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LOCAL AND GLOBAL EXISTENCE AND UNIQUENESS RESULTS FOR SECOND AND HIGHER ORDER IMPULSIVE FUNCTIONAL DIFFERENTIAL EQUATIONS WITH INFINITE DELAY

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ABSTRACT. In this paper, we discuss the local and global existence and uniqueness results for second and higher order impulsive functional differential equations with infinite delay. We shall rely on a nonlinear alternative of Leray-Schauder. For the global existence and uniqueness we apply a recent Frigon and Granas nonlinear alternative of Leray-Schauder type in Fréchet spaces.

Key words and phrases: Impulsive functional differential equations, Infinite delay, Higher order, Neutral, Contraction, Fixed point.

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1. INTRODUCTION

This paper is concerned with the existence of solutions to second and higher order impulsive functional and neutral functional differential equations with infinite delay. In particular, in Section 3, we will consider the class of second order functional differential equations with impulsive effects,

$$y''(t) = f(t, y_t),\ a.e.\ t \in J := [0, b],\ t \neq t_k,\ k = 1, \dots, m, \tag{1.1}$$

$$y(t_k^+) - y(t_k^-) = I_k(y(t_k^-)),\ \ t = t_k,\ k = 1, \dots, m, \tag{1.2}$$

$$y'(t_k^+) - y'(t_k^-) = \overline{I}_k(y(t_k^-)),\ \ t = t_k,\ k = 1, \dots, m, \tag{1.3}$$

$$y(t) = \phi(t), \quad t \in (-\infty, 0], \quad y'(0) = \eta, \tag{1.4}$$

where $\eta \in \mathbb{R}^n,\ 0 = t_0 < t_1 < \dots < t_m < t_{m+1} = b$, $f : J \times B \to \mathbb{R}^n$, ($B$ is called a *phase space* that will be defined later) $I_k, \overline{I}_k \in C(\mathbb{R}^n, \mathbb{R}^n),\ k = 1, 2, \dots, m,$ are given functions satisfying some assumptions that will be specified later, $y(t_k^+) = \lim\limits_{h \to 0^+} y(t_k + h)$ and $y(t_k^-) = \lim\limits_{h \to 0^-} y(t_k - h)$ represent the right and left limits of $y(t)$ at $t = t_k$, and $\phi \in B$.

For any function y defined on $(-\infty, b]$ and any $t \in [0, \infty)$, we denote by y_t the element of B defined by $y_t(\theta) = y(t + \theta),\ \ \theta \in (-\infty, 0]$. Here $y_t(\cdot)$ represents the history of the state from time $t - \theta$ up to the present time t.

Section 4 is devoted to second order impulsive neutral functional differential equations,

$$\frac{d}{dt}[y'(t) - g(t, y_t)] = f(t, y_t),\ t \in J,\ t \neq t_k,\ k = 1, .., m, \tag{1.5}$$

$$y(t_k^+) - y(t_k^-) = I_k(y(t_k^-)),\ \ t = t_k,\ k = 1, \dots, m, \tag{1.6}$$

$$y'(t_k^+) - y'(t_k^-) = \overline{I}_k(y(t_k^-)),\ \ t = t_k,\ k = 1, \dots, m, \tag{1.7}$$

$$y_0 = \phi \in B, \quad y'(0) = \eta, \tag{1.8}$$

where $\eta, f, I_k, \overline{I}_k,\ B$ are as in problem (1.1)-(1.4), and $g : J \times B \to \mathbb{R}^n$ is a given function. In the least section, for $n \geq 2,$ we consider the higher order problem,

$$y^{(n)}(t) = f(t, y_t),\ a.e.\ t \in J = [0, b],\ t \neq t_k,\ \ k = 1, \dots, m, \tag{1.9}$$

$$y^{(i)}(t_k^+) - y^{(i)}(t_k^-) = I_{k,i}(y(t_k^-)),\ t = t_k,\ k = 1, \dots, m, i = 1, \dots, n-1, \tag{1.10}$$

$$y^{(i)}(0) = y_i, \ \ i = 1, 2, \dots, n-1, \tag{1.11}$$

$$y(t) = \phi(t),\ \ t \in (-\infty, 0], \tag{1.12}$$

where f and ϕ are as in problem (1.1)–(1.4), and $I_{k,i} \in C(\mathbb{R}^n, \mathbb{R}^n),\ k = 1, \dots, m,\ i = 1, \dots, n-1.$ In the literature devoted to equations with finite. delay, the state space is much of the time the space of all continuous function on $[-r, 0],\ r > 0,$ endowed with the uniform norm topology; see the book of Hale and Lunel [22]. When the delay is infinite, the selection of the state B (i.e. phase space) plays an important role in the study of both qualitative and quantitative theory. A usual choice is a semi-normed space satisfying suitable axioms, which was introduced by Hale and Kato [21] (see also Kappel and Schappacher [24] and Schumacher [31]) and the papers of Hale [19, 20] and Sawano [30]. For a detailed discussion on this topic we refer the reader to the book by Hino *et al* [23]. For the case where the impulses are absent, an extensive theory for first order functional differential equations has been developed. We refer to Hale and Kato [21], Hale and Lunel [22], Corduneanu and Lakshmikantham [11], Hino *et al* [23], Lakshmikantham *et al* [26] and Shin [32].

Impulsive differential equations have become more important in recent years in some mathematical models of real processes and phenomena studied in control, physics, chemistry, population dynamics, biotechnology and economics. There has been a significant development in impulse theory, in recent years, especially in the area of impulsive differential equations with fixed moments; see the monographs of Bainov and Simeonov [3], Lakshmikantham *et al* [25] and Samoilenko and Perestyuk [29] and the papers of Agur *et al* [1], Ballinger and Liu [4], Benchohra *et al* [5, 6], Franco *et al* [13] and the references therein.

The goal of this paper is to give existence and uniqueness results for higher order impulsive functional differential equations with infinity delay. Very recently, Benchohra *et al*, [7, 8, 9] studied local and global existence for first order impulsive functional differential equations with infinite delay. The mains theorems of this paper extend to the infinite delay problems consider by Benchohra *et al* [5, 6]. Our approach here is based on the Leray-Schauder alternative [12], Banach fixed point theorem and a recent Frigon and Gransas nonlinear alternative of Leary-Schauder type in Fréchet spaces [14].

2. PRELIMINARIES

In this short section, we introduce notations and definitions which are used throughout the paper.

$C([0,b],\mathbb{R}^n)$ is the Banach space of all continuous functions from $[0,b]$ into $\mathbb{R}^n$ with the norm

$$\|y\|_\infty = \sup\{\|y(t)\| : 0 \le t \le b\}.$$

$L^1([0,b],\mathbb{R}^n)$ denotes the Banach space of measurable functions $y : [0,b] \longrightarrow \mathbb{R}^n$ which are Lebesgue integrable and normed by

$$\|y\|_{L^1} = \int_0^b \|y(t)\|dt \quad \text{for all} \quad y \in L^1([0,b],\mathbb{R}^n).$$

Definition 2.1. The map $f : [0,b] \times B \longrightarrow \mathbb{R}^n$ is said to be L^1-Carathéodory if

(i) $t \longmapsto f(t,x)$ is measurable for each $x \in B$;

(ii) $x \longmapsto f(t,x)$ is continuous for almost all $t \in [0,b]$;

(iii) For each $q > 0$, there exists $h_q \in L^1([0,b],\mathbb{R}_+)$ such that

$$\|f(t,x)\| \le h_q(t) \quad \text{for all} \quad \|x\|_B \le q \quad \text{and for almost all} \quad t \in [0,b].$$

3. LOCAL EXISTENCE AND UNIQUENESS RESULT

In order to define the phase space and the solution of (1.1)–(1.4) we shall consider the space

$$\begin{aligned} PC &= \Big\{ y : (-\infty,b] \to \mathbb{R}^n,\ \ y(t_k^-), y(t_k^+),\ \text{exist with}\ \ y(t_k) = y(t_k^-), \\ & y(t) = \phi(t), t \le 0,\ y_k \in C(J_k,\mathbb{R}^n) \Big\}, \end{aligned}$$

where y_k is the restriction of y to $J_k = (t_k, t_{k+1}],\ k = 0,\dots,m$. Let $\|\cdot\|_{PC}$ be the norm in PC defined by

$$\|y\|_{PC} = \sup\{\|y(s)\| : 0 \le s \le b\},\ y \in PC.$$

We will assume that B satisfies the following axioms:

(A) If $y : (-\infty,b] \to \mathbb{R}^n, b > 0$ and $y_0 \in B$, and $y(t_k^-),\ y(t_k^+)$, exist with $y(t_k) = y(t_k^-),\ k = 1,\dots,m$ then for every t in $[0,b)\backslash\{t_1\dots,t_m\}$ the following conditions hold:

(i) y_t is in B; and y_t is continuous on $[0,b]\backslash\{t_1,\dots,t_m\}$
(ii) $\|y_t\|_B \leq K(t)\sup\{\|y(s)\| : 0 \leq s \leq t\} + M(t)\|y_0\|_B$,
(iii) $\|y(t)\| \leq H\|y_t\|_B$
where $H \geq 0$ is a constant, $K : [0,\infty) \to [0,\infty)$ is continuous, $M : [0,\infty) \to [0,\infty)$ is locally bounded and H, K, M are independent of $y(\cdot)$.

