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POSITIVE SOLUTIONS FOR SYSTEMS OF THREE-POINT NONLINEAR BOUNDARY VALUE PROBLEMS

J. HENDERSON AND S. K. NTOUYAS

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DEPARTMENT OF MATHEMATICS, BAYLOR UNIVERSITY, WACO, TEXAS 76798-7328 USA.
Johnny_Henderson@baylor.edu
URL: http://www3.baylor.edu/~Johnny_Henderson

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF IOANNINA, 451 10 IOANNINA, GREECE.
sntouyas@cc.uoi.gr
URL: http://www.math.uoi.gr/~sntouyas

ABSTRACT. Values of λ are determined for which there exist positive solutions of the system of three-point boundary value problems, $u''(t) + \lambda a(t)f(v(t)) = 0,\ v''(t) + \lambda b(t)g(u(t)) = 0$, for $0 < t < 1$, and satisfying, $u(0) = 0, u(1) = \alpha u(\eta), v(0) = 0, v(1) = \alpha v(\eta)$. A Guo-Krasnosel'skii fixed point theorem is applied.

Key words and phrases: Three-point boundary value problem, system of differential equations, eigenvalue problem.

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1. INTRODUCTION

We are concerned with determining values of λ (eigenvalues) for which there exist positive solutions for the system of three-point boundary value problems,

$$\begin{aligned} u''(t) + \lambda a(t) f(v(t)) = 0, \quad 0 < t < 1, \\ v''(t) + \lambda b(t) g(u(t)) = 0, \quad 0 < t < 1, \end{aligned} \tag{1.1}$$

$$\begin{aligned} u(0) = 0, \quad u(1) = \alpha u(\eta), \\ v(0) = 0, \quad v(1) = \alpha v(\eta), \end{aligned} \tag{1.2}$$

where $0 < \eta < 1, 0 < \alpha < 1/\eta$, and

(A) $f, g \in C([0, \infty), [0, \infty))$,
(B) $a, b \in C([0, 1], [0, \infty))$, and each does not vanish identically on any subinterval,
(C) All of

$$f_0 := \lim_{x \to 0^+} \frac{f(x)}{x}, \quad g_0 := \lim_{x \to 0^+} \frac{g(x)}{x},$$

$$f_\infty := \lim_{x \to \infty} \frac{f(x)}{x} \quad \text{and} \quad g_\infty := \lim_{x \to \infty} \frac{g(x)}{x}$$

exist as positive real numbers.

For several years now, there has been a great deal of activity in studying positive solutions of boundary value problems for ordinary differential equations. Interest in such solutions is high from both a theoretical sense [4, 7, 10, 21] and as applications for which only positive solutions are meaningful [2, 5, 14, 15]. These considerations are caste primarily for scalar problems, but good attention has been given to boundary value problems for systems of differential equations [11, 12, 13, 17, 20, 23]. Of equal interest has been the intersection of questions involving positive solutions and nonlocal boundary value problems; see, for example [1, 6, 16, 18, 19, 21, 22, 23].

Recently Benchohra *et al.* [3] and Henderson and Ntouyas [9] studied the existence of positive solutions of systems of nonlinear eigenvalue problems. Here we extend these results to eigenvalue problems for systems of three-point boundary value problems.

The main tool in this paper is an application of the Guo-Krasnosel'skii fixed point theorem for operators leaving an annular-like region in a Banach space cone invariant [7]. A Green's function plays a fundamental role in defining an appropriate operator on a suitable cone.

2. SOME PRELIMINARIES

In this section, we state some preliminary lemmas and the well-known Guo-Krasnosel'skii fixed point theorem.

