CERTAIN INEQUALITIES FOR $p$-VALENT MEROMORPHIC FUNCTIONS WITH ALTERNATING COEFFICIENTS BASED ON INTEGRAL OPERATOR

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ABSTRACT. In this paper we introduce the class $\sigma_p^*(\beta)$ of functions $f(z) = Az^{-p} + \sum_{n=p}^{\infty} (-1)^{n-1}a_n z^n$ regular and multivalent in the $\Delta^* = \{z : 0 < |z| < 1\}$ and satisfying

$$Re \left\{ \frac{z[\mathcal{J}(f(z))]'}{\mathcal{J}(f(z))} \right\} < -\beta$$

where $\mathcal{J}$ is a linear operator.

Coefficient inequalities, distortion bounds, weighted mean and arithmetic mean of functions for this class have been obtained.

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1. Introduction

Let $\Sigma_p$ be the class of functions of the form

\[ f(z) = Az^{-p} + \sum_{n=p}^{\infty} a_n z^n, \quad A \geq 0 \]

that are regular in the punctured disk $\Delta^* = \{ z : 0 < |z| < 1 \}$ and $\sigma_p$ be the subclass of $\Sigma_p$ consisting of functions with alternating coefficients of the type

\[ f(z) = Az^{-p} + \sum_{n=p}^{\infty} (-1)^{n-1} a_n z^n, \quad a_n \geq 0, \quad A \geq 0. \]

Let

\[ \Sigma_p^*(\beta) = \left\{ f \in \Sigma_p : Re \left( \frac{z[J(f(z))']}{J(f(z))} \right) < -\beta, 0 \leq \beta < p \right\} \]

and let $\sigma_p^*(\beta) = \Sigma_p^*(\beta) \cap \sigma_p$ where

\[ J(f(z)) = (\gamma - p + 1) \int_0^1 (u^\gamma) f(uz) du, \quad p < \gamma \]

is a linear operator.

With a simple calculation we obtain

\[ J(f(z)) = \begin{cases} 
Az^{-p} + \sum_{n=p}^{\infty} (-1)^{n-1} \left( \frac{2-p+1}{\gamma + n + 1} \right) a_n z^n & f(z) \in \sigma_p, \\
Az^{-p} + \sum_{n=p}^{\infty} \left( \frac{2-p+1}{\gamma + n + 1} \right) a_n z^n & f(z) \in \Sigma_p.
\end{cases} \]

For more details about meromorphic $p$-valent functions, we can see the recent works of many authors in [1, 2, 3].

Also Urallegadi and Ganigi [4] worked on meromorphic univalent functions with alternating coefficients.

2. Coefficient Estimates

**Theorem 2.1.** Let $f(z) = Az^{-p} + \sum_{n=p}^{\infty} a_n z^n \in \Sigma_p$. If

\[ \sum_{n=p}^{\infty} (n + \beta) \left( \frac{\gamma - p + 1}{\gamma + n + 1} \right) |a_n| \leq A(p - \beta) \]

then $f(z) \in \Sigma_p^*(\beta)$.

**Proof.** It is enough to show that

\[ M = \frac{z[J(f(z))']}{|J(f(z))|} + p < 1 \text{ for } |z| < 1. \]

But by [1, 5]

\[ M = \left| \frac{-pAz^{-p} + \sum_{n=p}^{\infty} n \left( \frac{2-p+1}{\gamma + n + 1} \right) a_n z^n + pAz^{-p} + \sum_{n=p}^{\infty} p \left( \frac{2-p+1}{\gamma + n + 1} \right) a_n z^n}{-pAz^{-p} + \sum_{n=p}^{\infty} n \left( \frac{2-p+1}{\gamma + n + 1} \right) a_n z^n - (p - 2\beta)Az^{-p} - \sum_{n=p}^{\infty} (p - 2\beta) \left( \frac{2-p+1}{\gamma + n + 1} \right) a_n z^n} \right| \]
in (2.3) and letting \( z \)

The last expression is less than or equal to 1 provided

\[
\sum_{n=p}^{\infty} \left( n + p \right) \left( \frac{\gamma - p + 1}{\gamma + n + 1} \right) |a_n| \leq 2A(p - \beta) - \sum_{n=p}^{\infty} (n - p + 2\beta) \left( \frac{\gamma - p + 1}{\gamma + n + 1} \right) |a_n|
\]

which is equivalent to

\[
\sum_{n=p}^{\infty} (n + \beta) \left( \frac{\gamma - p + 1}{\gamma + n + 1} \right) |a_n| \leq A(p - \beta)
\]

which is true by (2.1) so the proof is complete.  

