The Australian Journal of Mathematical Analysis and Applications

AJMAA

Volume 4, Issue 2, Article 7, pp. 1-5, 2007

A NOTE ON INEQUALITIES DUE TO MARTINS, BENNETT AND ALZER

JÓZSEF SÁNDOR

Received 3 November, 2006; accepted 23 March, 2007; published 23 August, 2007.

BABEŞ-BOLYAI UNIVERSITY OF CLUJ, DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCES, KOGĂLNICEANU NR.1, CLUJ-NAPOCA, ROMANIA.
jjsandor@hotmail.com
jsandor@member.ams.org

ABSTRACT. A short history of certain inequalities by Martins, Bennett as well as Alzer, is provided. It is shown that, the inequality of Alzer for negative powers [6], or Martin's reverse inequality [7] are due in fact to Alzer [2]. Some related results, as well as a conjecture, are stated.

Key words and phrases: Martins' inequality, Bennett's inequality, Alzer's inequality, Inequalities for the sum of powers of the first n positive integers.

2000 *Mathematics Subject Classification.* 26D15.

ISSN (electronic): 1449-5910

1. Introduction

By investigating a question on Lorentz sequence spaces, in 1988 Martins [11] discovered certain inequalities for the sum of rth powers $(r > 0)$ of the first positive integers n. Put

$$S_r(n) := \sum_{i=1}^{n} i^r \quad (r > 0,\ n \geq 1)$$

Then one of his results states that

$$L_r(n) := \left[\frac{(n+1)S_r(n)}{nS_r(n+1)}\right]^{1/r} \leq x_n, \tag{1.1}$$

where $x_n := \sqrt[n]{n!}/\sqrt[n+1]{(n+1)!}$ $(n \geq 1)$.

In 1993 Alzer [1] established the reverse inequality

$$L_r(n) \geq y_n, \tag{1.2}$$

where $y_n := \dfrac{n}{n+1}$ $(n \geq 1)$.

Because of $\lim\limits_{r\to 0} L_r(n) = x_n$, $\lim\limits_{r\to\infty} L_r(n) = y_n$ (see e.g. [9], p.15), it follows that both bounds in 1.1 and 1.2 are best possible.

In 1992 Bennett [4] proved the inequalities

$$L_r(n) \leq y_{n+1} \text{ for } r \geq 1 \tag{1.3}$$

and

$$L_r(n) \geq y_{n+1} \text{ for } 0 < r \leq 1 \tag{1.4}$$

Since $x_n > y_{n+1}$ for all $n \geq 1$ (see e.g. [9] or [19]), and $y_{n+1} > y_n$, relations 1.3 and 1.4 are refinements of 1.1 and 1.2 for $r \geq 1$, and respectively $0 < r \leq 1$.

The proofs of 1.2, as well as 1.3-1.4 are quite involved. The author has obtained in 1995 a proof of 1.2, based on mathematical induction and Cauchy's mean value theorem of differential calculus (see [14]). The same method, based on Lagrange's mean value theorem has been applied for 1.3 and 1.4 (see [15]). Since then, many new proofs and extensions of 1.2 have been given (see e.g. [21]).

Let

$$P_r(n) := \sum_{i=1}^{n} i^{-r} \quad (n \geq 1,\ r > 0),$$

and define

$$Q_r(n) = \left[\frac{(n+1)P_r(n)}{nP_r(n+1)}\right]^{1/r}.$$

In the above mentioned paper [4], Bennett proved also the following remarkable companion of relation 1.1:

$$Q_r(n) \leq \frac{1}{y_{n+1}} \quad (n \geq 1,\ r > 0) \tag{1.5}$$

He gave also an interesting application of his results 1.3 and 1.5, by deriving a sharp lower bound for the so-called power means matrices (for details, see [3], [4]).

In 1994 Alzer [2] has improved Bennett's result 1.5 to

$$Q_r(n) \leq \frac{1}{x_n} \tag{1.6}$$

As $\frac{1}{x_n} < \frac{1}{y_{n+1}}$ (which follows also by the fact that the function $\frac{f(x)}{x+1}$ is strictly decreasing for $x \geq 1$, where $f(x) = (\Gamma(x+1))^{1/x}$, see [13, 19]), 1.6 offers indeed an improvement to 1.5.