Lemma 2.1. [8] *Let $0 < \eta < 1, 0 < \alpha < 1/\eta$; then, for any $y \in C[0, 1]$, the boundary value problem*

$$u''(t) + y(t) = 0, \quad 0 < t < 1, \tag{2.1}$$

$$u(0) = 0, \quad u(1) = \alpha u(\eta), \tag{2.2}$$

has a unique solution

$$u(t) = \int_0^1 k(t, s) y(s) ds$$

where $k(t,s):[0,1]\times[0,1]\to\mathbb{R}^+$ *is defined by*

$$k(t,s)=\begin{cases}\frac{t(1-s)}{1-\alpha\eta}-\frac{\alpha t(\eta-s)}{1-\alpha\eta}-(t-s), & 0\le s\le t\le 1 \text{ and } s\le\eta,\\ \frac{t(1-s)}{1-\alpha\eta}-\frac{\alpha t(\eta-s)}{1-\alpha\eta}, & 0\le t\le s\le\eta\ ,\\ \frac{t(1-s)}{1-\alpha\eta}, & \text{if } 0\le t\le s\le 1 \text{ and } s\ge\eta,\\ \frac{t(1-s)}{1-\alpha\eta}-(t-s), & \eta\le s\le t\le 1.\end{cases} \tag{2.3}$$

Notice that

$$u(t)=\frac{t}{1-\alpha\eta}\int_0^1(1-s)y(s)ds-\frac{\alpha t}{1-\alpha\eta}\int_0^\eta(\eta-s)y(s)ds-\int_0^t(t-s)y(s)ds. \tag{2.4}$$

From (2.4) obviously we have (see [16]) that

$$u(t)\le\frac{t}{1-\alpha\eta}\int_0^1(1-s)y(s)ds, \tag{2.5}$$

and

$$u(\eta)\ge\frac{\eta}{1-\alpha\eta}\int_\eta^1(1-s)y(s)ds. \tag{2.6}$$

Lemma 2.2. [16] *Let* $0<\alpha<1/\eta$ *and assume (A) and (B) hold. Then, the unique solution of (1.1)- (1.2) satisfies*

$$\inf_{t\in[\eta,1]}u(t)\ge\gamma\|u\|,$$

where $\gamma=\min\left\{\alpha\eta,\frac{\alpha(1-\eta)}{1-\alpha\eta},\eta\right\}$.

We note that a pair $(u(t),v(t))$ is a solution of eigenvalue problem (1.1), (1.2) if, and only if,

$$u(t)=\lambda\int_0^1 k(t,s)a(s)f\left(\lambda\int_0^1 k(s,r)b(r)g(u(r))dr\right)ds,\quad 0\le t\le 1,$$

where

$$v(t)=\lambda\int_0^1 k(t,s)b(s)g(u(s))ds,\quad 0\le t\le 1.$$

Values of λ for which there are positive solutions (positive with respect to a cone) of (1.1), (1.2) will be determined via applications of the following fixed point theorem.

Theorem 2.3. *Let* $\mathcal{B}$ *be a Banach space, and let* $\mathcal{P}\subset\mathcal{B}$ *be a cone in* $\mathcal{B}$. *Assume* Ω_1 *and* Ω_2 *are open subsets of* $\mathcal{B}$ *with* $0\in\Omega_1\subset\overline{\Omega}_1\subset\Omega_2$, *and let*

$$T:\mathcal{P}\cap(\overline{\Omega}_2\setminus\Omega_1)\to\mathcal{P}$$

be a completely continuous operator such that, either

(i) $||Tu||\le||u||, u\in\mathcal{P}\cap\partial\Omega_1$, *and* $||Tu||\ge||u||, u\in\mathcal{P}\cap\partial\Omega_2$, *or*
(ii) $||Tu||\ge||u||, u\in\mathcal{P}\cap\partial\Omega_1$, *and* $||Tu||\le||u||, u\in\mathcal{P}\cap\partial\Omega_2$.

Then T *has a fixed point in* $\mathcal{P}\cap(\overline{\Omega}_2\setminus\Omega_1)$.

