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CONSTRUCTION OF LYAPUNOV FUNCTIONALS IN FUNCTIONAL DIFFERENTIAL EQUATIONS WITH APPLICATIONS TO EXPONENTIAL STABILITY IN VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS

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ABSTRACT. Non-negative definite Lyapunov functionals are employed to obtain sufficient conditions that guarantee the exponential asymptotic stability and uniform exponential asymptotic stability of the zero solution of nonlinear functional differential systems. The theory is applied to Volterra integro-differential equations in the form of proposition examples.

Key words and phrases: Non-negative definite, Lyapunov functionals, Exponential asymptotic stability, Uniform exponential asymptotic stability, Volterra integro-differential equations.

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1. INTRODUCTION

Consider the Volterra integro-differential equation

$$x'(t) = Ax(t) + \int_0^t B(t,s)f(x(s))ds, \tag{1.1}$$

where $x : [0,t] \to D$ and D be a set in $\mathbb{R}^n$ that includes the origin. Suppose there exist a continuously differentiable Lyapunov functional $V : \mathbb{R}^+ \times D \to \mathbb{R}^+$, where $\mathbb{R}^+$ is the set of non-negative real numbers, that satisfies

$$W_1(|x|) \le V(t, x(\cdot)) \le W_2(|x|) + \int_0^t \varphi_1(t,s)W_3(|x(s)|)ds \tag{1.2}$$

and

$$V'(t, x(\cdot)) \le -\eta(t)V(t, x(\cdot)) + F(t) \tag{1.3}$$

Here the function $F : [0,t] \to \mathbb{R}^n$ is continuous and $W_i : [0,\infty) \to [0,\infty)$ are continuous in x with $W_i(0) = 0$, $W_i(s) > 0$ if $s > 0$ and W_i is strictly increasing. Such a function W_i is called a wedge. (In this paper wedges are always denoted by W or W_i, where i is a positive integer). The function η is continuous and non-negative. Let $t_0 \ge 0$, then for each continuous function $\phi : [0,t_0] \to \mathbb{R}^n$, there is at least one continuous function $x(t) = x(t,t_0,\phi)$ on an interval $[t_0, I]$ satisfying (1.1) for $t_0 \le t \le I$ and such that $x(t,t_0,\phi) = \phi(t)$ for $0 \le t \le t_0$. From (1.3) one obtains the variational of parameters formula

$$V(t, x(\cdot)) \le \Big[V(t_0,\phi) + \int_{t_0}^t |F(s)|exp\ (\int_{t_0}^s \eta(u)du)ds\Big]exp\ (-\int_{t_0}^t \eta(s)ds). \tag{1.4}$$

Now, if $W_1 = ||x||^p$, for some positive constant p, where $||\cdot||$ is the Euclidean norm, then by (1.2) and (1.4) we arrive at

$$||x|| \le \Big[V(t_0,\phi) + \int_{t_0}^t |F(s)|exp\ (\int_{t_0}^s \eta(u)du)ds\Big]^{1/p} exp\ \frac{1}{p}(-\int_{t_0}^t \eta(s)ds). \tag{1.5}$$

Thus, if

$$\int_{t_0}^t |F(s)|exp\ (\int_{t_0}^s \eta(u)du)ds \le K,$$

for some positive constant K, then (1.5) yields that the zero solution of (1.1) is exponentially asymptotically stable, provided that $\int_{t_0}^t \eta(s)ds \to \infty$, as $t \to \infty$. The variational of parameters formula (1.4) was easily obtained from (1.3). However, finding a Lyapunov functional V such that (1.3) is satisfied is extremely difficult. The purpose of this paper is to present a systematic approach to the construction of such a Lyapunov functional.

2. EXPONENTIAL ASYMPTOTIC STABILITY

In this paper we present six theorems and two propositions that provide an easy way of constructing Lyapunov functionals that meet condition (1.3), which in returns the exponential asymptotic stability of functional differential equations can be deduced. We make use of non-negative definite Lyapunov functionals and obtain sufficient conditions that guarantee the exponential asymptotic stability of the zero solution of the system of functional differential equations

$$x'(t) = G(t, x(s);\ 0 \le s \le t) \stackrel{def}{=} G(t, x(\cdot)) \tag{2.1}$$

where $x \in \mathbb{R}^n$, $G : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ is a given nonlinear continuous function in t and x with $G(t,0) = 0$. Let $t_0 \ge 0$, then for each continuous function $\phi : [0,t_0] \to \mathbb{R}^n$, there is at least one

continuous function $x(t) = x(t, t_0, \phi)$ on an interval $[t_0, I]$ satisfying (2.1) for $t_0 \leq t \leq I$ and such that $x(t, t_0, \phi) = \phi(t)$ for $0 \leq t \leq t_0$. It is assumed that at $t = t_0$, $x'(t)$ is the right hand derivative of $x(t)$. For conditions ensuring existence, uniqueness and continuability of solutions of (2.1) we refer the reader to [6] and [12] .
In [11], the author studied the boundedness of solutions of systems of differential equations. On the other hand, the author, in [12] studied the boundedness of solutions of (2.1) by making use of non-negative definite Lyapunov functionals.
A stereotype of equation (2.1) is equation (1.1). We apply our results to Volterra integro-differential equations of the form (1.1) with $f(x) = x^n$, where n is positive and rational. At the end of the paper we will compare our theorems to those obtained in [13] and show that our results are different when it comes to applications. For more on the boundedness and stability of solutions of (1.1), we refer the interested reader to [4], [5], [7], [8], [14], [9] and [10].

