad b_{8, 3} = \left( \sqrt{43 + 24\sqrt{3}} - \sqrt{42 + 24\sqrt{3}} \right)^{\frac{1}{2}},
\]
ON SOME REMARKABLE PRODUCT OF THETA-FUNCTION

\begin{equation}
(4.3) \quad b_{20,3} = \left(2 - \frac{\sqrt{15}}{2}\right)^{\frac{3}{2}} \left(47 - 21\sqrt{5}\right)^{\frac{1}{2}},
\end{equation}

\begin{equation}
(4.4) \quad b_{22,3} = 6 + \sqrt{33} - 2\sqrt{17} + 3\sqrt{33},
\end{equation}

\begin{equation}
(4.5) \quad b_{34,3} = 33 - 8\sqrt{17},
\end{equation}

\begin{equation}
(4.6) \quad b_{38,3} = 22 + 3\sqrt{57} - 2\sqrt{249} + 33\sqrt{57},
\end{equation}

\begin{equation}
(4.7) \quad b_{42,3} = \frac{2 + \sqrt{2}}{3} - \sqrt{\frac{1 + 2\sqrt{2}}{3}},
\end{equation}

\begin{equation}
(4.8) \quad b_{46,3} = \sqrt{3057 + 1248\sqrt{6}} - \sqrt{3056 + 1248\sqrt{6}},
\end{equation}

\begin{equation}
(4.9) \quad b_{60,3} = \frac{33}{3} + \sqrt{42 + 10\sqrt{33}} \left(\frac{1}{24}\sqrt{66} - \frac{3}{8}\sqrt{2}\right),
\end{equation}

\begin{equation}
(4.10) \quad b_{70,3} = \sqrt{54105 + 5280\sqrt{105}} - \sqrt{54104 + 5280\sqrt{105}},
\end{equation}

\begin{equation}
(4.11) \quad b_{110,3} = \sqrt{2537329 + 540960\sqrt{22}} - \sqrt{2537328 + 540960\sqrt{22}}
\end{equation}

and

\begin{equation}
(4.12) \quad b_{174,3} = \frac{11\sqrt{6}}{3} + \left(8\sqrt{6} - 23\right) \sqrt{\frac{99 + 42\sqrt{6}}{29}}.
\end{equation}

Proof of (4.1) : From the table in Chapter 34 of Ramanujan’s notebooks [3, pp.190, 341], we find that

\begin{equation}
(4.13) \quad G_{15}G_{\frac{3}{2}} = \sqrt{2}.
\end{equation}

Using Entries 12 (vi) and (vii) of Chapter 17 of Ramanujan’s notebooks [2, p.124] in Entry 5(ii) of Chapter 19 of Ramanujan’s notebooks [2, p.230], we find that

\begin{equation}
(4.14) \quad g_n^2 g_{9n}^2 = \frac{1}{2G_n G_{9n}} \left[G_n^3 G_{9n}^3 + \sqrt{G_n^6 G_{9n}^6 - 2}\right].
\end{equation}

Using (4.13) in (4.14) with \(n = \frac{3}{2}\), we find that

\begin{equation}
(4.15) \quad g_\frac{3}{2} g_{15} = \frac{\sqrt{3} + 1}{2}.
\end{equation}

From Theorem 4.1(i) in [6], we have

\begin{equation}
(4.16) \quad 2\sqrt{2} \left[g_n^3 g_{9n}^3 + g_n^{-3} g_{9n}^{-3}\right] = \frac{g_{9n}^6}{g_n^6} - \frac{g_n^6}{g_{9n}^6}.
\end{equation}

Using (4.15) in (4.16) with \(n = \frac{3}{2}\), we deduce that

\begin{equation}
(4.17) \quad \frac{g_{15}^6}{g_{\frac{3}{2}}^6} = \sqrt{\frac{1710 - 945\sqrt{3}}{4}} + \sqrt{\frac{1706 - 945\sqrt{3}}{4}}.
\end{equation}