The converse of the Theorem 2.1 is also true for functions in \( \sigma^*_p(\beta) \), where \( p \) is an odd number.

**Theorem 2.2.** A function \( f(z) \) in \( \sigma_p \) is in \( \sigma^*_p(\beta) \) if and only if

\[
\sum_{n=p}^{\infty} (n + \beta) \left( \frac{\gamma - p + 1}{\gamma + n + 1} \right) a_n \leq A(p - \beta).
\]

**Proof.** According to Theorem 2.1, it is enough to prove the “only if” part. Suppose that

\[
\Re \left( \frac{z(Jf(z))^\gamma}{(Jf(z))} \right) = \Re \left( \frac{-Apz^{-p} + \sum_{n=p}^{\infty} \left( n(-1)^{n-1} \left( \frac{\gamma - p + 1}{\gamma + n + 1} \right) a_n z^n \right)}{Az^{-p} + \sum_{n=p}^{\infty} \left( n(-1)^{n-1} \left( \frac{\gamma - p + 1}{\gamma + n + 1} \right) a_n z^n \right)} \right) < -\beta.
\]

By choosing values of \( z \) on the real axis so that \( (z(Jf(z))^\gamma) / (Jf(z)) \) is real and clearing the denominator in (2.3) and letting \( z \rightarrow -1 \) through real values we obtain

\[
Ap - \sum_{n=p}^{\infty} n \left( \frac{\gamma - p + 1}{\gamma + n + 1} \right) a_n \geq \beta \left( A + \sum_{n=p}^{\infty} \left( \frac{\gamma - p + 1}{\gamma + n + 1} \right) a_n \right)
\]

which is equivalent to

\[
\sum_{n=p}^{\infty} (n + \beta) \left( \frac{\gamma - p + 1}{\gamma + n + 1} \right) a_n \leq A(p - \beta).
\]

**Corollary 2.3.** If \( f(z) \in \sigma^*_p(\beta) \) then

\[
a_n \leq \frac{A(p - \beta)(\gamma + n + 1)}{(n + \beta)(\gamma - p + 1)} \text{ for } n = p, p + 1, \ldots.
\]

The result is sharp for the functions of the type

\[
f_n(z) = Az^{-p} + (-1)^{n-1} \frac{A(p - \beta)(\gamma + n + 1)}{(n + \beta)(\gamma - p + 1)} z^n.
\]
3. Distortion Bounds and Important Properties of $\sigma^*_p(\beta)$

In this section we obtain distortion bounds for functions in the class $\sigma^*_p(\beta)$ and prove some important properties of this class, where $p$ is an odd number.

**Theorem 3.1.** Let $f(z) = Az^{-p} + \sum_{n=p}^{\infty} (-1)^{n-1}a_n z^n$, $a_n \geq 0$ be in the class $\sigma^*_p(\beta)$ and $\beta \geq \gamma + 1$ then

\[
A_r^{-p} - \frac{A(p - \beta)}{\gamma - p + 1} r^p \leq |f(z)| \leq A_r^{-p} + \frac{A(p - \beta)}{\gamma - p + 1} r^p.
\]

**Proof.** Since $\beta \geq \gamma + 1$ so $\frac{n + \beta}{\gamma + n + 1} \geq 1$ then

\[
(\gamma - p + 1) \sum_{n=p}^{\infty} a_n \leq \sum_{n=p}^{\infty} \left( \frac{n + \beta}{\gamma + n + 1} \right) (\gamma - p + 1)a_n \leq A(p - \beta).
\]

We have

\[
|f(z)| = |Az^{-p} + \sum_{n=p}^{\infty} (-1)^{n-1}a_n z^n| \\
\leq \frac{A}{r^p} + r^p \sum_{n=p}^{\infty} a_n \leq \frac{A}{r^p} + r^p \left( A(p - \beta) \frac{1}{\gamma - p + 1} \right).
\]