(A-1) For the function $y(\cdot)$ in (A), y_t is a B-valued continuous function on $[0,b)\backslash\{t_1,\dots,t_m\}$.
(A-2) The space B is complete.

Set

$$B_b = \{y : (-\infty, b] \to \mathbb{R}^n y \in PC \cap B\},$$

and let $\|\cdot\|_b$ be the seminorm in B_b defined by

$$\|y\|_b := \|y_0\|_B + \sup\{\|y(t)\| : 0 \leq s \leq b\},\ y \in B_b.$$

Let us start by defining what we mean by a solution of problem (1.1)–(1.4).

Definition 3.1. A function $y \in B_b$, is said to be a solution of (1.1)–(1.4) if y satisfies (1.1)–(1.4).

We will need the following auxiliary result in order to prove our main existence theorems.

Lemma 3.1. *y is the unique solution of the problem (1.1)-(1.4) if and only if y is a solution of the problem,*

$$y'(t) = \eta + \int_0^t f(s, y_s)ds + \sum_{0<t_k<t} \overline{I}_k(y(t_k^-)),\ t \in J,\ t \neq t_k,\ \ k = 1,\dots,m, \tag{3.1}$$

$$y(t_k^+) - y(t_k^-) = I_k(y(t_k^-)),\ \ k = 1,\dots,m, \tag{3.2}$$

$$y(t) = \phi(t), \quad t \in (-\infty, 0]. \tag{3.3}$$

Proof. Let y be a solution of the problem (1.1)–(1.4). Then

$$y''(t) = f(t, y_t) \ \text{ for }\ t \in [0,b]\backslash\{t_1 \dots, t_m\}.$$

An integration from 0 to t (here $t \in (0, t_1]$) of both sides of the above equality yields

$$\begin{aligned}\int_0^t y''(t)ds &= \int_0^t f(s, y_s)ds\\ y'(t) - y'(0) &= \int_0^t f(s, y_s)ds.\end{aligned}$$

Thus for $t \in [0, t_1]$, we have

$$y'(t) = \eta + \int_0^t f(s, y_s)ds.$$

If $t \in (t_1, t_2]$, then we have

$$\begin{aligned}\int_0^t y''(s)ds &= \int_0^t f(s, y_s)ds\\ \int_0^{t_1} y''(s)ds + \int_{t_1}^t y''(s)ds &= \int_0^t f(s, y_s)ds\\ y'(t_1^-) - y'(0) + y'(t) - y'(t_1^+) &= \int_0^t f(s, y_s)ds\\ y'(t) - \overline{I}_1(y(t_1^-)) - \eta &= \int_0^t f(s, y_s)ds.\end{aligned}$$

Thus for $t \in (t_1, t_2]$ we have

$$y'(t) = \eta + \overline{I}_1(y(t_1^-)) + \int_0^t f(s, y_s)ds.$$

Continue to obtain for $t \in [0, b]$ that

$$y'(t) = \eta + \int_0^t f(s, y_s)ds + \sum_{0<t_k<t} \overline{I}_k(y(t_k^-)).$$

Conversely, we prove that if y satisfies the problem (3.1)–(3.3), then y is a solution of the problem (1.1)–(1.4). First, $y(t) = \phi(t)$ $t \in (-\infty, 0]$ and $y(t_k^+) - y(t_k^-) = I_k(y(t_k^-)),\ k = 1, \dots, m$. Let $t \in [0, b]\backslash\{t_1, \dots, t_m\}$ and

$$y'(t) = \eta + \int_0^t f(s, y_s)ds + \sum_{0<t_k<t} \overline{I}_k(y(t_k^-)).$$

Then,

$$y''(t) = f(t, y_t), \quad t \in [0, b]\backslash\{t_1, \dots, t_m\}.$$

∎

Theorem 3.2. *Let $f : J \times B \to \mathbb{R}^n$ be an L^1-Carathéodory function. Assume the condition,*

(H1) *There exist a continuous nondecreasing function $\psi : [0, \infty) \longrightarrow (0, \infty)$ and $p \in L^1([0, b], \mathbb{R}_+)$ such that*

$$\|f(t, x)\| \leq p(t)\psi(\|x\|_B) \text{ for a.e. } t \in [0, b] \text{ and each } x \in B,$$

with

$$K_b \int_0^b p(s)ds < \int_c^\infty \frac{dx}{\psi(x)},$$

where $K_b = \sup\{|K(t)| : t \in [0, b]\}$, $M_b = \sup\{|M(t)| : t \in [0, b]\}$ and $c = M_b\|\phi\|_B + K_b\|\phi(0)\|$.

Then the initial value problem (1.1)-(1.4) has at least one solution.

Proof. The proof will be given in several steps.
Step 1: Consider the problem,

$$y'(t) = \eta + \int_0^t f(s, y_s)ds,\ a.e.\ t \in [0, t_1], \tag{3.4}$$

$$y(t) = \phi(t),\ \ t \in (-\infty, 0]. \tag{3.5}$$

Transform the problem (3.4)–(3.5) into a fixed point problem. Consider the operator $N : B \cap C([0, t_1], \mathbb{R}^n) \longrightarrow B \cap C([0, t_1], \mathbb{R}^n)$ defined by,

$$N(y)(t) = \begin{cases} \phi(t), & t \in (-\infty, 0], \\ \phi(0) + t\eta + \displaystyle\int_0^t \int_0^s f(u, y_u)duds, & t \in [0, t_1]. \end{cases}$$

Let $x(\cdot) : (-\infty, t_1] \to \mathbb{R}^n$ be the function defined by

$$x(t) = \begin{cases} \phi(0), & \text{if } t \in [0, t_1], \\ \phi(t), & \text{if } t \in (-\infty, 0]. \end{cases}$$

Then $x_0 = \phi$. For each $z \in C([0,t_1], \mathbb{R}^n)$ with $z_0 = 0$, we denote by $\bar{z}$ the function defined by

$$\bar{z}(t) = \begin{cases} z(t), & \text{if } t \in [0,t_1], \\ 0, & \text{if } t \in (-\infty, 0]. \end{cases}$$

If $y(\cdot)$ satisfies the integral equation,

$$y(t) = \phi(0) + t\eta + \int_0^t \int_0^s f(u, y_u)duds,$$

we can decompose $y(\cdot)$ as $y(t) = \bar{z}(t) + x(t), 0 \leq t \leq t_1$, which implies $y_t = \bar{z}_t + x_t$, for every $0 \leq t \leq t_1$, and the function $z(\cdot)$ satisfies

$$z(t) = t\eta + \int_0^t \int_0^s f(u, \bar{z}_u + x_u)duds. \tag{3.6}$$

Set

$$C_0 = \{z \in C([0,t_1], \mathbb{R}^n) : z_0 = 0\}.$$

Let the operator $P : C_0 \to C_0$ be defined by

$$(Pz)(t) = t\eta + \int_0^t \int_0^s f(u, \bar{z}_u + x_u)duds, \quad t \in [0,t_1].$$

Obviously, that the operator N has a fixed point is equivalent to P has a fixed point, and so we turn to proving that P has a fixed point. We shall use the Leray-Schauder alternative to prove that P has fixed point.