3. POSITIVE SOLUTIONS IN A CONE

In this section, we apply Theorem 2.3 to obtain solutions in a cone (that is, positive solutions) of (1.1), (1.2). For our construction, let $\mathcal{B} = C[0,1]$ with supremum norm, $\|\cdot\|$, and define a cone $\mathcal{P} \subset \mathcal{B}$ by

$$\mathcal{P} = \left\{ x \in \mathcal{B} \mid x(t) \geq 0 \text{ on } [0,1], \text{ and } \min_{t\in[\eta,\,1]} x(t) \geq \gamma \|x\| \right\}.$$

For our first result, define positive numbers L_1 and L_2 by

$$L_1 := \max \left\{ \left[\frac{\gamma\eta}{1-\alpha\eta} \int_\eta^1 (1-r)a(r) f_\infty dr \right]^{-1}, \left[\frac{\gamma\eta}{1-\alpha\eta} \int_\eta^1 (1-r)a(r) g_\infty dr \right]^{-1} \right\},$$

and

$$L_2 := \min \left\{ \left[\frac{1}{1-\alpha\eta} \int_0^1 (1-r)a(r) f_0 dr \right]^{-1}, \left[\frac{1}{1-\alpha\eta} \int_0^1 (1-r)b(r) g_0 dr \right]^{-1} \right\}.$$

Theorem 3.1. *Assume conditions* (A), (B) *and* (C) *are satisfied. Then, for each* λ *satisfying*

$$L_1 < \lambda < L_2, \tag{3.1}$$

there exists a pair (u,v) *satisfying* (1.1), (1.2) *such that* $u(x) > 0$ *and* $v(x) > 0$ *on* $(0,1)$.

Proof. Let λ be as in (3.1) and $\epsilon > 0$ be chosen such that

$$\max \left\{ \left[\frac{\gamma\eta}{1-\alpha\eta} \int_\eta^1 (1-r)a(r)(f_\infty - \epsilon) dr \right]^{-1}, \left[\frac{\gamma\eta}{1-\alpha\eta} \int_\eta^1 (1-r)a(r)(g_\infty - \epsilon) dr \right]^{-1} \right\} \leq \lambda$$

and

$$\lambda \leq \min \left\{ \left[\frac{1}{1-\alpha\eta} \int_0^1 (1-r)a(r)(f_0 + \epsilon) dr \right]^{-1}, \left[\frac{1}{1-\alpha\eta} \int_0^1 (1-r)b(r)(g_0 + \epsilon) dr \right]^{-1} \right\}.$$

Define an integral operator $T : \mathcal{P} \to \mathcal{B}$ by

$$Tu(t) := \lambda \int_0^1 k(t,s)a(s) f\left(\lambda \int_0^1 k(s,r)b(r)g(u(r))dr \right) ds, \quad u \in \mathcal{P}. \tag{3.2}$$

We seek suitable fixed points of T in the cone $\mathcal{P}$.

By Lemma 2.2, $T\mathcal{P} \subset \mathcal{P}$. In addition, standard arguments show that T is completely continuous.

Now, from the definitions of f_0 and g_0, there exists an $H_1 > 0$ such that

$$f(x) \leq (f_0 + \epsilon)x \text{ and } g(x) \leq (g_0 + \epsilon)x, \quad 0 < x \leq H_1.$$

Let $u \in \mathcal{P}$ with $\|u\| = H_1$. We first have from (2.5) and choice of ϵ,

$$\lambda \int_0^1 k(s,r)b(r)g(u(r))dr \leq \lambda \frac{t}{1-\alpha\eta} \int_0^1 (1-r)b(r)g(u(r))dr$$

$$\begin{aligned}
&\leq \lambda\frac{t}{1-\alpha\eta}\int_0^1(1-r)b(r)(g_0+\epsilon)u(r)dr\\
&\leq \lambda\frac{1}{1-\alpha\eta}\int_0^1(1-r)b(r)dr(g_0+\epsilon)\|u\|\\
&\leq \|u\|\\
&= H_1.
\end{aligned}$$

As a consequence, we next have from (2.5), and choice of ϵ,

$$\begin{aligned}
Tu(t) &= \lambda\int_0^1 k(t,s)a(s)f\left(\lambda\int_0^1 k(s,r)b(r)g(u(r))dr\right)ds\\
&\leq \lambda\frac{t}{1-\alpha\eta}\int_0^1(1-s)a(s)f\left(\lambda\int_0^1 k(s,r)b(r)g(u(r))dr\right)ds\\
&\leq \lambda\frac{t}{1-\alpha\eta}\int_0^1(1-s)a(s)(f_0+\epsilon)\lambda\int_0^1 k(s,r)b(r)g(u(r))drds\\
&\leq \lambda\frac{1}{1-\alpha\eta}\int_0^1(1-s)a(s)(f_0+\epsilon)H_1ds\\
&\leq H_1\\
&= \|u\|.
\end{aligned}$$