From this point forward, if a function is written without its argument, then the argument is assumed to be t. Let $\phi : [0, t_0] \to \mathbb{R}^n$ be continuous, we define $|\phi| = \sup\{||\phi(s)|| : 0 \leq s \leq t_0\}$. Next, we state the following definition.

Definition 1. *We say that the zero solution of system* (2.1) *is exponentially asymptotically stable if for a positive constant M, any solution $x(t, t_0, \phi)$ of* (2.1) *satisfies*

$$||x(t, t_0, \phi)|| \leq C\Big(|\phi|, t_0\Big)e^{-M(t-t_0)}, \quad \textit{for all } t \geq t_0,$$

where $C\Big(|\phi|, t_0\Big)$ is a constant that depends on $|\phi|$ and t_0 and ϕ is a given continuous and bounded initial function. We say that solutions of system (2.1) *are uniformly exponentially asymptotically stable if C is independent of t_0.*

If $x(t)$ is any solution of system (2.1), then for a continuously differentiable function

$$V : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^+,$$

we define the derivative V' of V by

$$V'(t, x(\cdot)) = \frac{\partial V(t, x(\cdot))}{\partial t} + \sum_{i=1}^{n} \frac{\partial V(t, x(\cdot))}{\partial x_i} G_i(t, x).$$

Theorem 2.1. *Let D be a set in $\mathbb{R}^n$ containing the origin. Suppose there exist a continuously differentiable Lyapunov functional $V : \mathbb{R}^+ \times D \to \mathbb{R}^+$ that satisfies*

$$\lambda_1||x||^p \leq V(t, x(\cdot)) \leq \lambda_2 W_2(|x|) + \lambda_2 \int_0^t \varphi_1(t, s) W_3(|x(s)|) ds \tag{2.2}$$

and

$$V'(t, x(\cdot)) \leq -\lambda_3 W_4(|x|) - \lambda_3 \int_0^t \varphi_2(t, s) W_5(|x(s)|) ds + Le^{-\delta t} \tag{2.3}$$

for some positive constants $p, \lambda_1, \lambda_2, \lambda_3, \delta$ and L. The functions $\varphi_i(t, s) \geq 0$ are scalar-valued and continuous for $0 \leq s \leq t < \infty, i = 1, 2$. If the inequality

$$W_2(|x|) - W_4(|x|) + \int_0^t \Big(\varphi_1(t, s) W_3(|x(s)|) - \varphi_2(t, s) W_5(|x(s)|)\Big) ds \leq \gamma e^{-\delta t} \tag{2.4}$$

holds for some positive γ and $\int_0^t \varphi_1(t, s) ds \leq B$ for some positive constant B, then the zero solution of (2.1) *is uniformly exponentially asymptotically stable.*

Proof. Let $0 < M = \lambda_3/\lambda_2 < \delta$. For any initial time $t_0 \geq 0$, let $x(t)$ be any solution of (2.1) with $x(t) = \phi(t)$, for $0 \leq t \leq t_0$. Then,

$$\frac{d}{dt}\Big(V(t, x(\cdot))e^{M(t-t_0)}\Big) = \Big[V'(t, x(\cdot)) + MV(t, x(\cdot))\Big]e^{M(t-t_0)}.$$

For $x(t) \in \mathbb{R}^n$, using (2.2) we get

$$\begin{aligned}
\frac{d}{dt}\Big(V(t, x(\cdot))e^{M(t-t_0)}\Big) &\leq \Big[-\lambda_3 W_4(|x|) - \lambda_3 \int_0^t \varphi_2(t,s)W_5(|x(s)|)ds + Le^{-\delta t} \\
&+ M\lambda_2 W_2(|x|) + M\lambda_2 \int_0^t \varphi_1(t,s)W_3(|x(s)|)ds\Big]e^{M(t-t_0)} \\
&= \lambda_3\Big[W_2(|x|) - W_4(|x|) + Le^{-\delta t} \\
&+ \int_0^t \Big(\varphi_1(t,s)W_3(|x(s)|) - \varphi_2(t,s)W_5(|x(s)|)\Big)ds\Big]e^{M(t-t_0)} \\
&\leq (\lambda_3\gamma + L)e^{-\delta t}e^{M(t-t_0)} \\
&\leq (\lambda_3\gamma + L)e^{-\delta(t-t_0)}e^{M(t-t_0)} \\
&= (\lambda_3\gamma + L)e^{(M-\delta)(t-t_0)}. \qquad (2.5)
\end{aligned}$$