As a corollary (stated also in [2]), from 1.2, 1.1, and 1.6 we can write the following chain of inequalities:

$$y_n \leq L_r(n) \leq x_n \leq \frac{1}{Q_r(n)}. \tag{1.7}$$

2. MAIN REMARKS

Relations 1.7 sharpens the inequality of Minc and Sathre [12]:

$$\frac{n}{n+1} < \frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}} \quad (n \geq 1) \tag{2.1}$$

See also [13, 16] for related results.

Another remark is that, the first and last terms of 1.7 are

$$y_n \leq \frac{1}{Q_n(r)} \tag{2.2}$$

Since $\frac{1}{Q_n(r)}$ is nothing else, than $L_{-r}(n)$; i.e., when "r" is replaced with "$-r$" in Alzer's inequality, 2.2 is in fact "Alzer's inequality for negative powers"! For this result, Chen and Qi [5, 6] gave in 2003 and 2004 a proof based on mathematical induction and convex functions. A proof, similar to the one of [14] is given by the author in [18]. In [20] however, this result is generalized to convex function, by a method of Ch. Kuang [10].

Alzer's classical inequality 1.2 has been rediscovered in 1998 by Dragomir and van der Hoek, too (see [8]), in the form:

$$G_r(n) := \frac{S_r(n)}{n^r} \geq \frac{(n+1)^r}{(n+1)^{r+1} - n^{r+1}}. \tag{2.3}$$

It is easy to see that, 2.3 is equivalent to 1.2, as well as to another inequality, having applications in "guessing theory" (for details, see [17]).

We note here that, in a similar manner, inequality 1.1 of Martins can be rewritten as follows:

$$\frac{S_r(n)}{(n+1)^r} \leq \frac{\sqrt[n]{n!}}{(n+1)\sqrt[n+1]{(n+1)!} - n\sqrt[n]{n!}} \tag{2.4}$$

In the recent paper [7] Chen, Qi and Dragomir have studied the reverse of Martins' inequality as follows:

$$x_n \leq \left(\frac{1}{n} \sum_{i=1}^{n} i^s \Big/ \frac{1}{n+1} \sum_{i=1}^{n+1} i^s \right)^{1/s} \tag{2.5}$$

where $s < 0$. Put $s = -r$, where $r > 0$. Then a simple computation shows that inequality 2.5 is in fact equivalent to the last inequality of 1.7 (i.e. to 1.6).

Relation 1.7 improves also the interesting inequality

$$L_r(n)Q_r(n) \leq 1, \tag{2.6}$$

or written equivalently:

$$\frac{\left(\sum_{i=1}^{n+1} i^r\right)\left(\sum_{i=1}^{n+1} i^{-r}\right)}{\left(\sum_{i=1}^{n} i^r\right)\left(\sum_{i=1}^{n} i^{-r}\right)} \geq \frac{(n+1)^{2r}}{n^{2r}} \tag{2.7}$$

Let

$$A_r(n) := \left[\frac{S_r(n)}{n}\right]^{1/r}.$$

As an application of 1.7, the following additive analogue of Martins' inequality holds true (see [2]):

$$A_r(n+1) - A_r(n) \geq z_n, \tag{2.8}$$

where $z_n = \sqrt[n+1]{(n+1)!} - \sqrt[n]{n!}$ $(n \geq 1)$ is the additive analogue of $\frac{1}{x_n}$. Since the author [13] has proved that $f(x) = \Gamma(x+1)^{1/x}$ is strictly concave for $x \geq 7$, it follows that $z_n > z_{n+1}$ for $n \geq 7$. A direct computation shows that $z_n > z_{n+1}$ for $1 \leq n \leq 6$, too. Hence (z_n) is a strictly decreasing sequence for all $n \geq 1$. (The sequence (z_n) is called also as the Traian Lalescu sequence, see [13, 18, 19]). Since it is well-known that $\lim_{n\to\infty} z_n = \frac{1}{e}$, by 2.8 we get the sharp inequality:

$$A_r(n+1) - A_r(n) > \frac{1}{e} \quad (r > 0,\ n \geq 1) \tag{2.9}$$

Finally, we mention a conjecture by Alzer (see [2]): Put

$$B_r(n) = \left(\frac{1}{n}\sum_{i=1}^{n} i^{-r}\right)^{1/r} \quad (r > 0)$$

Prove or disprove that

$$B_r(n+1) - B_r(n) < \frac{1}{\sqrt[n]{n!}} - \frac{1}{\sqrt[n+1]{(n+1)!}} \quad (r > 0,\ n \geq 1).$$

REFERENCES

[1] H. ALZER, On an inequality of H. Minc and L. Sathre, *J. Math. Anal. Appl.*, **179**(1993), pp. 396-402.

[2] H. ALZER, Refinement of an inequality of G. Bennett, *Discrete Math.*, **135**(1994), pp. 39-46.

[3] G. BENNETT, Lower bounds for matrices, *Linear Algebra Appl.*, **82**(1986), pp. 81-98.

[4] G. BENNETT, Lower bounds for matrices, II, *Canad. J. Math.*, **44** (1992), pp. 54-74.

[5] C. P. CHEN and F. QI, The inequality of Alzer for negative powers, *Octogon Math. Mag.*, **11**(2003), no. 2, pp. 442-445.

[6] C. P. CHEN and F. QI, Extension of an inequality of Alzer for negative powers, *Tamkang J. Math.*, **36**(2004), no. 1, pp. 69-72.

[7] C. P. CHEN, F. QI, S. S. DRAGOMIR, Reverse of Martins' inequality, *Australian J. Math. Anal. Appl.*, **2**(2005), no. 1, pp. 1-5.

[8] S. S. DRAGOMIR and J. VAN DER HOEK, Some new analytic inequalities and their applications in guessing theory, *J. Math. Anal. Appl.*, **225**(1998), pp. 542-556.

[9] G. H. HARDY, J. E. LITTLEWOOD and G. PÓLYA, *Inequalities*, Cambridge Univ. Press, 1952.

[10] J. CH. KUANG, Some extensions and refinements of Minc-Sathre inequality, *Math. Gaz.*, **83**(1999), pp. 123-127.

[11] J. S. MARTINS, Arithmetic and geometric means, an application to Lorenz sequence spaces, *Math. Nachr.*, **139**(1988), pp. 281-288.

[12] H. MINC and L. SATHRE, Some inequalities involving $(r!)^{1/r}$, *Edinburgh Math. Soc.*, **14**(1964-1965), pp. 41-46.

[13] J. SÁNDOR, Sur la fonction gamma, *Publ. Centre Rech. Math. Pures* (Neuchâtel), Série I, Fasc. **21**(1989), pp. 4-7.

[14] J. SÁNDOR, On an inequality of Alzer, *J. Math. Anal. Appl.*, **192** (1995), pp. 1034-1035.

[15] J. SÁNDOR, On an inequality of Bennett, *General Math.*, **3**(1995), no. 3-4, pp. 121-125.

[16] J. SÁNDOR, On the gamma function II, *Publ. Centre Rech. Math. Pures* (Chambéry), Série I, Fasc.**28**(1997), pp. 10-12.

[17] J. SÁNDOR, Comments on an inequality for the sum of powers of positive integers, *RGMIA, Reports Collection, Victoria Univ.*, **2**(1999), no. 2, pp. 259-261.

[18] J. SÁNDOR, On an inequality of Alzer II, *Octogon Math. Mag.*, **11**(2003), no. 2, pp. 554-555.

[19] J. SÁNDOR, *Selected Chapters of Geometry, Analysis and Number Theory*, RGMIA Monographs in Mathematics, Victoria Univ. of Tehnology, Australia, 2005.

[20] J. SÁNDOR, On an inequality of Alzer for negative powers, *RGMIA*, **9**(2006), no. 4.

[21] *RGMIA* (Research group on mathematical inequalities and applications) Research Reports Collection (Australia), 1994-2006.