Claim 1: *P is continuous.*

Let $\{z_n\}$ be a sequence such that $z_n \to z$ in C_0. Then

$$\|(Pz_n)(t) - (Pz)(t)\| \leq t_1 \int_0^{t_1} \|f(s, \bar{z}_{n_s} + x_s) - f(s, \bar{z}_s + x_s)\|ds.$$

Since f is L^1-Carathéodory, then we have

$$\|P(z_n) - P(z)\|_\infty \leq t_1\|f(\cdot, \bar{z}_{n_{(\cdot)}} + x_{(\cdot)}) - f(\cdot, \bar{z}_{(\cdot)} + x_{(\cdot)})\|_{L^1} \to 0 \text{ as } n \to \infty.$$

Claim 2: *P maps bounded sets into bounded sets in C_0.*

Indeed, it is enough to show that for any $q > 0$, there exists a positive constant ℓ such that for each $z \in \mathcal{B}_q = \{z \in C_0 : \|z\|_\infty \leq q\}$ one has $\|P(z)\|_\infty \leq \ell$. Let $z \in \mathcal{B}_q$. Since f is an L^1-Carathéodory function, we have for each $t \in [0,t_1]$

$$\|(Pz)\|_\infty \leq t_1\|\eta\| + t_1 \int_0^{t_1} h_{q_*}(s)ds := \ell,$$

where

$$\|\bar{z}_s + x_s\|_B \leq \|\bar{z}_s\|_B + \|x_s\|_B \leq K_b q + K_b\|\phi(0)\| + M_b\|\phi\|_B := q_*.$$

Claim 3: *P maps bounded sets into equicontinuous sets of C_0.*

Let $l_1, l_2 \in [0,t_1]$, $l_1 < l_2$ and let $\mathcal{B}_q$ be a bounded set of C_0 as in Claim 2. Let $z \in \mathcal{B}_q$. Then for each $t \in [0,t_1]$ we have

$$\begin{aligned} \|(Pz)(l_2) - (Pz)(l_1)\| &\leq |l_1 - l_2||\eta| + \int_{l_1}^{l_2} \int_0^s \|f(u, \bar{z}_u + x_u)\|dsdu \\ &\leq |l_1 - l_2||\eta| + |l_2 - l_1|\|h_{q_*}\|_{L^1}. \end{aligned}$$

We see that $\|(Pz)(l_2) - (Pz)(l_1)\|$ tends to zero independently of $z \in \mathcal{B}_q$, as $l_2 - l_1 \to 0$. As a consequence of Claims 1 to 3, together with the Arzelá-Ascoli theorem, we can conclude that $P : C_0 \longrightarrow C_0$ is continuous and completely continuous.

Claim 4: *There exist a priori bounds on solutions.*
Let z be a possible solution of the equation $z = \lambda P(z)$ and $z_0 = \lambda\phi$, for some $\lambda \in (0,1)$. Then

$$\|z(t)\| \le t_1|\eta| + t_1\int_0^t \|f(s,\bar{z}_s + x_s)\|ds \le t_1|\eta| + t_1\int_0^t p(s)\psi(\|\bar{z}_s + x_s\|_B)ds. \tag{3.7}$$

But

$$\begin{aligned}\|\bar{z}_s + x_s\|_B &\le \|\bar{z}_s\|_B + \|x_s\|_B \\ &\le K(t)\sup\{\|z(s)\| : 0 \le s \le t\} + M(t)\|z_0\|_B \\ &\quad +K(t)\sup\{\|x(s)\| : 0 \le s \le t\} + M(t)\|x_0\|_B \\ &\le K_b\sup\{\|z(s)\| : 0 \le s \le t\} + M_b\|\phi\|_B + K_bM\|\phi(0)\|.\end{aligned} \tag{3.8}$$

If we name $w(t)$ the right hand side of (3.8), then we have

$$\|\bar{z}_s + x_s\|_B \le w(t),$$

and therefore (3.7) becomes

$$\|z(t)\| \le t_1\|\eta\| + t_1\int_0^t p(s)\psi(w(s))ds, \quad t \in [0,t_1]. \tag{3.9}$$

Using (3.9) in the definition of w, we have that

$$w(t) \le t_1K_b\int_0^t p(s)\psi(w(s))ds + t_1|\eta| + M_b\|\phi\|_B + K_b\|\phi(0)\|, \quad t \in [0,t_1].$$

Denoting by $\beta(t)$ the right hand side of the last inequality we have

$$w(t) \le \beta(t), \quad t \in [0,t_1],$$

$$\beta(0) = t_1|\eta| + M_b\|\phi\|_B + K_b\|\phi(0)\|,$$

and

$$\begin{aligned}\beta'(t) &= t_1K_bp(t)\psi(w(t)) \\ &\le t_1K_bp(t)\psi(\beta(t)), \quad t \in [0,t_1].\end{aligned}$$

This implies that for each $t \in [0,t_1]$

$$\int_{\beta(0)}^{\beta(t)} \frac{ds}{\psi(s)} \le t_1K_b\int_0^{t_1} p(s)ds < \int_c^{\infty}\frac{ds}{\psi(s)}.$$

Thus from (H1) there exists a constant K_* such that $\beta(t) \le K_*,\ t \in [0,t_1]$, and hence $\|\bar{z}_t + x_t\|_B \le w(t) \le K_*,\ t \in [0,t_1]$. From equation (3.9) we have that

$$\|z\|_\infty \le \int_0^{t_1} p(s)\psi(K_*)ds := \widetilde{K}_1.$$

Set

$$U_0 = \{z \in C_0 : \sup\{\|z(t)\| : 0 \le t \le t_1\} < \widetilde{K}_1 + 1\}.$$

$P : \overline{U}_0 \to C_0$ is continuous and completely continuous. From the choice of U_0, there is no $z \in \partial U_0$ such that $z = \lambda P(z)$, for some $\lambda \in (0,1)$. As a consequence of the nonlinear alternative of Leray-Schauder type [12], we deduce that P has a fixed point z in U_0. Hence N has a fixed point y which is a solution to problem (3.4)–(3.5). Denote this solution by y_0.

Step 2: Consider now the problem,

$$y'(t) = \int_{t_1}^{t} f(s, y_s)ds + y_0'(t_1^-) + \overline{I}_1(y_0(t_1^-)),\ a.e.\ t \in (t_1, t_2], \tag{3.10}$$

$$y(t_1^+) = y_0(t_1^-) + I_1(y_0(t_1)),\ \ y(t) = y_0(t),\ t \in (-\infty, t_1]. \tag{3.11}$$

Let

$$C_1 = \{y \in C((t_1, b], \mathbb{R}^n) : y(t_1^+) \text{ exists}\}.$$

Set $C_* = B \cap C([0, t_1], \mathbb{R}^n) \cap C_1$. Consider the operator $N_1 : C_* \to C_*$ defined by:

$$N_1(y)(t) = \begin{cases} y_0(t), & (-\infty, t_1], \\ y_0(t_1^-) + I_1(y_0(t_1^-)) + (t - t_1)[y_0'(t_1^-) + \overline{I}_1(y_0(t_1^-))] & \\ + \displaystyle\int_{t_1}^{t}\int_{t_1}^{s} f(u, \bar{z}_u + x_u)dsdu, & t \in (t_1, t_2]. \end{cases}$$

Let $x(\cdot) : (-\infty, t_2] \to \mathbb{R}^n$ be the function defined by

$$x(t) = \begin{cases} y_0(t_1^-) + I_1(y_0(t_1)), & \text{if } t \in (t_1, t_2], \\ y_0(t), & \text{if } t \in (-\infty, t_1]. \end{cases}$$

Then $x_{t_1} = y_0$. For each $z \in C_*$ with $z_{t_1} = 0$, we denote by $\bar{z}$ the function defined by

$$\bar{z}(t) = \begin{cases} z(t), & \text{if } t \in [t_1, t_2], \\ 0, & \text{if } t \in (-\infty, t_1]. \end{cases}$$

If $y(\cdot)$ satisfies the integral equation,

$$y(t) = y_0(t_1^-) + I_1(y_0(t_1^-)) + (t - t_1)[y_0'(t_1^-) + \overline{I}_1(y_0(t_1^-))] + \int_{t_1}^{t}\int_{t_1}^{s} f(u, y_u)duds,$$

we can decompose it as $y(t) = \bar{z}(t) + x(t), t_1 \le t \le t_2$, which implies $y_t = \bar{z}_t + x_t$, for every $t_1 \le t \le t_2$, and the function $z(\cdot)$ satisfies

$$z(t) = (t - t_1)[y_0'(t_1^-) + \overline{I}_1(y_0(t_1^-))] + \int_{t_1}^{t}\int_{t_1}^{s} f(u, \bar{z}_u + x_u)dsdu. \tag{3.12}$$