Integrating (2.5) from t_0 to t we obtain,

$$\begin{aligned}
V(t, x(\cdot))e^{M(t-t_0)} &\leq V(t_0, \phi) + \frac{\lambda_3\gamma + L}{M-\delta}e^{(M-\delta)(t-t_0)} - \frac{\lambda_3\gamma + L}{M-\delta} \\
&\leq V(t_0, \phi) + \frac{\lambda_3\gamma + L}{\delta - M}.
\end{aligned}$$

Consequently,

$$V(t, x(\cdot)) \leq \Big(V(t_0, \phi) + \frac{\lambda_3\gamma + L}{\delta - M}\Big)e^{-M(t-t_0)}.$$

From condition (2.2) we have $\lambda_1||x||^p \leq V(t, x(\cdot))$, which implies that

$$\begin{aligned}
||x|| \leq \{\frac{1}{\lambda_1}\}^{1/p} \Big(\lambda_2 W_2(|\phi|) \\
+\lambda_2 W_3(|\phi|)\int_0^{t_0} \varphi_1(t_0, s)ds + \frac{\lambda_3\gamma + L}{\delta - M}\Big)^{\frac{1}{p}} e^{-\frac{M}{p}(t-t_0)}, \text{ for all } t \geq t_0.
\end{aligned}$$

∎

Remark 2.1. Condition (2.4) can be easily satisfied if $W_2 = W_4$, $W_3 = W_5$ and with the appropriate growth condition on the functions φ_1 and φ_2, as the next proposition shows.

Proposition 2.2. *For $1 < \delta$ and a bounded continuous given initial function ϕ, consider the scalar nonlinear Volterra integro-differential equation*

$$x' = \sigma(t)x(t) + e^{-\delta t}\int_0^t B(t,s)x^{2/3}(s)ds, \quad t \geq 0, \qquad (2.6)$$

with $x(t) = \phi(t)$ for $0 \leq t \leq t_0$. If

$$2\sigma(t) + e^{-\delta t}\int_0^t |B(t,s)|ds + \int_t^\infty e^{-\delta u}|B(u,t)|du \leq -1,$$

$$\int_0^t \int_t^\infty e^{-\delta u}|B(u,s)|duds,\ \int_0^t |B(t,s)|ds < \infty,$$

and

$$\frac{e^{-\delta t}|B(t,s)|}{3} \geq \int_t^\infty e^{-\delta u}|B(u,s)|du,$$

then the zero solution of (2.6) *is uniformly exponentially asymptotically stable.*

Proof. To see this we let

$$V(t,x(\cdot)) = x^2 + \int_0^t \int_t^\infty e^{-\delta u}|B(u,s)|dux^2(s)ds.$$

Then along solutions of (2.5) we have

$$\begin{aligned} V'(t,x(\cdot)) &= 2xx' + \int_t^\infty e^{-\delta u}|B(u,t)|x^2(t)du - \int_0^t e^{-\delta t}|B(t,s)|x^2(s)ds \\ &\leq 2\sigma(t)x^2 + 2e^{-\delta t}\int_0^t |B(t,s)|\ |x(t)|x^{2/3}(s)ds \\ &\quad + \int_t^\infty e^{-\delta u}|B(u,t)|x^2(t)du - \int_0^t e^{-\delta t}|B(t,s)|x^2(s)ds. \end{aligned}$$

Using the fact that $ab \leq a^2/2 + b^2/2$, the above inequality simplifies to

$$\begin{aligned} V'(t,x(\cdot)) &\leq 2\sigma(t)x^2 + e^{-\delta t}\int_0^t |B(t,s)|(x^2(t) + x^{4/3}(s))ds \\ &+ \int_t^\infty e^{-\delta u}|B(u,t)|x^2(t)du - \int_0^t e^{-\delta t}|B(t,s)|x^2(s)ds. \end{aligned}$$

(2.7)

To further simplify (2.7) we make use of Young's inequality, which says for any two nonnegative real numbers w and z, we have

$$wz \leq \frac{w^e}{e} + \frac{z^f}{f},\ \text{ with } 1/e + 1/f = 1.$$

Thus, for $e = 3/2$ and $f = 3$, we get

$$\begin{aligned} \int_0^t |B(t,s)|x^{4/3}(s)ds &= \int_0^t |B(t,s)|^{1/3}|B(t,s)|^{2/3}x^{4/3}(s)ds \\ &\leq \int_0^t \Big(\frac{|B(t,s)|}{3} + \frac{2}{3}|B(t,s)|x^2(s)\Big)ds \end{aligned}$$

By substituting the above inequality into (2.6), we arrive at

$$\begin{aligned} V'(t,x(\cdot)) &\leq \Big(2\sigma(t) + e^{-\delta t}\int_0^t |B(t,s)|ds + \int_t^\infty e^{-\delta u}|B(u,t)|du\Big)x^2(t) \\ &\quad -e^{-\delta t}\int_0^t \Big(|B(t,s)| - \frac{2}{3}|B(t,s)|\Big)x^2(s)ds + \frac{e^{-\delta t}}{3}\int_0^t |B(t,s)|ds \\ &\leq -x^2(t) - \int_0^t \frac{e^{-\delta t}|B(t,s)|}{3}x^2(s)ds + L\, e^{-\delta t}, \end{aligned}$$

