EXISTENCE OF SOLUTIONS FOR THIRD ORDER NONLINEAR BOUNDARY VALUE PROBLEMS

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ABSTRACT. In this paper, the existence of solution for a class of third order quasilinear ordinary differential equations with nonlinear boundary value problems

$$(\Phi_p(u^{''}))'=f(t,u,u',u''), \quad u(0)=A, \quad u'(0)=B, \quad R(u'(1),u''(1))=0$$

is established. The results are obtained by using upper and lower solution methods.

Key words and phrases: Quasilinear ordinary differential equations; Boundary value problem; Upper and lower solutions.

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1. Introduction

We consider the nonlinear equation
\begin{equation}
(\Phi_p(u'))' = f(t, u, u'), \quad t \in I = [0, 1]
\end{equation}
satisfying the conditions
\begin{equation}
u(0) = A, \quad u'(0) = B, \quad R(u'(1), u''(1)) = 0,
\end{equation}
where $A$ and $B$ are constants, and $\Phi_p(s) = |s|^{p-2}s, p > 1$. Equations of the above form are mathematical models occurring in studies of the $p$-Laplace equation, generalized reaction-diffusion theory \[1\], non-Newtonian fluid theory, and the turbulent flow of a gas in porous medium \[2\]. In the non-Newtonian fluid theory, the quantity $p$ is characteristic of the medium. Media with $p > 2$ are called dilatant fluids and those with $p < 2$ are called pseudopastics. If $p = 2$, they are Newtonian fluids.

For the equation
\begin{equation}
(\Phi_p(u'))' = f(t, u, u'), \quad t \in I = [0, 1]
\end{equation}
with different boundary conditions has been studied by many authors, see, for example, \[3\]-\[11\] and references therein. Our results were motivated by the paper \[11\] and references therein. Our results were motivated by the paper \[3\], \[14\]-\[17\], \[21\] which studied periodic and Neumann with nonlinear boundary conditions for Eq. (1.3). On the contrary, it seems that little is known about the result for problem (1.1)-(1.2). When $p = 2$, the related some results have been obtained by \[18\]-\[21\] for problem (1.1)-(1.2). In this paper, we obtain the existence of solutions to the problem (1.1)-(1.2), extended to results and complement to the results by \[18\]-\[21\].

2. Preliminaries

In this section, we present results for second order Volterra type integro-differential equation, which help to prove our main results.

Let us consider the following boundary value problem
\begin{equation}
(\Phi_p(u'))' = f(t, u, Tu, u')
\end{equation}
\begin{equation}
u(0) = D, \quad R(u(1), u'(1)) = 0,
\end{equation}
where $Tu(t) = \phi(t) + \int_0^t K(t, s)u(s)ds$, function $K(t, s) \in C([0, 1] \times [0, 1]), \phi(t) \in C[0, 1]$, $K(t, s) \geq 0$ on $[0, 1] \times [0, 1]$, and $D$ is a constant.

As in \[12\], we give the following definition:

**Definition 2.1.** We say that a function $\alpha(t) \in C^1[0, 1]$ is a lower solution of Eq. (2.1) if $\Phi_p(\alpha') \in C^1(0, 1)$ and satisfies
\begin{equation}
(\Phi_p(\alpha'))' \geq f(t, \alpha, T\alpha, \alpha'), \quad \text{for} \quad t \in I = [0, 1].
\end{equation}

Analogously, we say that $\beta \in C^1[0, 1]$ is a upper solution of Eq. (2.1) if $\Phi_p(\beta') \in C^1(0, 1)$ and satisfies
\begin{equation}
(\Phi_p(\beta'))' \leq f(t, \beta, T\beta, \beta'), \quad \text{for} \quad t \in I = [0, 1].
\end{equation}

Assume that $f(t, u, v, w)$ satisfies the following conditions:
\begin{enumerate}
\item[(H1)] $f(t, u, v, w)$ is nonincreasing in $v$.
\item[(H2)] $f(t, u, v, w) \in C([0, 1] \times R^3, R)$, for any positive constants $r_1, r_2 > 0$, there exist positive function $h(x) \in C[0, \infty)$ satisfying
\begin{equation}
\int_0^\infty \Phi_p^{-1}(u)/h(\Phi_p^{-1}(u))du = \infty.
\end{equation}
\end{enumerate}
and while $0 \leq t \leq 1, |u| \leq r_1, |v| \leq r_2, w \in \mathbb{R}, |f(t, u, v, w)| \leq h(|w|)$.