So, $\|Tu\|\leq\|u\|$. If we set

$$\Omega_1=\{x\in\mathcal{B}\mid \|x\|<H_1\},$$

then

$$\|Tu\|\leq\|u\|,\ \text{for } u\in\mathcal{P}\cap\partial\Omega_1. \tag{3.3}$$

Next, from the definitions of f_∞ and g_∞, there exists $\overline{H}_2>0$ such that

$$f(x)\geq(f_\infty-\epsilon)x \text{ and } g(x)\geq(g_\infty-\epsilon)x,\quad x\geq\overline{H}_2.$$

Let

$$H_2=\max\left\{2H_1,\frac{\overline{H}_2}{\gamma}\right\}.$$

Let $u\in\mathcal{P}$ and $\|u\|=H_2$. Then,

$$\min_{t\in[\eta,1]}u(t)\geq\gamma\|u\|\geq\overline{H}_2.$$

Consequently, from (2.6) and choice of ϵ,

$$\begin{aligned}
\lambda\int_0^1 k(s,r)b(r)g(u(r))dr &\geq \lambda\frac{\eta}{1-\alpha\eta}\int_\eta^1(1-r)b(r)g(u(r))dr\\
&\geq \lambda\frac{\eta}{1-\alpha\eta}\int_\eta^1(1-r)b(r)g(u(r))dr\\
&\geq \lambda\frac{\eta}{1-\alpha\eta}\int_\eta^1(1-r)b(r)(g_\infty-\epsilon)u(r)dr\\
&\geq \lambda\frac{\eta}{1-\alpha\eta}\int_\eta^1(1-r)b(r)(g_\infty-\epsilon)dr\gamma\|u\|\\
&\geq \|u\|\\
&= H_2.
\end{aligned}$$

And so, we have from (2.6) and choice of ϵ,

$$\begin{aligned} Tu(\eta) &\geq \lambda \frac{\eta}{1-\alpha\eta} \int_\eta^1 (1-s)a(s)f\left(\lambda \int_\eta^1 k(s,r)b(r)g(u(r))dr\right) ds \\ &\geq \lambda \frac{\eta}{1-\alpha\eta} \int_\eta^1 (1-s)a(s)(f_\infty - \epsilon)\lambda \int_\eta^1 k(s,r)b(r)g(u(r))drds \\ &\geq \lambda \frac{\eta}{1-\alpha\eta} \int_\eta^1 (1-s)a(s)(f_\infty - \epsilon)H_2 ds \\ &\geq \lambda \frac{\gamma\eta}{1-\alpha\eta} \int_\eta^1 (1-s)a(s)(f_\infty - \epsilon)H_2 ds \\ &\geq H_2 \\ &= \|u\|. \end{aligned}$$

Hence, $\|Tu\| \geq \|u\|$. So, if we set

$$\Omega_2 = \{x \in \mathcal{B} \mid \|x\| < H_2\},$$

then

$$\|Tu\| \geq \|u\|, \text{ for } u \in \mathcal{P} \cap \partial\Omega_2. \tag{3.4}$$

Applying Theorem 2.3 to (3.3) and (3.4), we obtain that T has a fixed point $u \in \mathcal{P} \cap (\overline{\Omega}_2 \setminus \Omega_1)$. As such, and with v defined by

$$v(t) = \lambda \int_0^1 k(t,s)b(s)g(u(s))ds,$$

the pair (u, v) is a desired solution of (1.1), (1.2) for the given λ. The proof is complete. □