where $L = \frac{1}{3}\int_0^t |B(t,s)|ds$. By taking $W_2 = W_4 = x^2(t),\ W_3 = W_5 = x^2(s),\ \lambda_1 = \lambda_2 = \lambda_3 = 1$ and $\varphi_1(t,s) = \int_t^\infty e^{-\delta u}|B(u,s)|du,\ \varphi_2(t,s) = \frac{e^{-\delta t}|B(t,s)|}{3},$ we see that conditions (2.2) and (2.3) of Theorem 2.1 are satisfied with $M = 1$. It remains to show that condition (2.4) holds. Since $\frac{e^{-\delta t}|B(t,s)|}{3} \geq \int_t^\infty e^{-\delta u}|B(u,s)|du$ we have that

$$\begin{aligned}
& W_2(|x|) - W_4(|x|) + \int_0^t \Big(\varphi_1(t,s)W_3(|x(s)|) - \varphi_2(t,s)W_5(|x(s)|)\Big)ds \\
= \; & x^2(t) - x^2(t) + \int_0^t \Big(\int_t^\infty e^{-\delta u}|B(u,s)|du - \frac{e^{-\delta t}|B(t,s)|}{3}\Big)x^4(s)ds \\
= \; & \int_0^t \Big(\int_t^\infty e^{-\delta u}|B(u,s)|du - \frac{e^{-\delta t}|B(t,s)|}{3}\Big)x^4(s)ds \leq 0.
\end{aligned}$$

Thus, condition (2.4) is satisfied for $\gamma = 0$. By Theorem 2.1, the zero solution of (2.6) is uniformly exponentially asymptotically stable. ∎

Note that, if we take $B(t,s) = 1$, $\sigma(t) = \frac{-(1+te^{-t}+(1/\delta)e^{-\delta t})}{2}$, then the first two conditions of Proposition 2.2 are satisfied. Also, by taking $\delta = 3$, the condition

$$\frac{e^{-\delta t}|B(t,s)|}{3} \geq \int_t^\infty e^{-\delta u}|B(u,s)|du,$$

is satisfied. Thus, we have shown that that the zero solution of

$$x' = \frac{-(1+te^{-t}+(1/\delta)e^{-\delta t})}{2}x(t) + \int_0^t e^{-\delta t}x^{2/3}(s)ds, \quad t \geq 0,$$

is uniformly exponentially asymptotically stable.

In the next theorem we show that the zero solution is exponentially asymptotically stable.

Theorem 2.3. *Let D be a set in $\mathbb{R}^n$ containing the origin. Suppose that for positive constants L, p and δ, there exist a continuously differentiable Lyapunov function $V : \mathbb{R}^+ \times D \to \mathbb{R}^+$ that satisfies*

$$\lambda_1(t)||x||^p \leq V(t,x(\cdot)) \leq \lambda_2(t)W_2(|x|) + \lambda_2(t)\int_0^t \varphi_1(t,s)W_3(|x(s)|)ds \tag{2.8}$$

and

$$V'(t,x(\cdot)) \leq -\lambda_3(t)W_4(|x|) - \lambda_3(t)\int_0^t \varphi_2(t,s)W_5(|x(s)|)ds + Le^{-\delta t} \tag{2.9}$$

for some positive continuous functions $\lambda_1(t), \lambda_2(t),$ and $\lambda_3(t)$, where $\lambda_1(t)$ is nondecreasing and $\varphi_i(t,s) \geq 0$ is a scalar-valued function which is continuous for $0 \leq s \leq t < \infty, i = 1, 2,$. If the inequality

$$W_2(|x|) - W_4(|x|) + \int_0^t \Big(\varphi_1(t,s)W_3(|x(s)|) - \varphi_2(t,s)W_5(|x(s)|)\Big)ds \leq \gamma e^{-\delta t} \tag{2.10}$$

holds for some positive constant γ, $\int_0^t \varphi_1(t,s)ds \leq B$ and $\lambda_2(t) \leq N$ for some positive constants B and N for all $t \geq 0$, then the zero solution of (2.1) *is uniformly exponentially asymptotically stable.*

Proof. Let

$$M = \inf_{t\in\mathbb{R}^+} \frac{\lambda_3(t)}{\lambda_2(t)} < \delta.$$

For any initial time t_0, let $x(t)$ be any solution of (2.1) with $x(t_0) = \phi(t_0)$. Then,

$$\begin{aligned}
\frac{d}{dt}\Big(V(t,x(\cdot))e^{M(t-t_0)}\Big) &\leq \Big[-\lambda_3(t)W_4(|x|) - \lambda_3(t)\int_0^t \varphi_2(t,s)W_5(|x(s)|)ds + Le^{-\delta t} \\
&+ M\lambda_2(t)W_2(|x|) + M\lambda_2(t)\int_0^t \varphi_1(t,s)W_3(|x(s)|)ds\Big]e^{M(t-t_0)}.
\end{aligned}$$