From [12], we give the following Lemma

**Lemma 2.1.** Let $\alpha(t)$ and $\beta(t)$ be a lower and an upper solution of Eq. (2.1), respectively, with $\alpha \leq \beta$ in $I$. Assume that hypotheses $(H_1)-(H_2)$ are satisfied. Then boundary value problem

$$(\Phi_p(u'))' = f(t, u, Tu, u'), \quad u(0) = A, \quad u(1) = B$$

has at least one solution $\alpha(t) \leq u(t) \leq \beta(t)$ for all $\alpha(0) \leq A \leq \beta(0)$, $\alpha(1) \leq B \leq \beta(1)$.

**Proof.** First, we assume that $\alpha(1) = \beta(1), \alpha(0) = \beta(0)$. Then, by $\alpha(t) \leq \beta(t)$, it follows that $\alpha'(1) = \beta'(1)$. On the other hand, it is clear that $\alpha'(1) \leq \beta'(1)$ from $R(\alpha(1), \alpha'(1)) \leq R(\beta(1), \beta'(1))$, which means $\alpha'(1) = \beta'(1)$. Then the problem

$$(\Phi_p(u'))' = f(t, u, Tu, u'), \quad u(0) = D, \quad u(1) = \alpha(1)$$

has at least one solution $\alpha(t) \leq u(t) \leq \beta(t)$, which is a solution of problem (2.3).

Next we consider that $\alpha(1) < \beta(1)$ and $\alpha(0) < \beta(0)$ (if $\alpha(1) = \beta(1), \alpha(0) < \beta(0)$ or $\alpha(1) < \beta(1), \alpha(0) = \beta(0)$ similar be proved). Applying Lemma 2.1 we know that the problem

$$(\Phi_p(u'))' = f(t, u, Tu, u'), \quad u(0) = D, \quad u(1) = \alpha(1)$$

has at least one solution $\alpha(0)(t)$, and $\alpha(t) \leq \alpha(0)(t) \leq \beta(t)$, it follows that $\alpha'(0)(1) \leq \alpha'(1)$. From the assumptions in $R$, we see that

$$R(\alpha(0)(1), \alpha'(0)(1)) \leq R(\alpha(1), \alpha'(1)) \leq 0.$$  

If $=0$ in (2.4) is true, then $\alpha(0)(t)$ is a solution of problem (2.3). Thus the proof is complete.

Otherwise, we consider the following boundary value problem:

$$(\Phi_p(u'))' = f(t, u, Tu, u'), \quad u(0) = D, \quad u(1) = \beta(1).$$

Clearly, the same reasoning gets to a solution $\beta(0)(t)$ and such that

$$\alpha(0)(t) \leq \beta(0)(t) \leq \beta(t), \quad 0 \leq t \leq 1,$$

$$R(\beta(0)(1), \beta'(0)(1)) \geq R(\beta(1), \beta'(1)) \geq 0.$$  

Consequently, if $=0$ in (2.5) is true, then the proof is completed. Otherwise we choose $d_1 = (\beta(0)(1) + \alpha(0)(1))/2$, and we consider the problem

$$(\Phi_p(u'))' = f(t, u, Tu, u'), \quad u(0) = D, \quad u(1) = d_1.$$  

Applying Lemma 2.1 we obtain a solution $u_1(t)$ from the above problem, and $\alpha_0 \leq u_1(t) \leq \beta_0$. If $R(u_1(1), u_1'(1)) = 0$, then the proof is completed. If $R(u_1(1), u_1'(1)) > 0$, then let $\alpha_1(t) = \alpha_0(t), \beta_1(t) = u_1(t)$; if $R(u_1(1), u_1'(1)) < 0$, then let $\alpha_1(t) = u_1(t), \beta_1(t) = \beta_0(t)$. Hence $\beta_1(1) - \alpha_1(1) = \frac{1}{2}(|\beta_0(1) - \alpha_0(1)|)$. By induction method, that we have obtained $\alpha_n(t), \beta_n(t)(n = 1, 2, \cdots, m)$, which satisfy

$$\alpha_{n-1}(t) \leq \alpha_n(t) \leq \beta_n(t) \leq \beta_{n-1}(t), \quad 0 \leq t \leq 1,$$
From (2.10) and (2.12), it follows that

\[ \alpha_m(t) \leq \alpha_{m+1}(t) \leq \beta_{m+1}(t) \leq \beta_m(t), \quad 0 \leq t \leq 1, \]

\[ \beta_{m+1}(1) - \alpha_{m+1}(1) = \frac{1}{2}[\beta_m(1) - \alpha_m(1)]. \]

Hence, we prove the relations (2.6) and (2.7) for every \( n \).