Set

$$C_{t_1} = \{z \in C_* : z(t_1) = 0\}.$$

Let the operator $P_1 : C_{t_1} \to C_{t_1}$ be defined by,

$$(P_1 z)(t) = \begin{cases} 0, & t \in (-\infty, t_1], \\ (t - t_1)[y_0'(t_1^-) + \overline{I}_1(y_0(t_1^-))] + \displaystyle\int_{t_1}^{t}\int_{t_1}^{s} f(u, \bar{z}_u + x_u)dsdu, & t \in [t_1, t_2]. \end{cases}$$

As in Step 1 we can show that P_1 is continuous and completely continuous, and if z is a possible solution of the equations $z = \lambda P_1(z)$ and $z_0 = \lambda y_0$, for some $\lambda \in (0, 1)$, there exists $K_{*_1} > 0$ such that

$$\|z\|_\infty \le K_{*_1}.$$

Set

$$U_1 = \{z \in C_{t_1} : \sup\{\|z(t)\| : t_1 \le t \le t_2\} \le K_{*_1} + 1\}.$$

As a consequence of the nonlinear alternative of Leray-Schauder type [12], we deduce that P_1 has a fixed point z in U_1. Thus N_1 has a fixed point y which is an solution to problem (3.10)–(3.11). Denote this solution by y_1.

Step 3: We continue this process and taking into account that $y_m := y\Big|_{[t_m,b]}$ is a solution to the problem

$$y'(t) = \int_{t_m}^{t} f(s, y_s)ds + y'_{m-1}(t_m^-) + \overline{I}_{m-1}(y(t_m^-)),\ a.e.\ t \in (t_m, b], \tag{3.13}$$

$$y(t_m^+) = y_{m-1}(t_{m-1}^-) + I_m(y_{m-1}(t_m^-)),\ \ y(t) = y_{m-1}(t),\ t \in (-\infty, t_{m-1}]. \tag{3.14}$$

The solution y of the problem (1.1)-(1.4) is then defined by

$$y(t) = \begin{cases} y_0(t), & \text{if } t \in (-\infty, t_1], \\ y_1(t), & \text{if } t \in (t_1, t_2], \\ \dots \\ y_m(t), & \text{if } t \in (t_m, b]. \end{cases}$$

The proof is complete. ∎

We next introduce some additional conditions that lead to uniqueness of the solution of (1.1)–(1.4).

(A1) There exists $l \in L^1([0,b], \mathbb{R}_+)$ such that

$$\|f(t,x) - f(t,\overline{x})\| \le l(t)\|x - \overline{x}\|_B \ \text{ for all } x,\ \overline{x} \in B \text{ and } t \in J.$$

(A2) There exist constants $0 < c_k < 1,\ 0 < d_k < 1,\ k = 1, \dots, m,$ such that

$$\|I_k(y) - I_k(x)\| \le c_k\|x - \overline{x}\|, \quad |\overline{I}_k(y) - \overline{I}_k(x)| \le d_k\|x - \overline{x}\|, \text{ for each } \ x, \overline{x} \in \mathbb{R}^n.$$

Theorem 3.3. *Assume that hypotheses (A1)-(A2) hold. Then the IVP (1.1)–(1.4) has a unique solution.*

Proof. The proof will given in several steps.
Step 1 We prove that the problem (3.4)–(3.5) has unique solution. Then we prove only that the operator P defined in Theorem 3.2 has a unique fixed point. We shall show that P is a contraction operator. Indeed, consider $z, z^* \in C_0$. Thus for each $t \in [0, t_1]$,

$$(Pz)(t) = t\eta + \int_0^t \int_0^s f(u, \bar{z}_u + x_u)duds.$$

Then,

$$\begin{aligned} \|P(z)(t) - P(z^*)(t)\| &\le \int_0^t \int_0^s \|f(u, \bar{z}_u + x_u)) - f(u, \bar{z}_u^* + x_u)\|duds \\ &\le \int_0^t t_1 l(s)\|\bar{z}_s - \bar{z}_s^*\|_B ds \\ &\le \int_0^t l(s)K_b \sup_{s\in[0,t]} \|z(s) - z^*(s)\|ds \\ &\le \int_0^t \frac{1}{\tau}\widetilde{l}(s)e^{\tau\widehat{l}(s)}e^{-\tau\widehat{l}(s)} \sup_{s\in[0,t]} \|z(s) - z^*(s)\|ds \\ &\le \int_0^t \frac{1}{\tau}\widetilde{l}(t)e^{\tau\widehat{l}(t)}ds\|z - z^*\|_{B_*} \end{aligned}$$

$$\begin{aligned} &\leq \frac{1}{\tau}\int_0^t (e^{\tau\widehat{l}(s)})' ds \|z - z^*\|_{B_*} \\ &\leq \frac{1}{\tau} e^{\tau\widehat{l}(t)} ds \|z - z^*\|_{B_*}. \end{aligned}$$

Thus

$$e^{-\tau\widehat{l}(t)} \|P(z)(t) - P(z^*)(t)\| \leq \frac{1}{\tau}\|z - z^*\|_{B_*}.$$

Therefore,

$$\|P(z) - P(z^*)\|_{B_*} \leq \frac{1}{\tau}\|z - z_*\|_{B_*},$$

where $\widehat{l}(t) = \int_0^t \widetilde{l}(s)ds, \ \widetilde{l}(t) = t_1 K_b l(t)$ and $\|\cdot\|_{B_*}$ is the Bielecki-type norm on C_0 defined by

$$\|z\|_{B_*} = \max\{\|z(t)\|e^{-\tau\widehat{l}(t)} : t \in [0, t_1]\}.$$

As a consequence of the Banach fixed point theorem, we deduce that P has a unique fixed point which is a solution to (3.4)–(3.5). Denote this solution by y_0.

Step 2 By analogies of Step 1 we can prove that the problem (3.10)–(3.11) has a unique solution. We denote this solution by y_1.

We continue this process and taking into account that y_m is the unique solution of the problem (3.13)–(3.14). The solution y of the problem (1.1)-(1.4) is then defined by

$$y(t) = \begin{cases} y_0(t), & \text{if } t \in (-\infty, t_1], \\ y_1(t), & \text{if } t \in (t_1, t_2], \\ \dots \\ y_m(t), & \text{if } t \in (t_m, b]. \end{cases}$$

Let x, y be a two solutions of the problem (1.1)–(1.4). If $t \in (t_k, t_{k+1}], \ k = 0, \dots, m$, then $x(t) = y(t)$. If $t = t_k^+, \ k = 1, \dots, m$, then from (A2) we have

$$|I_k(x(t_k)) - I_k(y(t_k))| \leq c_k |x(t_k) - y(t_k)|$$

and

$$|\overline{I}_k(x(t_k)) - \overline{I}_k(y(t_k))| \leq d_k |x(t_k) - y(t_k)|.$$

Then

$$|x(t_k^+) - y(t_k^+))| \leq 0, \ \ |x'(t_k^+) - y'(t_k^+))| \leq 0,$$

and these inequalities imply that $x(t_k^+) = y(t_k^+)$ and $x'(t_k^+) = y'(t_k^+)$. Thus, there is a unique solution of the problem (1.1)–(1.4). ∎

3.1. **Global Existence and Uniqueness Result.** In this subsection, we are concerned with an application of a recent nonlinear alternative for contraction maps in Fréchet spaces, due to Frigon and Granas [14], to the existence and uniqueness of a problem, with infinitely many impulses and infinit delay. More precisely we consider the problem,

(3.15) $$y''(t) = f(t, y_t) \ a.e. \ t \in J_* := [0, \infty)\backslash\{t_1, t_2, \dots\},$$