Using (4.17) in (2.23) with \(M = 5\), we obtain the required result (4.1).
Proof of (4.2). From Theorem 4.5(i) in [6], we have

\[ g_{\frac{8}{3}} g_{24} = \sqrt{\frac{3}{8}} + 1. \]

Using (4.18) in (4.16) with \( n = \frac{8}{3} \), we find that

\[ \frac{g_{\frac{12}{3}}}{g_{\frac{12}{3}}} = \left( 44 + 27 \sqrt{3} \right) + \left( 33 + 18 \sqrt{3} \right) \sqrt{2}. \]

Using (4.19) in (2.23) with \( M = 8 \), we obtain the required result (4.2).

Proof of (4.4). From the table in Chapter 34 of Ramanujan’s notebooks [3, p.201], we have

\[ g_{66} = \left( \sqrt{3} + \sqrt{2} \right) \frac{1}{4} \left( \sqrt{7 + \sqrt{33}} \pm \sqrt{\frac{33}{8} - \sqrt{33} - 1} \right)^{\frac{1}{2}}. \]

Using (4.20) in (4.16) with \( n = \frac{22}{3} \), we find that

\[ g_{\frac{22}{3}} = \left( \sqrt{3} + \sqrt{2} \right) \frac{1}{4} \left( \sqrt{7 + \sqrt{33}} \pm \sqrt{\frac{33}{8} - \sqrt{33} - 1} \right)^{\frac{1}{2}}. \]

Using (4.20) and (4.21) in (2.23) with \( M = 22 \), we obtain the required result (4.4).

Proof of the identity (4.3) is similar to the proof of the identity (4.1) and proofs of the identities (4.5)-(4.12) being similar to the proof of the identity (4.4). So we omit the details.

Theorem 4.2. We have

\[ b_{6,5} = \left( \sqrt{2} - 1 \right)^2, \]

\[ b_{38,5} = \left( 17 - 12 \sqrt{2} \right)^2 \]

and

\[ b_{62,5} = \left( 28 + 9 \sqrt{10} - 3 \sqrt{177 + 56 \sqrt{10}} \right)^2. \]

Proof of (4.22). From the table in Chapter 34 of Ramanujan’s notebooks [3, p.200], we have

\[ g_{30} = \left( 2 + \sqrt{5} \right)^\frac{1}{3} \left( 3 + \sqrt{10} \right)^\frac{1}{5}. \]

From Theorem 4.1(ii) in [6], we have

\[ 2 \left[ g_n^2 g_{25n}^2 + g_n^{-2} g_{25n}^{-2} \right] = \frac{g_{25n}^3}{g_n^3} = \frac{g_n^3}{g_{25n}^3}. \]

Using (4.25) in (4.26) with \( n = \frac{6}{5} \), we find that

\[ g_{\frac{6}{5}} = \left( 2 + \sqrt{5} \right) \left( -3 + \sqrt{10} \right). \]

As the proofs of the identities (4.23)-(4.24) being similar to the proof of the identity (4.22). So we omit the details.
Theorem 4.3. We have

(4.28) \( b_{6, 7} = 5 - 2\sqrt{6}, \)

(4.29) \( b_{10, 7} = \left( \sqrt{10} - 3 \right)^2, \)

(4.30) \( b_{14, 7} = -\sqrt{\frac{2}{9}} + \sqrt{\frac{7 + 2\sqrt{14}}{49}} \)

and

(4.31) \( b_{18, 7} = \left( \sqrt{34 + 24\sqrt{2}} - \sqrt{33 + 24\sqrt{2}} \right)^2. \)

Proof of (4.28). From the table in Chapter 34 of Ramanujan’s notebooks [3, p.201], we have

(4.32) \( g_{42} = \left( 2\sqrt{2} + \sqrt{7} \right)^\frac{1}{9} \left( \frac{\sqrt{3} + \sqrt{7}}{2} \right)^\frac{1}{2}. \)