In view of the fact choosing \( \alpha_n(t) \) and \( \beta_n(t) \), it easy to see that

\[ R(\alpha_n(1), \alpha_n'(1)) < 0, \quad R(\beta_n(1), \beta_n'(1)) > 0. \]

From (2.8), we imply that

\[ \beta_n(1) - \alpha_n(1) = \frac{1}{2n}[\beta_0(1) - \alpha_0(1)]. \]

In addition, Nagumo condition shows that \( \{\alpha_n(t)\}, \{\beta_n(t)\}, \{\alpha_n'(t)\}, \{\beta_n'(t)\} \) are equicontinuous and uniformly bounded in \( 0 \leq t \leq 1 \). Therefore, applying the Arzela-Ascoli theorem to the sequences \( \{\alpha_n(t)\} \) and \( \{\beta_n(t)\} \), there exist two subsequences \( \{\beta_{n_j}(t)\} \) and \( \{\alpha_{n_i}(t)\} \) such that as \( j \to \infty, \beta_{n_j}(t) \to u_0(t), \beta_{n_j}'(t) \to u_0'(t) \), uniformly on \( [0,1] \), and as \( i \to \infty, \alpha_{n_i}(t) \to \bar{u}_0(t), \alpha_{n_i}'(t) \to \bar{u}_0'(t) \), uniformly on \( [0,1] \). Therefore \( u_0(t) \) and \( \bar{u}_0(t) \) satisfies (2.1), and we have from (2.8)

\[ R(u_0(1), u_0'(1)) \geq 0, \quad R(\bar{u}_0(1), \bar{u}_0'(1)) \leq 0. \]

From (2.6), it is obvious that

\[ u_0(t) \geq \bar{u}_0(t), \quad 0 \leq t \leq 1. \]

On the other hand, by (2.9), we can show that \( \bar{u}_0(1) = u_0(1) \). Thus, we have from (2.11)

\[ \bar{u}_0(1) \geq u_0'(1). \]

From (2.10) and (2.12), it follows that

\[ 0 \leq R(u_0(1), u_0'(1)) \leq R(\bar{u}_0(1), \bar{u}_0'(1)) \leq 0. \]

From (2.13), it is easy to show the following relations

\[ 0 = R(u_0(1), u_0'(1)) = R(\bar{u}_0(1), \bar{u}_0'(1)). \]

Hence, we complete the proof. \( \blacksquare \)

3. MAIN RESULTS

In this section, we discuss the existence of solutions for boundary value problem (1.1)-(1.2).

**Definition 3.1.** We say that a function \( \alpha(t) \in C^2[0,1] \) is a lower solution of Eq. (1.1) if \( \Phi_p(\alpha'') \in C^1(0,1) \) and satisfies

\[ (\Phi_p(\alpha''))' \geq f(t, \alpha, \alpha'), \quad \text{for} \quad t \in I. \]

Analogously, we say that \( \beta \in C^2[0,1] \) is a upper solution of Eq. (1.1) if \( \Phi_p(\beta'') \in C^1(0,1) \) and satisfies

\[ (\Phi_p(\beta''))' \leq f(t, \beta, \beta'), \quad \text{for} \quad t \in I. \]

We obtain the following main theorem.
Theorem 3.1. Assume that
(i) \( f(t, u, u', u'') \) is nonincreasing in \( u \) and continuous on \([0, 1] \times \mathbb{R}^3\); 
(ii) Nagumo Condition, for \((t, u, u')\) on \([0, 1] \times \mathbb{R}^2\),
\[
f(t, u, u', u'') = O(|u''|^2), \quad \text{as } |u''| \to \infty;
\]
(iii) \( R(u, v) \) is nondecreasing in \( v \) and continuous on \( \mathbb{R}^2 \);
(iv) there exists an upper solution \( \beta(t) \) and a lower solution \( \alpha(t) \) of Eq. (1.1) on \( I = [0, 1] \) such that
\[
\alpha(t) \leq \beta(t), \quad \alpha'(t) \leq \beta'(t), 0 \leq t \leq 1,
\]
\[
\alpha(0) \leq A \leq \beta(0), \quad \alpha'(0) \leq B \leq \beta'(0),
\]
\[
R(\alpha'(1), \alpha''(1)) \leq 0 \leq R(\beta'(1), \beta''(1)),
\]
then the boundary value problem (1.1)-(1.2) has a solution \( u(t) \) such that \( \alpha(t) \leq u(t) \leq \beta(t) \).