But $\frac{\lambda_3(t)}{\lambda_2(t)} \geq M$, which implies that $-\lambda_3(t) \leq -M\lambda_2(t)$, and hence the above inequality becomes after invoking (2.11),

$$\begin{aligned}
\frac{d}{dt}\Big(V(t,x(\cdot))e^{M(t-t_0)}\Big) &\leq \Big[M\lambda_2(t)\Big(-W_4(|x|) - \int_0^t \varphi_2(t,s)W_5(|x(s)|)ds \\
&+ W_2(|x|) + \int_0^t \varphi_1(t,s)W_3(|x(s)|)ds\Big) + Le^{-\delta t}\Big]e^{M(t-t_0)} \\
&\leq (MN\gamma + L)e^{(M-\delta)(t-t_0)}.
\end{aligned}$$

An integration of the above inequality from t_0 to t yields,

$$V(t,x(\cdot)) \leq \Big(V(t_0,\phi) + \frac{MN\gamma + L}{\delta - M}\Big)e^{-M(t-t_0)}.$$

Since $\lambda_1(t)$ is nondecreasing we have for $t \geq t_0 \geq 0$ that $\lambda_1(t) \geq \lambda_1(t_0)$.Thus, by (2.9) we have $\lambda_1(t)||x||^p \leq V(t,x(\cdot))$, which implies that

$$\begin{aligned}
||x|| &\leq \{\frac{1}{\lambda_1(t_0)}\}^{1/p}\Big(\lambda_2(t_0)W_2(|\phi|) \\
&+\lambda_2(t_0)W_3(|\phi|)\int_0^{t_0}\varphi_1(t_0,s)ds + \frac{MN\gamma + L}{\delta - M}\Big)^{\frac{1}{p}}e^{-\frac{M}{p}(t-t_0)},
\end{aligned} \tag{2.11}$$

for all $t \geq t_0$. ∎

The next theorem is a special case of Theorem 2.3.

Theorem 2.4. *Suppose the hypothesis of Theorem 2.3 hold except the condition λ_1 is nondecreasing is replaced by*

there exists a positive constant $a < M$ such that $\lambda_1(t) \geq e^{-at}, \forall t \geq t_0 \geq 0,$

then the zero solution of (2.1) *is uniformly exponentially asymptotically stable.*

Proof. The proof is nearly identical to the proof of Theorem 2.3. It follows from inequality (2.11) that

$$\begin{aligned}
||x|| &\leq \{\frac{1}{\lambda_1(t)}\}^{1/p}\Big(\lambda_2(t_0)W_2(|\phi|) \\
&+\lambda_2(t_0)W_3(|\phi|)\int_0^{t_0}\varphi_1(t_0,s)ds + \frac{MN\gamma + L}{\delta - M}\Big)^{\frac{1}{p}}e^{-\frac{M}{p}(t-t_0)} \\
&\leq \Big(\lambda_2(t_0)W_2(|\phi|) \\
&+\lambda_2(t_0)W_3(|\phi|)\int_0^{t_0}\varphi_1(t_0,s)ds + \frac{MN\gamma + L}{\delta - M}\Big)^{\frac{1}{p}}e^{-\frac{(M-a)}{p}(t-t_0)}
\end{aligned} \tag{2.12}$$

for all $t \geq t_0$. ∎

Theorem 2.5. *If the condition $\lambda_2(t) \leq N, \forall t \geq 0$ for some positive constant N does not hold and $\gamma = 0$ in* (2.4) *and* (2.11) *then either Theorem 2.4 or Theorem 2.5 implies that the zero solution of* (2.1) *is exponentially asymptotically stable.*

Proof. The proof is easily deduced from either (2.11) or (2.12). To see this, inequality (2.12) with $\gamma = 0$ implies that

$$\begin{aligned} ||x|| &\leq \{\frac{1}{\lambda_1(t_0)}\}^{1/p} \Big(\lambda_2(t_0)W_2(|\phi|) \\ &\quad +\lambda_2(t_0)W_3(|\phi|)\int_0^{t_0}\varphi_1(t_0,s)ds + \frac{L}{\delta - M}\Big)^{\frac{1}{p}} e^{-\frac{M}{p}(t-t_0)}, \end{aligned}$$

for all $t \geq t_0$. The same is true if we consider (2.12). ∎

As an application of the previous Theorem, we furnish the following proposition.

Proposition 2.6. *Suppose* $1 < \delta = k_1 + k_2$ *for positive constants* k_1, k_2 *with* $k_2 < 1$. *For a given bounded continuous initial function* ϕ*, consider the scalar nonlinear Volterra integro-differential equation*

$$x' = \sigma(t)x(t) + e^{-k_1 t}\int_0^t B(t,s)x^{2/3}(s)ds, \ \ t \geq 0, \tag{2.13}$$

with $x(t) = \phi(t)$ *for* $0 \leq t \leq t_0$. *If*

$$2\sigma(t) - k_2 + \int_0^t |B(t,s)|ds + \int_t^\infty |B(u,t)|du \leq -1,$$

$$\int_0^t\int_t^\infty |B(u,s)|duds, \ \int_0^t |B(t,s)|ds < \infty,$$

and

$$\frac{|B(t,s)|}{3} \geq \int_t^\infty |B(u,s)|du$$

then the zero solution of (2.13) *is uniformly exponentially asymptotically stable.*