Prior to our next result, we define positive numbers L_3 and L_4 by

$$L_3 := \max\left\{ \left[\frac{\gamma\eta}{1-\alpha\eta}\int_\eta^1 (1-r)a(r)f_0 dr\right]^{-1}, \left[\frac{\gamma\eta}{1-\alpha\eta}\int_\eta^1 (1-r)a(r)g_0 dr\right]^{-1} \right\},$$

and

$$L_4 := \min\left\{ \left[\frac{1}{1-\alpha\eta}\int_0^1 (1-r)a(r)f_\infty dr\right]^{-1}, \left[\frac{1}{1-\alpha\eta}\int_0^1 (1-r)b(r)g_\infty dr\right]^{-1} \right\}.$$

Theorem 3.2. *Assume conditions* (A)–(C) *are satisfied. Then, for each* λ *satisfying*

$$L_3 < \lambda < L_4, \tag{3.5}$$

there exists a pair (u, v) *satisfying* (1.1), (1.2) *such that* $u(x) > 0$ *and* $v(x) > 0$ *on* $(0, 1)$.

Proof. Let λ be as in (3.5). And let $\epsilon > 0$ be chosen such that

$$\max\left\{ \left[\frac{\gamma\eta}{1-\alpha\eta}\int_\eta^1 (1-r)a(r)(f_0-\epsilon)dr\right]^{-1}, \left[\frac{\gamma\eta}{1-\alpha\eta}\int_\eta^1 (1-r)a(r)(g_0-\epsilon)dr\right]^{-1} \right\} \leq \lambda$$

and

$$\lambda \leq \min\left\{ \left[\frac{1}{1-\alpha\eta}\int_0^1 (1-r)a(r)(f_\infty+\epsilon)dr\right]^{-1}, \right.$$

$$\left[\frac{1}{1-\alpha\eta}\int_0^1(1-r)b(r)(g_\infty+\epsilon)dr\right]^{-1}\Bigg\}.$$

Let T be the cone preserving, completely continuous operator that was defined by (3.2).

From the definitions of f_0 and g_0, there exists $H_1 > 0$ such that

$$f(x) \geq (f_0-\epsilon)x \text{ and } g(x) \geq (g_0-\epsilon)x, \quad 0 < x \leq H_1.$$

Also, from the definition of g_0 it follows that $g(0) = 0$ and so there exists $0 < H_3 < \overline{H_3}$ such that

$$\lambda g(x) \leq \frac{\overline{H_3}}{\frac{1}{1-\alpha\eta}\int_0^1(1-r)b(r)dr}, \quad 0 \leq x \leq H_3.$$

Choose $u \in \mathcal{P}$ with $\|u\| = H_3$. Then

$$\begin{aligned}
\lambda\int_0^1 k(s,r)b(r)g(u(r))dr &\leq \lambda\frac{t}{1-\alpha\eta}\int_0^1(1-r)b(r)g(u(r))dr \\
&\leq \lambda\frac{1}{1-\alpha\eta}\int_0^1(1-r)b(r)g(u(r))dr \\
&\leq \frac{\frac{1}{1-\alpha\eta}\int_0^1(1-r)b(r)\overline{H_3}dr}{\frac{1}{1-\alpha\eta}\int_0^1(1-r)b(s)ds} \\
&\leq \overline{H_3}.
\end{aligned}$$

Then, by (2.6)

$$\begin{aligned}
Tu(\eta) &\geq \lambda\frac{\eta}{1-\alpha\eta}\int_\eta^1(1-s)a(s)f\left(\lambda\frac{\eta}{1-\alpha\eta}\int_\eta^1(1-r)b(r)g(u(r))dr\right)ds \\
&\geq \lambda\frac{\eta}{1-\alpha\eta}\int_\eta^1(1-s)a(s)(f_0-\epsilon)\lambda\frac{\eta}{1-\alpha\eta}\int_\eta^1(1-r)b(r)g(u(r))drds \\
&\geq \lambda\frac{\eta}{1-\alpha\eta}\int_\eta^1(1-s)a(s)(f_0-\epsilon)\lambda\frac{\gamma\eta}{1-\alpha\eta}\int_\eta^1(1-r)b(r)(g_0-\epsilon)\|u\|drds \\
&\geq \lambda\frac{\eta}{1-\alpha\eta}\int_\eta^1(1-s)a(s)(f_0-\epsilon)\|u\|ds \\
&\geq \lambda\frac{\gamma\eta}{1-\alpha\eta}\int_\eta^1(1-s)a(s)(f_0-\epsilon)\|u\|ds \\
&\geq \|u\|.
\end{aligned}$$