(3.16) $$y(t_k^+) - y(t_k^-) = I_k(y(t_k^-)), \quad k = 1, \dots,$$

(3.17) $$y'(t_k^+) - y'(t_k^-) = \overline{I}_k(y(t_k^-)), \quad k = 1, \dots,$$

(3.18) $$y(t) = \phi(t), \ \ t \in (-\infty, 0], \ \ y'(0) = \eta,$$

where $f : J_* \times B \to \mathbb{R}^n$ and $I_k,\ \overline{I}_k \in C(\mathbb{R}^n, \mathbb{R}^n),\ k = 1, \ldots,$ and $0 = t_0 < t_1 < \ldots < t_m < \ldots,\ \lim_{n\to\infty} t_n = \infty$, $y(t_k^+) = \lim_{h\to 0^+} y(t_k + h)$ and $y(t_k^-) = \lim_{h\to 0^-} y(t_k - h)$ represent the right and left limits of $y(t)$ at $t = t_k$. For any function y defined on $(-\infty, \infty)$ and any $t \in [0, \infty)$, we denote by y_t the element of B defined by $y_t(\theta) = y(t + \theta),\ \ \theta \in (-\infty, 0]$.
As we know, investigation of many properties of solutions for a given equation, such as stability, oscillation, needs its guarantee of global existence. Thus it is important and necessary to establish sufficient conditions for global existence of solutions for impulsive differential equations. The global existence results for impulsive differential equations with different conditions were studied by Cheng and Yan [10], Graef and Ouahab [15], Guo [16, 17], Guo and Liu [18], Ouahab [28], Marino *et al* [27], Stamov and Stamova [33], Weng [34], Yan [35, 36]. Very, recently this alternative was applied by Arara *et al* [2] for controllability of functional semilinear differential equations, and by Graef and Ouahab [15] for functional impulsive differential equations with variable times. In [8] Benchohra *et al*, obtained some results on global existence of first order impulsive functional differential equations with infinity delay and boundary conditions.
For more details on the following notions we refer to [14]. Let X be a Fréchet space with a family of semi-norms $\{\|\cdot\|_n),\ n \in \mathbb{N}\}$. Let $Y \subset X$. We say that Y is bounded if for every $n \in \mathbb{N}$, there exists $M_n > 0$ such that

$$\|y\|_n \leq M_n \ \text{ for all } \ y \in Y.$$

To X, we associate a sequence of Banach spaces $\{(X^n, \|\cdot\|_n)\}$ as follows. For every $n \in \mathbb{N}$, we consider the equivalence relation $\sim_n$ defined by $x \sim_n y$ if and only if $\|x - y\|_n = 0$ for all $x, y \in X$. We denote $X^n = (X/\sim_n, \|\cdot\|)$ the quotient space, the completion of X^n with respect to $\|\cdot\|_n$. To every $Y \subset X$, we associate a sequence $\{Y^n\}$ of subsets $Y^n \subset X^n$ as follows. For every $x \in X$, we denote by $[x]_n$ the equivalence class of x of subsets X^n, and we define $Y^n = \{[x]_n : x \in Y\}$. We denote by $\overline{Y}^n$, $\text{int}_n(Y^n)$ and $\partial_n Y^n$, respectively, the closure, the interior and the boundary of Y^n with respect to $\|\cdot\|_n$ in X^n. We assume that the family of semi-norms $\{\|\cdot\|_n\}$ verifies

$$\|x\|_1 \leq \|x\|_2 \leq \|x\|_3 \leq \ldots \ \text{ for every } \ x \in X.$$

Definition 3.2. A function $f : X \to X$ is said to be a contraction if for each $n \in \mathbb{N}$ there exists $k_n \in (0, 1)$ such that

$$\|f(x) - f(y)\|_n \leq k_n \|x - y\|_n \text{ for all } x, y \in X.$$

Theorem 3.4. *(Nonlinear Alternative,* [14]*). Let X be a Fréchet space and $Y \subset X$ a closed subset in Y and let $N : Y \to X$ be a contraction such that $N(Y)$ is bounded. Then one of the following statements holds:*

(C1) *N has a unique fixed point;*
(C2) *there exists $\lambda \in [0, 1),\ n \in \mathbb{N}$, and $x \in \partial_n Y^n$ such that $\|x - \lambda N(x)\|_n = 0$.*

Definition 3.3. The map $f : [0, \infty) \times B \longrightarrow \mathbb{R}^n$ is said to be L^1-Carathéodory if

(i) $t \longmapsto f(t, x)$ is measurable for each $x \in B$;
(ii) $x \longmapsto f(t, x)$ is continuous for almost all $t \in [0, \infty)$;
(iii) For each $q > 0$, there exists $h_q \in L^1([0, \infty), \mathbb{R}_+)$ such that

$$\|f(t, x)\| \leq h_q(t) \quad \text{for all } \ \|x\|_B \leq q \quad \text{and for almost all } \ t \in [0, \infty).$$

In order to define the phase space and the solution of (3.15)–(3.18) we shall consider the space

$$\begin{aligned} PC_* &= \Big\{ y : (-\infty,\infty) \to \mathbb{R}^n \mid y(t_k^-), y(t_k^+) \text{ exist with } y(t_k) = y(t_k^-), \\ & y(t) = \phi(t), t \leq 0, \ y_k \in C(J_k, \mathbb{R}^n), \ k = 1, \dots \Big\}, \end{aligned}$$

where y_k is the restriction of y to $J_k = (t_k, t_{k+1}], \ k = 0, \dots .$

We will assume that B satisfies the following axioms:

(A) If $y : (-\infty, \infty) \to \mathbb{R}^n$, and $y_0 \in B$, then for every t in $[0, \infty)$ the following conditions hold:

(i) y_t is in B;

(ii) $\|y_t\|_B \leq K(t) \sup\{|y(s)| : 0 \leq s \leq t\} + M(t)\|y_0\|_B$,

(iii) $|y(t)| \leq H\|y_t\|_B$

where $H \geq 0$ is a constant, $K : [0, \infty) \to [0, \infty)$ is continuous, $M : [0, \infty) \to [0, \infty)$ is locally bounded and H, K, M are independent of $y(\cdot)$.

(A-1) For the function $y(\cdot)$ in (A), y_t is a B-valued continuous function on $[0, \infty)\backslash\{t_1, \dots ..\}$.

(A-2) The space B is complete.

Set

$$B_* = \{y : (-\infty, \infty) \to \mathbb{R}^n : y \in PC_* \cap B\}$$

we consider

$$B_k = \{y \in B_* : \sup_{t \in J_k^*} |y(t)| < \infty\}, \ J_k^* = (-\infty, t_k],$$

let $\| \cdot \|_k$ be the seminorm in B_k defined by

$$\|y\|_k = \|y_0\|_B + \sup\{|y(t)| : 0 \leq s \leq t_k\}, \ y \in B_k.$$

Let us start by defining what we mean by a solution of problem (3.15)–(3.18).

Definition 3.4. A function $y \in B_*$, is said to be a solution of (3.15)–(3.18) if y satisfies (3.15)–(3.18).

Theorem 3.5. *Assume that:*

(H2) *There exist constants* $d_k, \bar{d}_k > 0, k = 1, 2, \dots,$ *such that for all* $x, \overline{x} \in \mathbb{R}^n$,

$$\|I_k(x) - I_k(\overline{x})\| \leq d_k\|x - \overline{x}\|, \ \|\overline{I}_k(x) - \overline{I}_k(\overline{x})\| \leq \bar{d}_k\|x - \overline{x}\|, \ \textit{for each } k = 1, \dots;$$

(H3) *For all* $R > 0$ *there exist* $l_R \in L^1_{loc}([0, \infty), \mathbb{R}_+)$ *such that*

$$\|f(t, x) - f(t, \overline{x})\| \leq l(t)\|x - \overline{x}\|_B,$$

for each $t \in [0, \infty)$, *and all* $x, \overline{x} \in B$ *with* $\|x\|, \|\overline{x}\| \leq R$; *for a.e.* $t \in [0, \infty J_*.;$

(H4) *There exist a continuous non-decreasing function* $\psi : [0, \infty) \longrightarrow (0, \infty)$, $p \in L^1([0, \infty), \mathbb{R}_+)$ *such that*

$$\|f(t, u)\| \leq p(t)\psi(\|u\|_B) \ \textit{ for each } (t, u) \in [0, \infty) \times B,$$

$$K_k \int_0^\infty p(t)dt < \int_0^\infty \frac{du}{\psi(u)},$$

where

$$K_k = \sup\{|K(t)| : \ t \in [0, t_k]\}, \ \ k = 1, \dots, .$$

If $\sum_{k=1}^{\infty} d_k < 1, \ \sum_{k=1}^{\infty} \bar{d}_k < \infty$, *then the initial value problem (3.15)–(3.18) has unique solution.*

Proof. We consider the problem,

$$y'(t) = \eta + \int_0^t f(s, y_s)ds + \sum_{0<t_k<t} \overline{I}_k(y(t_k^-) \; a.e. \; t \in J_* := [0, \infty)\backslash\{t_1, t_2, \ldots\}, \tag{3.19}$$

$$y(t_k^+) - y(t_k^-) = I_k(y(t_k^-)), \quad k = 1, \ldots, \tag{3.20}$$

$$y(t) = \phi(t), \;\; t \in (-\infty, 0]. \tag{3.21}$$

∎

Remark 3.1. We can easily prove that if y is a solution of the problem (3.19)–(3.21) if and only if y is a solution of (3.15)–(3.18).