Proof. Let

$$V(t,x(\cdot)) = e^{-k_2 t}\Big(x^2 + \int_0^t\int_t^\infty |B(u,s)|dux^2(s)ds\Big).$$

Then along solutions of (2.13) we have after using the inequality $ab \leq a^2/2 + b^2/2$,

$$\begin{aligned} V'(t,x(\cdot)) &\leq \big(2\sigma(t) - k_2\big)x^2(t)e^{-k_2 t} - k_2\int_0^t\int_t^\infty |B(u,s)|dux^2(s)ds \\ &\quad +2e^{-(k_1+k_2)t}\int_0^t |B(t,s)||x(t)|x^{2/3}(s)ds + e^{-k_2 t}\int_t^\infty |B(u,t)|x^2(t)du \\ &\quad -e^{-k_2 t}\int_0^t |B(t,s)|x^2(s)ds \\ &\leq \big(2\sigma(t) - k_2\big)x^2(t)e^{-k_2 t} - k_2\int_0^t\int_t^\infty |B(u,s)|dux^2(s)ds \\ &\quad +e^{-(k_1+k_2)t}\int_0^t |B(t,s)|dsx^2(t) + e^{-(k_1+k_2)t}\int_0^t |B(t,s)|x^{4/3}(s)ds \\ &\quad +e^{-k_2 t}\int_t^\infty |B(u,t)|x^2(t)du - e^{-k_2 t}\int_0^t |B(t,s)|x^2(s)ds \end{aligned}$$

By using Young's inequality we arrive at

$$\begin{aligned} e^{-(k_1+k_2)t}\int_0^t |B(t,s)|x^{4/3}(s)ds &= e^{-(k_1+k_2)t}\int_0^t |B(t,s)|^{1/3}|B(t,s)|^{2/3}x^{4/3}(s)ds \\ &\le e^{-(k_1+k_2)t}\int_0^t \Big(\frac{|B(t,s)|}{3}+\frac{2}{3}|B(t,s)|x^2(s)\Big)ds \end{aligned}$$

Substituting the above inequality into the inequality satisfied by V', we arrive at

$$\begin{aligned} V'(t,x(\cdot)) &\le \Big(2\sigma(t)-k_2+\int_0^t |B(t,s)|ds+\int_t^\infty |B(u,t)|du\Big)e^{-k_2 t}x^2(t) \\ &\quad -\int_0^t \frac{|B(t,s)|}{3}x^2(s)ds \\ &\le -x^2(t)-e^{-k_2 t}\int_0^t \frac{B(t,s)|}{3}x^2(s)ds+L\,e^{-(k_1+k_2)t}, \end{aligned}$$

where $L=\frac{1}{3}\int_0^t |B(t,s)|ds$. By taking $W_2=W_4=x^2(t),\ W_3=W_5=x^2(s),\ \lambda_1(t)=\lambda_2(t)=\lambda_3(t)=e^{-k_2 t}$ and $\varphi_1(t,s)=\int_t^\infty |B(u,s)|du,\ \varphi_2(t,s)=\frac{|B(t,s)|}{3},$ we see that conditions (2.8), (2.9) (2.10) of 2.4 are satisfied with $M=1$ and $\gamma=0$. If we take $k_2<M$, then the hypothesis of Theorem 2.3 is satisfied with $a=k_2$ and hence the zero solution of (2.13) is uniformly exponentially asymptotically stable. ∎

Theorem 2.7. *Assume $D\subset\mathbb{R}^n$ contains the origin and there exists a type I Lyapunov function $V: D\to[0,\infty)$ such that for all $(t,x)\in[0,\infty)\times D$:*

(2.14) $$\lambda_1\|x\|^p\le V(x),$$

(2.15) $$\dot V(t,x)\le -\lambda_3 V(x)+Le^{-\delta t};$$

where $\lambda_1,\lambda_3,p,\delta>0, L\ge 0$ are constants and $0<\varepsilon<min\{\lambda_3,\delta\}$. Then the trivial solution of (2.1) *is uniformly exponentially asymptotically stable.*

Proof. For any initial time t_0, let $x(t)$ be any solution of (2.1) in D with $x(t_0)=\phi(t_0)$. Define ε such that $0<\varepsilon<min\{\lambda_3,\delta\}$. Then,

$$\begin{aligned} \frac{d}{dt}\Big(V(t,x(\cdot))e^{\varepsilon t}\Big) &= V'(t,x(t))e^{\varepsilon t}+\varepsilon V(x(t))e^{\varepsilon t}, \\ &\le \Big(-\lambda_3 V(x(t))+L^{-\delta t}+\varepsilon V(x(t))\Big)e^{\varepsilon t}, \quad \text{by ((2.15))}, \\ &= e^{\varepsilon t}[\varepsilon V(x(t))-\lambda_3 V(x(t))+Le^{-\delta t}] \\ &\le Le^{(\varepsilon-\delta)t}. \end{aligned}$$