Similarly,

\[
|f(z)| \geq \frac{A}{r^p} - \sum_{n=p}^{\infty} a_n r^n \geq \frac{A}{r^p} - r^p \sum_{n=p}^{\infty} a_n \geq \frac{A}{r^p} - \frac{A(p - \beta)}{\gamma - p + 1} r^p.
\]

**Theorem 3.2.** Let

\[
f(z) = Az^{-p} + \sum_{n=p}^{\infty} a_n z^n \quad \text{and} \quad g(z) = Az^{-p} + \sum_{n=p}^{\infty} (-1)^{n-1}b_n z^n
\]

be in the class $\sigma^*_p(\beta)$ then the weighted mean of $f$ and $g$ defined by

\[
W_j(z) = \frac{1}{2} [(1 - j)f(z) + (1 + j)g(z)]
\]

is also in the same class.

**Proof.** Since $f$ and $g$ belong to $\sigma^*_p(\beta)$ so by (2.2) we have

\[
\begin{cases}
\sum_{n=p}^{\infty} (n + \beta) \left( \frac{\gamma - p + 1}{\gamma + n + 1} \right) a_n \leq A(p - \beta), \\
\sum_{n=p}^{\infty} (n + \beta) \left( \frac{\gamma - p + 1}{\gamma + n + 1} \right) b_n \leq A(p - \beta).
\end{cases}
\]

After a simple calculation we obtain

\[
W_j(z) = Az^{-p} + \sum_{n=p}^{\infty} \left[ \frac{1 - j}{2} a_n + \frac{1 + j}{2} b_n \right] (-1)^{n-1} z^n.
\]
But
\[
\sum_{n=p}^{\infty} (n + \beta) \left( \frac{\gamma - p + 1}{\gamma + n + 1} \right) \left[ \frac{1 - j}{2} a_n + \frac{1 + j}{2} b_n \right]
\]
\[
= \left( \frac{1 - j}{2} \right) \sum_{n=p}^{\infty} (n + \beta) \left( \frac{\gamma - p + 1}{\gamma + n + 1} \right) a_n + \left( \frac{1 + j}{2} \right) \sum_{n=p}^{\infty} (n + \beta) \left( \frac{\gamma - p + 1}{\gamma + n + 1} \right) b_n
\]
\[
\leq \left( \frac{1 - j}{2} \right) A(p - \beta) + \left( \frac{1 + j}{2} \right) A(p - \beta) = A(p - \beta).
\]

Hence by Theorem 2.2, \( W_j(z) \in \sigma_p^*(\beta). \)

**Theorem 3.3.** Let
\[
f_k(z) = Az^{-p} + \sum_{n=p}^{\infty} (-1)^{n-1} a_{n,k} z^n \in \sigma_p^*(\beta), \quad k = 1, 2, \ldots, m
\]
then the arithmetic mean of \( f_k(z) \) defined by
\[
F(z) = \frac{1}{m} \sum_{k=1}^{m} f_k(z)
\]
is also in the same class.

**Proof.** Since \( f_k(z) \in \sigma_p^*(\beta) \) so by (2.2) we have
\[
\sum_{n=p}^{\infty} (n + \beta) \left( \frac{\gamma - p + 1}{\gamma + n + 1} \right) a_{n,k} \leq A(p - \beta) \quad (k = 1, 2, \ldots, m).
\]

After a simple calculation we obtain
\[
F(z) = \frac{1}{m} \sum_{k=1}^{m} \left( Az^{-p} + \sum_{n=p}^{\infty} (-1)^{n-1} a_{n,k} z^n \right)
\]
\[
= Az^{-p} + \sum_{n=p}^{\infty} (-1)^{n-1} \left( \frac{1}{m} \sum_{k=1}^{m} a_{n,k} \right) z^n.
\]

But
\[
\sum_{n=p}^{\infty} (n + \beta) \left( \frac{\gamma - p + 1}{\gamma + n + 1} \right) \left( \frac{1}{m} \sum_{k=1}^{m} a_{n,k} \right) \leq \frac{1}{m} \sum_{k=1}^{m} A(p - \beta) = A(p - \beta)
\]
which in view of Theorem 2.2 yields the proof of Theorem 3.3.

**REFERENCES**