From Theorem 4.1(iii) in [6], we have

(4.33) \( 16\sqrt{2} \left[ g_n^{9} g_{49n} + g_n^{-9} g_{49n}^{-9} \right] + 168 \left[ g_n^6 g_{49n}^6 + g_n^{-6} g_{49n}^{-6} \right] \)

\( + 336\sqrt{2} \left[ g_n^3 g_{49n}^3 + g_n^{-3} g_{49n}^{-3} \right] + 658 \cdot \frac{g_{49n}^{12}}{g_n^{12}} + \frac{g_{49n}^{12}}{g_n^{12}}. \)

Using (4.32) in (4.33) with \( n = \frac{6}{7}, \) we find that

(4.34) \( g_{47} = \left( 2\sqrt{2} + \sqrt{7} \right)^\frac{1}{9} \left( \frac{-\sqrt{3} + \sqrt{7}}{2} \right)^\frac{1}{2}. \)

Using (4.32) and (4.34) in (2.30) with \( M = 6, \) we obtain the required result (4.28). As the proofs of the identities (4.29)-(4.31) being similar to the proof of the identity (4.28). So we omit the details.

Theorem 4.4. We have

(4.35) \( b_{10, 9} = \left( \sqrt{10 + 4\sqrt{6}} - \sqrt{9 + 4\sqrt{6}} \right)^2, \)

(4.36) \( b_{22, 9} = \left( \sqrt{253 + 44\sqrt{33}} - \sqrt{252 + 44\sqrt{33}} \right)^2 \)

and

(4.37) \( b_{58, 9} = \left( \sqrt{117370 + 47916\sqrt{6}} - \sqrt{117369 + 47916\sqrt{6}} \right)^2. \)

Proof of (4.35). From the table in Chapter 34 of Ramanujan’s notebooks [3, p.202], we have

(4.38) \( g_{90} = \left( 2 + \sqrt{5} \right) \left( \sqrt{5 + \sqrt{6}} \right)^\frac{1}{2} \left( \frac{3 + \sqrt{6}}{4} + \sqrt{\frac{\sqrt{6} - 1}{4}} \right). \)
Using (4.38) in an identity from a page 145 of Chapter 4 in \cite{5} eq(4.7.12),p.145 with changing $q$ to $-q$, we obtain

$$
(4.39) \quad g_{\frac{10}{q}} = \left[ (2 + \sqrt{5}) (\sqrt{5} + \sqrt{6}) \right]^\frac{1}{2} \left( \frac{3 + \sqrt{6}}{4} - \sqrt{\frac{\sqrt{6} - 1}{4}} \right).
$$

Using (4.38) and (4.39) in (2.32) with $M = 10$, we obtain the required result (4.35). As the proofs of the identities (4.36)-(4.37) being similar to the proof of the identity (4.35). So we omit the details.

**Theorem 4.5.** We have

$$
(4.40) \quad b_{6,13} = \left( 3 - 2\sqrt{2} \right)^2
$$

and

$$
(4.41) \quad b_{10,13} = \left( \sqrt{65} - 8 \right)^2.
$$

**Proof of (4.40).** From the table in Chapter 34 of Ramanujan’s notebooks \cite{3} p.202, we have

$$
(4.42) \quad g_{78} = \left( \frac{3 + \sqrt{13}}{2} \right)^\frac{1}{2} \left( 5 + \sqrt{26} \right)^\frac{1}{6}.
$$

Using (4.42) in Entry 41 of Chapter 38 of Ramanujan’s notebooks \cite{3} p.378], we find that

$$
(4.43) \quad g_{\frac{6}{13}} = \left( \frac{\sqrt{13} - 3}{2} \right)^\frac{1}{2} \left( 5 + \sqrt{26} \right)^\frac{1}{6}.
$$

Using (4.42) and (4.43) in (2.34) with $M = 6$, we obtain the required result (4.40). As the proof of the identity (4.41) being similar to the proof of the identity (4.40). So we omit the details.

**Remark:** $b_{M,N}$ are units in some quadratic field. We retain the details for our future paper.

**REFERENCES**