Integrating both sides of the above inequality from t_0 to t we obtain

$$\begin{aligned} V(x(t))e^{\varepsilon t} &\le V(\phi)e^{\varepsilon t_0}+\frac{L}{\varepsilon-\delta}e^{(\varepsilon-\delta)t}-\frac{L}{\varepsilon-\delta}e^{(\varepsilon-\delta)t_0} \\ &\le V(t_0,\phi)e^{\varepsilon t_0}+\frac{L}{\delta-\varepsilon}e^{(\varepsilon-\delta)t_0} \\ &\le \Big(V(t_0,\phi)+\frac{L}{\delta-\varepsilon}\Big)e^{\varepsilon t_0}. \end{aligned}$$

Dividing both sides of the above inequality by $e^{\varepsilon t}$ yields

$$V(x(t)) \le \Big(V(t_0,\phi)+\frac{L}{\delta-\varepsilon}\Big)e^{-\varepsilon(t-t_0)}.$$

The proof is completed by invoking condition (2.14). ∎

Proposition 2.8. *To illustrate the application of Theorem 2.7, for a bounded continuous given initial function ϕ, we consider the scalar nonlinear Volterra integro-differential equation*

$$x'(t) = \sigma(t)\, x(t) + \int_0^t B(t,s)\, f(s,x(s))ds + g(t,x(t)) \tag{2.16}$$

with $x(t) = \phi(t)$ for $0 \le t \le t_0$, where $\sigma(t)$ is continuous for $t \ge 0$ and $B(t,s)$ is continuous for $0 \le s \le t < \infty$. We assume $f(t,x(t))$ and $g(t,x(t))$ are continuous in x and t and satisfy

$$|g(t,x(t))| \le \beta(t)\, |x(t)|^{1/2}, \tag{2.17}$$

and

$$|f(t,x(t))| \le \gamma(t)\, |x(t)|, \tag{2.18}$$

where $\gamma(t)$ and $\beta(t)$ are positive and bounded. Suppose there exist constants $k > 1$ and $\lambda_3 > 0$ such that

$$\sigma(t) + \frac{1}{2} + k\int_t^\infty |B(u,t)|du\gamma(t) \le -\lambda_3 < 0 \tag{2.19}$$

and let $k = 1+\epsilon$ for some $\epsilon > 0$ and suppose

$$|B(t,s)| \ge \lambda \int_t^\infty |B(u,s)|du \tag{2.20}$$

where $\lambda \ge \dfrac{k\lambda_3}{\epsilon} > 0,\ 0 \le s < t \le u < \infty$, and

$$\int_0^{t_0}\int_{t_0}^\infty |B(u,s)|du\gamma(s)ds \le \rho < \infty \quad \textit{for all } t_0 \ge 0. \tag{2.21}$$

Then all solutions of (2.16) *are uniformly exponentially asymptotically stable.*

Proof. Define

$$V(t,x(\cdot)) = |x(t)| + k\int_0^t \int_t^\infty |B(u,s)|du|f(s,x(s))|ds \tag{2.22}$$

Using (2.17)–(2.20), along the solutions of (2.16) we have,

$$\begin{aligned}
V'(t,x(\cdot)) &= \frac{x(t)}{|x(t)|}\, x'(t) + k\int_t^\infty |B(u,t)|du|f(t,x(t))| - k\int_0^t |B(t,s)||f(s,x(s))|ds \\
&\le \sigma(t)|x(t)| + \int_0^t |B(t,s)||f(s,x(s))|ds + |g(t,x(t))| \\
&\quad + k\int_t^\infty |B(u,t)|du|f(t,x(t))| - k\int_0^t |B(t,s)||f(s,x(s))|ds \\
&\le \big[\sigma(t) + \frac{1}{2} + k\int_t^\infty |B(u,t)|du\gamma(t)\big]|x(t)| \\
&\quad + (1-k)\int_0^t |B(t,s)||f(s,x(s))|ds + \frac{\beta^2(t)}{2}
\end{aligned}$$

$$
\begin{aligned}
&\leq \quad -\lambda_3|x(t)| - \epsilon \int_0^t |B(t,s)||f(s,x(s))|ds + \frac{\beta^2(t)}{2} \\
&\leq \quad -\lambda_3|x(t)| - \epsilon\lambda \int_0^t \int_t^\infty |B(u,s)|du|f(s,x(s))|ds + \frac{\beta^2(t)}{2} \\
&\leq \quad -\lambda_3\big[|x(t)| + k \int_0^t \int_t^\infty |B(u,s)|du|f(s,x(s))|ds\big] + \frac{\beta^2(t)}{2} \\
&= \quad -\lambda_3 V(t,x(\cdot)) + \frac{\beta^2(t)}{2}.
\end{aligned}
$$

If $\beta^2(t) \leq e^{-\delta t}$ for positive constant δ, then by taking $\lambda_1 = 1$, $p = 1$ and $L = 1/2$, the hypothesis of Theorem 2.7 is satisfied and the proof is concluded. ∎

We end this paper by discussing how our theorems fair against some of the results in the literature. In particular, Burton and Somolinos [3] considered the scalar Volterra integro-differential equation

$$x'(t) = -h(t)x(t) - b(t)x^3(t) + \int_0^t C(at-s)x(s)ds,\ t \geq 0 \tag{2.23}$$

where $h(t), b(t)$ and $C(at-s)$ are continuous on their respective domains. They obtained conditions that guaranteed the uniform asymptotic stability of the zero solution of (2.23) for $a > 1$ and only asymptotic stability for $0 < a < 1$. They had to assume that $b(t) \geq b_0 > 0, \forall t \geq 0$ for some positive constant b_0. For the sake of completeness we state one of their major theorems.