Transform the problem (3.19)–(3.21) into a fixed point problem. Consider the operator $\overline{N} : B_* \longrightarrow B_*$ defined by,

$$\overline{N}(y)(t) = \begin{cases} \phi(t), & t \in (-\infty, 0], \\ \phi(0) + t\eta + \int_0^t \int_0^s f(u, y_u)ds & \\ + \sum_{0<t_k<t} [I_k(y(t_k^-)) + (t - t_k)\overline{I}_k(y(t_k^-))], & t \in [0, \infty). \end{cases}$$

Let $x(\cdot) : (-\infty, \infty) \to \mathbb{R}^n$ be the function defined by

$$x(t) = \begin{cases} \phi(0), & \text{if } \; t \in [0, \infty), \\ \phi(0) + \sum_{0<t_k<t} [I_k(x(t_k^-)) + (t - t_k)\overline{I}_k(x(t_k^-))], & \text{if } \; t \in (-\infty, 0]. \end{cases}$$

Then $x_0 = \phi(0)$. For each $z \in C([0, \infty), \mathbb{R}^n)$ with $z(0) = 0$, we denote by $\bar{z}$ the function defined by

$$\bar{z}(t) = \begin{cases} z(t), & \text{if } t \in [0, \infty), \\ 0, & \text{if } t \in (-\infty, 0]. \end{cases}$$

If $y(\cdot)$ satisfies the integral equation,

$$y(t) = \phi(0) + \sum_{0<t_k<t} [I_k(y(t_k^-)) + (t - t_k)\overline{I}_k(y(t_k^-))] + \int_0^t \int_0^s f(u, y_u)duds,$$

we can decompose $y(\cdot)$ as $y(t) = \bar{z}(t) + x(t), 0 \leq t < \infty$, which implies $y_t = \bar{z}_t + x_t$, for every $0 \leq t < \infty$, and the function $z(\cdot)$ satisfies

$$\begin{aligned} z(t) &= t\eta + \int_0^t \int_0^s f(u, \bar{z}_u + x_u)duds \\ &\quad + \sum_{0<t_k<t} [I_k(\bar{z}(t_k^-) + x(t_k^-)) + (t - t_k)\overline{I}_k(\bar{z}(t_k^-) + x(t_k^-))]. \end{aligned} \tag{3.22}$$

Let

$$\begin{aligned} B_*^k &= \Big\{ z : (-\infty, t_k] \to \mathbb{R}^n \mid \; z(t_i^-), z(t_i^+) \text{ exist with } \; z(t_i) = z(t_i^-), \\ &\quad z_k \in C(J_i, \mathbb{R}^n), \; i = 1, \ldots, k-1, \text{ and } z_0 = 0 \Big\}. \end{aligned}$$

For any $z \in B_*^k$ we have

$$\|z\|_k = \|z_0\|_B + \sup\{\|z(s)\| : 0 \leq s \leq t_k\} = \sup\{\|z(s)\| : 0 \leq s \leq t_k\}.$$

Thus $(B^k_*, \|\cdot\|_k)$ is a Banach space. Set

$$\overline{C}_0 = \{z \in B_* : z_0 = 0\}.$$

$\overline{C}_0$ is a Fréchet space with a family of semi-norms $\|\cdot\|_k$. Let the operator $\overline{P} : \overline{C}_0 \to \overline{C}_0$ be defined by

$$(\overline{P}z)(t) = \begin{cases} 0, & t \in (-\infty, 0], \\ t\eta + \displaystyle\int_0^t \int_0^s f(u, \bar{z}_u + x_u)duds & \\ + \displaystyle\sum_{0<t_k<t} [I_k(\bar{z}(t_k^-) + x(t_k^-)) + (t - t_k)\overline{I}_k(\bar{z}(t_k^-) + x(t_k^-))], & t \in [0, \infty). \end{cases}$$

Obviously, that the operator $\overline{N}$ has a fixed point is equivalent to $\overline{P}$ has a fixed point, and so we turn to proving that $\overline{P}$ has a fixed point. We shall use the alternative to prove that $\overline{P}$ has fixed point.

Let z be a possible solutions of the problem, $z = \gamma\overline{P}(z)$ for some $0 < \gamma < 1$. This implies by $(H4)$ that for each $t \in [0, t_1]$ we have

$$z(t) = \gamma\Big[t\eta + \int_0^t \int_0^s f(u, \bar{z}_u + x_u)duds + \sum_{0<t_k<t} [I_k(\bar{z}(t_k^-) + x(t_k^-)) + (t - t_k)\overline{I}_k(\bar{z}(t_k^-) + x(t_k^-))].\Big].$$

As in Theorem 3.2 we can show that there exists $\overline{M}_1 > 0$ such that $\|z\|_1 \leq \overline{M}_1$. Now if $t \in (t_1, t_2]$, then

$$z(t) = \Big[t\eta + \int_0^t \int_0^s f(u, \bar{z}_u + x_u)duds + I_1(z(t_1^-) + x(t_1^-)) + (t - t_1)\overline{I}_1(z(t_1^-) + x(t_1^-))\Big].$$

Note that

$$|I_1(z(t_1^-) + x(t_1^-)) + (t - t_1)\overline{I}_1(z(t_1^-) + x(t_1^-))| \leq \sup_{x \in B(0,\overline{M}_1)} [|I_1(x)| + (t_2 - t_1)\overline{I}_1(x)] := \overline{M},$$

where

$$B(0, \overline{M}_1) = \{x \in \mathbb{R}^n : \|x\| \leq \overline{M}_1\}.$$

Hence

$$\|z(t)\| \leq M_1 + \overline{M} + t_2 \int_{t_1}^t p(s)\psi(\|\bar{z}_s + x_s\|_B)ds,$$

where

$$\|\bar{z}_s + x_s\|_B \leq K_2 \sup_{s \in [0,t]} \|z(s)\| + K_2 \sup_{s \in [0,t]} \|x(s)\| + M_2\|x_0\|_B := h(t)$$

$h(t_*) = K_2 \sup_{s \in [0,t_*]} \|z(s)\| + K_2 \sup_{s \in [0,t_*]} \|x(s)\| + M_2\|x_0\|_B$. By the previous inequality we have for $t \in [0, t_2]$

$$h(t) \leq K_2 t_2 \int_0^t p(s)\psi(h(s))ds + K_2[\overline{M}_1 + (t_2 - t_1)\overline{M}] + K_2 \sup_{s \in [0,t_*]} \|x(s)\| + M_k\|x_0\|_B.$$

Let us take the right-hand side of the above inequality as $v(t)$. Then we have

$$v(0) = K_2[\overline{M}_1 + (t_2 - t_1)\overline{M}] + K_2 \sup_{s \in [0,t_*]} \|x(s)\| + M_2\|x_0\|_B$$

and

$$v'(t) = K_2 t_2 p(t)\psi(h(t)), \quad \text{a.e. } t \in [0, t_2].$$

Using the increasing character of ψ we get

$$v'(t) \leq K_2 t_2 p(t)\psi(h(t)) \text{ a.e. } t \in [0, t_2].$$

Then for each $t \in [0, t_2]$ we have

$$\int_{v(0)}^{h(t)} \frac{du}{\psi(u)} \leq K_2 t_2 \int_0^{\infty} p(s)ds < +\infty.$$

Consequently, by $(H4)$, there exists a constant M_* such that $v(t) \leq M_*,\ t \in [0, t_2]$, and hence

$$\|z\|_2 \leq M_*/K_2 := \overline{M}_2.$$

We continue this process and also take into account that $t \in [t_k, t_{k+1}],\ k > 2$.