Proof. Set \( u' = z \), then we have \( u(t) = A + \int_0^t z(s) ds \). Thus, the boundary value problem (1.1)-(1.2) can be written as a boundary value problem for the second order integro-differential equation of Volterra type as below
\[
(\Phi_p(z'))' = f(t, A + \int_0^t z(s) ds, z, z'), \quad (3.1)
\]
\[
z(0) = B, \quad R(z(1), z'(1)) = 0. \quad (3.2)
\]

In order to employ Lemma 2.2, we construct the lower and upper solutions for the boundary value problem (3.1)-(3.2) by using \( \alpha(t), \beta(t) \) and hypotheses (i)-(iv). We set
\[
\bar{\alpha}(t) = \alpha(t) + \delta_1, \quad \bar{\beta}(t) = \beta(t) + \delta_2,
\]
where \( \delta_1 = A - \alpha(0), \delta_2 = \beta(0) - A \). Then, it is clear that \( \bar{\alpha}(0) = A = \bar{\beta}(0) \). Moreover, if we write
\[
\bar{\alpha}'(t) = \alpha_+(t), \quad \bar{\beta}'(t) = \beta_+(t),
\]
we have \( \alpha_+(t) \leq \beta_+(t) \) because of (3.4) and (iv).

Note that \( \bar{\alpha}(t) = A + \int_0^t \alpha_+(s) ds, \bar{\beta}(t) = A + \int_0^t \beta_+(s) ds \). Now, using (3.4)-(3.5), (iv), and the monotonicity of \( f \) from (i), we obtain
\[
(\Phi_p(\alpha_+'))' \geq f(t, A + \int_0^t \alpha_+(s) ds, \alpha_+(t), \alpha_+'(t)),
\]
\[
(\Phi_p(\beta_+'))' \leq f(t, A + \int_0^t \beta_+(s) ds, \beta_+(t), \beta_+'(t)),
\]
\[
\alpha_+(0) \leq B \leq \beta_+(0), \quad R(\alpha_+(1), \alpha_+'(1)) \leq 0 \leq R(\beta_+(1), \beta_+'(1)).
\]
Thus, we see that the functions \( \alpha_+(t) \) and \( \beta_+(t) \) are the lower and the upper solutions respectively for the boundary value problem (3.1)-(3.2). Hence, by Lemma 2.2, we have
\[
\alpha_+(t) \leq z(t) \leq \beta_+(t), \quad 0 \leq t \leq 1,
\]
where \( z(t) \) is a solution of the boundary value problem (3.1)-(3.2). Finally, from the relation \( z(t) = u'(t) \), we can recover \( u(t) = A + \int_0^t z(s) ds \).
Example 3.1. We consider the following third-order boundary value problem:

\begin{align}
(Φ_p(u'))' + (t - u)^2 - t(4 + t^2)u' - (u')^2 \sin(u') &= 0, \quad 0 < t < 1, \\
\phi(t) &= (3.7),
\end{align}


It is easily to prove that \( \alpha(t) = -t, \beta(t) = t \) are lower and upper solutions of BVP (3.6)-(3.7), respectively. \( f \) is continuous on \([0, 1] \times \mathbb{R}^3 \) and nonincreasing in \( u \) when \( \alpha(t) \leq u(t) \leq \beta(t), t \in [0, 1] \). \( R \) are continuous on \( \mathbb{R}^2 \), \( R(v, w) \) is increasing in \( w \). Furthermore, we obtain \( f \) satisfies Nagumo condition in

\[ D = \{(t, u, v, w) \in [0, 1] \times \mathbb{R}^3 : -t \leq u(t) \leq t, -1 \leq u'(t) \leq 1 \}. \]

Therefore, by Theorem 3.7 there exists at least one solution \( u(t) \) for BVP (3.6)-(3.7) such that

\[ -t \leq u(t) \leq t, \quad -1 \leq u'(t) \leq 1, \quad t \in [0, 1]. \]

REFERENCES