Theorem 2.9. [3] *Assume*

$$C \in L^1[0,\infty), a > 1, h(t) \geq 0, \tag{2.24}$$

$$b(t) \geq b_0 > 0, \tag{2.25}$$

$$2h(t) \geq [1 + (1/a)] \int_{(a-1)t}^\infty |C(v)|dv, \tag{2.26}$$

and

$$\int_t^\infty |C(u)|du \in L^1[0,\infty), \tag{2.27}$$

then the zero solution of (2.23) *is uniformly asymptotically stable.*

On page 5 of [3], the authors made the assertion that their results and in particular, Theorem [3] apply to functions $b(t)x^n$ when n is the quotient of odd positive integers.
Thus, for the sake of simplicity we consider the the scalar Volterra integro-differential equation

$$x'(t) = -h(t)x(t) - b(t)x^{1/3}(t) + \int_0^t C(at-s)x(s)ds,\ t \geq 0. \tag{2.28}$$

In the next theorem, by displaying a suitable Lyaponuv function and by making use of the results of Theorem 2.1, we show that the zero solution of (2.28) is uniformly exponentially asymptotically stable, where the condition (2.25) is not required. On the other hand, to arrive at our result, which gives a stronger type of stability, we will have to strengthen condition (2.26) and require that the function $b(t)$ decays exponentially. The exponential decay of $b(t)$ should be of no surprise to anyone since we are trying to make all solutions decay exponentially to zero.

Theorem 2.10. *Assume* (2.24) *and* (2.27) *hold. If for constants* $\delta, k > 1,$ *the inequalities*

$$-2h(t) + \frac{4|b(t)|}{3} + [1 + (k/a)] \int_{(a-1)t}^{\infty} |C(v)|dv \le -1, \tag{2.29}$$

$$(-1 + k) \int_0^t |C(at - s)|ds \ge (k/a) \int_{(a-1)t}^{\infty} |C(v)|dv, \tag{2.30}$$

and

$$|b(t)| \le e^{-\delta t} \tag{2.31}$$

hold, then the zero solution of (2.28) *is uniformly exponentially asymptotically stable.*

Proof. Let $x(t)$ be any solution of (2.28) with $x(t) = \phi(t)$ for $0 \le t \le t_0$, where $\phi(t)$ is a given continuous and bounded initial function.
Consider the Lyapunov functional

$$V(t, x(\cdot)) = x^2(t) + \frac{k}{a} \int_0^t \int_{at-s}^{\infty} |C(u)|dux^2(s))ds.$$

Then along the solutions of (2.28) we have

$$\begin{aligned} V'(t, x(\cdot)) \;\; &\le \;\; -2h(t)x^2(t) + 2|b(t)|x^{4/3}(t) + 2|x(t)| \int_0^t |C(at - s)||x(s)|ds \\ &\quad + \frac{k}{a} \int_{(a-1)t}^{\infty} |C(v)|dvx^2(t) - k \int_0^t |C(at - s)|x^2(s)ds. \end{aligned}$$

By noticing that

$$2|x(t)| \int_0^t |C(at - s)||x(s)|ds \le \int_0^t |C(at - s)|(x^2(t) + x^2(s)ds,$$

and using Young's inequality with $f = 3$ and $e = 3/2$ we arrive at

$$\begin{aligned} 2|b(t)|x^{4/3}(t) &= 2|b(t)|^{1/3}|b(t)|^{2/3}x^{4/3}(t) \\ &\le \frac{2|b(t)|}{3} + \frac{4|b(t)|}{3}x^2(t). \end{aligned}$$

Using these results we find that the bound for $V'(t, x(\cdot))$ reduces to

$$\begin{aligned} V'(t, x(\cdot)) \;\; &\le \;\; \Big[-2h(t) + \frac{4|b(t)|}{3} + [1 + (k/a)] \int_{(a-1)t}^{\infty} |C(v)|dv \Big] x^2(t) \\ &\quad - \;\; (-1 + k) \int_0^t |C(at - s)||x^2(s)|ds + \frac{2}{3}|b(t)| \\ &\le \;\; -x^2(t) - \int_0^t (-1 + k)|C(at - s)||x^2(s)|ds + \frac{2}{3}|b(t)|. \end{aligned}$$

It is easy to verify that all the conditions of Theorem 2.1 are satisfied for $L = \frac{4}{3}, M = 1, W_2 = W_4 = x^2(t),\ W_3 = W_5 = x^2(s),\ \lambda_1 = \lambda_2 = \lambda_3 = 1, \varphi_1(t, s) = (k/a) \int_{at-s}^{\infty} |C(u)|du,\ \varphi_2(t, s) = (-1 + k)|C(at - s)|$, and $\gamma = 0$. ∎

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