Then

$$z(t) = \gamma\Big[t\eta + \int_{t_k}^{t}\int_{t_k}^{s} f(u, \bar{z}_u + x_u)duds + \sum_{0<t_k<t} [I_k(z(t_k^-) + x(t_k^-) + (t - t_k)\overline{I}_k(z(t_k^-) + x(t_k^-))\Big].$$

We obtain that there exists a constant $\overline{M}_{k+1}$ such that

$$\sup\{\|z(t)\| : t \in [0, t_{k+1}]\} \leq \overline{M}_{k+1}.$$

$$Y = \{z \in \overline{C}_0 : \sup\{\|z(t)\| : 0 \leq t \leq t_k\} \leq \overline{M}_k + 1 \text{ for all } k \in= 1, 2, \ldots\}.$$

Clearly, Y is a closed subset of C_0. We shall show that $\overline{P} : Y \to B_*^k$ is a contraction maps. Indeed, consider $z, z^* \in Y$. Then we have for each $t \in [0, t_k]$ and $k \in \{1, 2, \ldots, \}$

$$\begin{aligned}
\|\overline{P}(z)(t) - \overline{P}(z^*)(t)\| &\leq \int_0^t \|f(s, \bar{z}_s + x_s)) - f(s, \bar{z}_s^* + x_s)\|ds \\
&\quad + \sum_{i=1}^{i=k} \|I_i(\bar{z}(t_i^-) + x(t_i)) - I_i(\bar{z}^*(t_i^-) + x(t_i^-)))\| \\
&\quad + \sum_{0<t_k<t}^{i=k} (t - t_k)\|I_i(\bar{z}(t_i^-) + x(t_i)) - I_i(\bar{z}^*(t_i^-) + x(t_i^-)))\| \\
&\leq \int_0^t l(s)\|\bar{z}_s - \bar{z}_s^*\|_B ds + \sum_{i=1}^{i=k} d_i\|z(t_i^-) - z^*(t_i^-)\| \\
&\quad + \sum_{0<t_k<t} (t - t_k)\overline{d}_k\|z(t_i^-) - z^*(t_i^-)\| \\
&\leq \int_0^t l(s)K_k \sup_{s\in[0,t]} \|z(s) - z^*(s)\|ds + \sum_{i=1}^{i=k} d_i\|z(t_i^-) - z^*(t_i^-)\| \\
&\quad + \sum_{0<t_k<t} (t - t_k)\overline{d}_k\|z(t_i^-) - z^*(t_i^-)\| \\
&\leq \int_0^t \frac{1}{\tau}\widehat{L}(s)e^{\tau\widehat{L}(s)}e^{-\tau\widehat{L}(s)} \sup_{s\in[0,t]} \|z(s) - z^*(s)\|ds \\
&\quad + \sum_{i=1}^{i=k} d_i e^{\tau\widehat{L}(t)}e^{-\tau\widehat{L}(t)} \sup_{s\in[0,t_k]} \|z(s) - z^*(s)\|
\end{aligned}$$

$$+\sum_{0<t_k<t}(t-t_k)\overline{d}_k e^{\tau\widehat{L}(t_k)}e^{-\tau\widehat{L}(t_k)}\sup_{s\in[0,t_k]}\|z(s)-z^*(s)\|.$$

Since $(t-s)\leq e^{(t-s)}$ for $0\leq s\leq t$, we find $(t-s)e^s\leq e^t$. Assume that τ is sufficient large

$$t-s=\int_s^t d\xi\leq\int_s^t l_*(\xi)d\xi=\widehat{L}(t)-\widehat{L}(s).$$

Then

$$\tau(t-s)\leq\tau\widehat{L}(t)-\tau\widehat{L}(s).$$

This implies that

$$\tau(t-s)e^{\tau\widehat{L}(s)}\leq e^{\tau\widehat{L}(t)}\Longrightarrow(t-s)e^{\tau\widehat{L}(s)}\leq\frac{e^{\tau\widehat{L}(t)}}{\tau}$$

where $\widehat{L}(t)=\displaystyle\int_0^t l_*(s)ds$ and

$$l_*(t)=\max(1,K_k l(t)).$$

Thus

$$e^{-\tau\widehat{L}(t)}\|\overline{P}(z)(t)-\overline{P}(z^*)(t)\| \leq \frac{1}{\tau}\|z-z^*\|_{B_*^k}+\sum_{i=1}^{i=k}d_i\|z-z^*\|_{B_*^k}+\frac{\sum_{i=1}^{i=k}\overline{d}_i}{\tau}\|z-z^*\|_{B_*^k}$$

$$e^{-\tau\widehat{L}(t)}\|\overline{P}(z)(t)-\overline{P}(z)(t)\| \leq \left(\frac{1}{\tau}+\sum_{i=1}^{i=k}d_i+\frac{\sum_{i=1}^{i=k}\overline{d}_i}{\tau}\right)\|z-z^*\|_{B_*^k}.$$

Therefore,

$$\|\overline{P}(z)-\overline{P}(z^*)\|_{B_*^k}\leq\left(\frac{1}{\tau}+\sum_{i=1}^{i=k}d_i+\frac{\sum_{i=1}^{i=k}\overline{d}_i}{\tau}\right)\|z-z_*\|_{B_*^k},$$

where $\|\cdot\|_{B_*^k}$ is the Bielecki-type norm on B_*^k
defined by

$$\|z\|_{B_*^k}=\max\{\|z(t)\|e^{-\tau\widehat{L}(t)}:t\in[0,t_k]\}.$$

From the choice of Y there is no $z\in\partial Y^n$ such that $z=\lambda\overline{P}(z)$ for some $\lambda\in(0,1)$. As a consequence of the nonlinear alternative type [12] we deduce that $\overline{P}$ has a unique fixed point which is a solution to (3.15)–(3.18).

4. NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS

This section is devoted to the existence of solutions for second order neutral functional differential equations with impulses and infinite delay (1.5)-(1.8). Much of the notation and spaces, etc., that appear in this section have been defined in previous sections.

Theorem 4.1. *Let $f,g:J\times B\to\mathbb{R}^n$ be L^1-Carathéodory functions and assume the following condition is satisfied:*

(B1) *There exist a continuous nondecreasing function* $\psi : [0,\infty) \longrightarrow (0,\infty)$ *and* $p \in L^1(J, \mathbb{R}_+)$ *such that*

$$\|f(t,x)\| \le p(t)\psi(\|x\|_B), \quad \|g(t,x)\| \le p(t)\psi(\|x\|_B) \textit{for a.e. } t \in [0,b] \textit{ and each } x \in B$$

with

$$(1+t_k)K_b \int_0^{t_k} p(s)ds < \int_0^{\infty} \frac{dx}{\psi(x)}.$$

Then the IVP (1.5)-(1.8) has at least one solution.

Proof. The proof will be given in several steps.
Step 1: Consider the problem,

$$y'(t) - g(t,y_t) = \eta + \int_0^t f(s,y_s)ds, \ a.e. \ t \in [0,t_1], \tag{4.1}$$

$$y(t) = \phi(t), \quad t \in (-\infty, 0]. \tag{4.2}$$

Consider the operator $N^* : B \cap C([0,t_1], \mathbb{R}^n) \to B \cap C([0,t_1], \mathbb{R}^n)$ defined by,

$$N^*(y)(t) = \begin{cases} \phi(t), & \text{if } t \in (-\infty, 0], \\ \phi(0) + t\eta + \int_0^t g(s,y_s)ds + \int_0^t \int_0^s f(u,y_u)duds, & \text{if } t \in [0,t_1]. \end{cases}$$

In analogy to Theorem 3.2, we consider the operator $P^* : C_0 \to C_0$ defined by

$$(P^*z)(t) = \begin{cases} 0 & t \in (-\infty, 0], \\ t\eta + \int_0^t g(s, \bar{z}_s + x_s)ds + \int_0^t \int_0^s f(u, \bar{z}_u + x_u)duds, & t \in [0,t_1]. \end{cases}$$

In order to use the Leray-Schauder alternative, we shall obtain a priori estimates for the solutions of the integral equation

$$z(t) = \lambda \left[t\eta + \int_0^t g(s, \bar{z}_s + x_s)ds + \int_0^t \int_0^s f(u, \bar{z}_u + x_u)duds \right],$$

where $z_0 = \lambda\phi$ for some $\lambda \in (0,1)$. Then

$$\begin{aligned} \|z(t)\| &\le t_1\|\eta\| + \int_0^t p(s)\psi(\|\bar{z}_s + x_s\|_B)ds + \int_0^t p(s)\psi(\|\bar{z}_s + x_s\|_B)ds \\ &\le t_1\|\eta\| + (1+t_1)\int_0^t p(s)\psi(\|\overline{z}_s + x_s\|_B)ds. \end{aligned}$$