So, $\|Tu\| \geq \|u\|$. If we put

$$\Omega_3 = \{x \in \mathcal{B} \mid \|x\| < H_3\},$$

then

$$\|Tu\| \geq \|u\|, \text{ for } u \in \mathcal{P}\cap\partial\Omega_3. \tag{3.6}$$

Next, by definitions of f_∞ and g_∞, there exists $\overline{H}_1$ such that

$$f(x) \leq (f_\infty+\epsilon)x \text{ and } g(x) \leq (g_\infty+\epsilon)x, \quad x \geq \overline{H}_4.$$

Clearly, since g_∞ is assumed to be a positive real number, it follows that g is unbounded at ∞, and so, there exists $\widetilde{H}_4 > \max\{2H_3, \overline{H}_4\}$ such that $g(x) \leq g(\widetilde{H}_4)$, for $0 < x \leq \widetilde{H}_4$.

Set

$$f^*(t) = \sup_{0\le s\le t} f(s), \quad g^*(t) = \sup_{0\le s\le t} g(s), \quad \text{for} \quad t \ge 0.$$

Clearly f^* anf g^* are nodecreasing real valued function for which it holds

$$\lim_{x\to\infty} \frac{f^*(x)}{x} = f_\infty, \quad \lim_{x\to\infty} \frac{g^*(x)}{x} = g_\infty.$$

Hence, there exists H_4 such that $f^*(x) \le f^*(H_4)$, $g^*(x) \le g^*(H_4)$ for $0 < x \le H_4$. Choosing $u \in \mathcal{P}$ with $\|u\| = H_4$, we have

$$\begin{aligned}
Tu(t) &\le \lambda \frac{1}{1-\alpha\eta} \int_0^1 (1-s)a(s) f\left(\lambda \frac{1}{1-\alpha\eta} \int_0^1 (1-r)b(r)g(u(r))dr\right) ds \\
&\le \lambda \frac{1}{1-\alpha\eta} \int_0^1 (1-s)a(s) f^*\left(\lambda \frac{1}{1-\alpha\eta} \int_0^1 (1-r)b(r)g(u(r))dr\right) ds \\
&\le \lambda \frac{1}{1-\alpha\eta} \int_0^1 (1-s)a(s) f^*\left(\lambda \frac{1}{1-\alpha\eta} \int_0^1 (1-r)b(r)g^*(u(r))dr\right) ds \\
&\le \lambda \frac{1}{1-\alpha\eta} \int_0^1 (1-s)a(s) f^*\left(\lambda \frac{1}{1-\alpha\eta} \int_0^1 (1-r)b(r)g^*(H_4)dr\right) ds \\
&\le \lambda \frac{1}{1-\alpha\eta} \int_0^1 (1-s)a(s) f^*\left(\lambda \frac{1}{1-\alpha\eta} \int_0^1 (1-r)b(r)(g_\infty+\epsilon)H_4 dr\right) ds \\
&\le \lambda \frac{1}{1-\alpha\eta} \int_0^1 (1-s)a(s) f^*(H_4) ds \\
&\le \lambda \frac{1}{1-\alpha\eta} \int_0^1 (1-s)a(s) ds (f_\infty+\epsilon)H_4 \\
&\le H_4 \\
&= \|u\|,
\end{aligned}$$

and so $\|Tu\| \le \|u\|$. For this case, if we let

$$\Omega_4 = \{x \in \mathcal{B} \mid \|x\| < H_4\},$$

then

$$\|Tu\| \le \|u\|, \text{ for } u \in \mathcal{P} \cap \partial\Omega_4. \tag{3.7}$$

Application of part (ii) of Theorem 2.3 yields a fixed point u of T belonging to $\mathcal{P} \cap (\overline{\Omega}_4 \setminus \Omega_3)$, which in turn yields a pair (u, v) satisfying (1.1), (1.2) for the chosen value of λ. The proof is complete. □

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