Let $\alpha = M_b\|\phi\|_B + K_b\|\phi(0)\| + M_b\|\phi\|_B$. We have

$$\|\bar{z}_t + x_t\|_B \le K_b \sup_{s\in[0,t]} \|z(s)\| + \alpha := w(t)$$

and

$$\|z(t)\| \le t_1\|\eta\| + (1+t_1)\int_0^t p(s)\psi(w(s))ds.$$

But

$$w(t) \le K_b t_1\|\eta\| + \alpha + (1+t_1)K_b \int_0^t p(s)\psi(w(s))ds.$$

Taking the right hand side as $\beta(t)$ we have

$$w(t) \le \beta(t), \quad t \in [0,t_1],$$

$$\beta(0) := \bar{c} = K_b t_1 \|\eta\|,$$
$$\beta'(t) = (1+t_1)K_b p(t)\psi(w(t)) \leq (1+t_1)K_b p(t)\psi(\beta(t)), \quad t \in [0, t_1],$$
and
$$\int_{\beta(0)}^{\beta(t)} \frac{ds}{\psi(s)} \leq (1+t_1)K_b \int_0^b p(s)ds < \int_{\bar{c}}^{\infty} \frac{ds}{\psi(s)}.$$
Therefore, there exists a constant K_* such that $\beta(t) \leq K_*,\ t \in [0, t_1]$, and hence $\|\bar{z}_t + x_t\|_B \leq w(t) \leq K_*,\ t \in [0, t_1]$, and

$$\|z(t)\| \leq K_b t_1 \|\eta\| + \alpha + K_b \int_0^{t_1} (1+t_1)p(s)\psi(K_*)ds := \overline{K}_2.$$

Set
$$U_1 = \{z \in C_0 : \sup\{\|z(t)\| : 0 \leq t \leq t_1\} < \overline{K}_2 + 1\}.$$
From the choice of U_1, there is no $z \in \partial U_0$ such that $z = \lambda P^*(z)$, for some $\lambda \in (0,1)$. As a consequence of the nonlinear alternative of Leray-Schauder type [12], we deduce that P^* has a fixed point z in U_0. Then N^* has a fixed point y_0 which is a solution to problem (4.1)–(4.2).
Step 2: Consider now the problem,

$$y'(t) - g(t, y_t) = \int_{t_1}^{t} f(s, y_s)ds + y_0(t_1^-) + \overline{I}_1(y_0(t_1^-)), \text{ a.e. } t \in (t_1, t_2], \tag{4.3}$$

$$y(t_1^+) = y_0(t_1^-) + I_1(y_0(t_1)), \ \ y(t) = y_0(t), \ \ t \in (-\infty, t_1]. \tag{4.4}$$

Let $N_1^* : C_* \to C_*$ be defined by
$$N_1^*(y)(t) = \begin{cases} y_0(t), & t \in (-\infty, t_1], \\ \\ y_0(t_1^-) + I_1(y_0(t_1^-)) + (t-t_1)[y_0'(t_1) + \overline{I}_1(y_0(t_1))] \\ + \displaystyle\int_{t_1}^{t} g(s, y_s)ds + \int_{t_1}^{t}\int_{t_1}^{s} f(u, y_u)duds, & t \in (t_1, t_2]. \end{cases}$$
In analogy to Theorem 3.5, we consider the operator $P_1^* : C_{t_1} \to C_{t_1}$ defined by
$$\begin{aligned}(P_1^* z)(t) &= (t-t_1)\left[\eta + y_0'(t_1) + \overline{I}_1(y_0(t_1^-))\right] \\ &\quad + \int_{t_1}^{t} g(s, \overline{z}_s + x_s)ds + \int_{t_1}^{t}\int_{t_1}^{s} f(u, \overline{z}_u + x_u)duds,\end{aligned}$$

and there exists $\overline{M} > 0$ such that, if z is a possible solutions of the integral equation

$$\begin{aligned}z(t) &= \lambda\Bigg[(t-t_1)\left[\eta + y_0'(t_1) + \overline{I}_1(y_0(t_1^-))\right] \\ &\quad + \int_{t_1}^{t} g(s, \overline{z}_s + x_s)ds + \int_{t_1}^{t}\int_{t_1}^{s} f(u, \overline{z}_u + x_u)duds\Bigg],\end{aligned}$$
where $z_0 = \lambda y_0$, for some $\lambda \in (0,1)$, we have
$$\|z\|_\infty \leq \overline{M}.$$
Set
$$U_1 = \{z \in C_{t_1} : \sup\{\|z(t)\| : t_1 \leq t \leq t_2\} < \overline{M} + 1\}.$$
From the choice of U_1, there is no $z \in \partial U_1$ such that $z = \lambda P_1^*(z)$, for some $\lambda \in (0,1)$. As a consequence of the nonlinear alternative of Leray-Schauder type [12], we deduce that P_1^* has a fixed point z in U_1. Then the problem (4.3)-(4.4) has at least one solution. Denote this solution

by y_1.

Step 3: We continue this process and taking into account that $y_m := y\Big|_{[t_m,b]}$ is a solution to the problem

$$y'(t) - g(t,y_t) = \int_{t_1}^{t} f(s,y_s)ds + y'_{m-1}(t_m^-) + \overline{I}_m(y_{m-1}(t_m^-)),\ a.e.\ t \in (t_m,b], \tag{4.5}$$

$$y(t_m^+) = y_{m-1}(t_{m-1}^-) + I_m(y_{m-1}(t_m)), y(t) = y_{m-1}(t)\, t \in [-\infty,t_1]. \tag{4.6}$$

The solution y of the problem (1.5)-(1.8) is then defined by

$$y(t) = \begin{cases} y_1(t), & \text{if } t \in (-\infty,t_1], \\ y_2(t), & \text{if } t \in (t_1,t_2], \\ \dots & \\ y_m(t), & \text{if } t \in (t_m,b]. \end{cases}$$

The proof is complete. ∎

In this second part of Section 4, the Banach fixed point theorem for principal contraction maps is used to investigate the existence and uniqueness of second order impulsive neutral functional differential equations with infinite delay (1.5)-(1.8).

Theorem 4.2. *Assume $(A1), (A2)$ and the condition:*

(B^*1) *There exists a function $\tilde{l} \in L^1([0,b], \mathbb{R}_+)$ such that*

$$\|g(t,u) - g(t,\overline{u})\| \leq \tilde{l}(t)\|u-\overline{u}\|_B, t \in [0,b].$$

are satisfied. Then the IVP (1.5)-(1.8) has unique solution.

Proof. Exactly the same ideas in Theorem 3.3 establish the result. ∎

4.1. **Global Existence and Uniqueness Result.** In this subsection, we present an existence and uniqueness result for second order neutral impulsive functional differential equations with infinite delay. More precisely we consider the problem,

$$\frac{d}{dt}[y'(t) - g(t,y_t)] = f(t,y_t),\ t \in [0,\infty),\ t \neq t_k,\ k = 1,\dots, \tag{4.7}$$

$$y(t_k^+) - y(t_k) = I_k(y(t_k)),\ \ t = t_k,\ k = 1,\dots, \tag{4.8}$$

$$y'(t_k^+) - y'(t_k) = \overline{I}_k(y(t_k)),\ \ t = t_k,\ k = 1,\dots, \tag{4.9}$$

$$y_0 = \phi \in B,\ y'(0) = \eta. \tag{4.10}$$

where $I_k, \overline{I}_k, B$ are as in problem (1.1)-(1.4), and $f, g : J \times B \to \mathbb{R}^n$ are given functions.

Theorem 4.3. *Assume that $(H2),\ (H3)$ and the following conditions are satisfied:*

(M1) *For all $R > 0$ there exists $l_R \in L^1_{loc}([0,\infty), \mathbb{R}_+)$ such that*

$$\|g(t,u) - g(t,\overline{u})\| \leq l_R(t)\|u-\overline{u}\|_B, t \in [0,\infty),\ \|u\| \leq R,\ \|\overline{u}\| \leq R.$$

(M2) *There exist a continuous nondecreasing function* $\psi : [0,\infty) \longrightarrow (0,\infty)$ *and* $p \in L^1(J,\mathbb{R}_+)$ *such that, for each* $x \in B$,

$$\|f(t,x)\| \le p(t)\psi(\|x\|_B), \ \ \|g(t,x)\| \le p(t)\psi(\|x\|_B) \text{ for a.e. } t \in [0,\infty),$$

with

$$(1+t_k)K_b\int_0^{t_k} p(s)ds < \int_0^\infty \frac{dx}{\psi(x)}.$$

Then the IVP (4.7)-(4.10) has unique solution.

Proof. Essentially the same reasoning as in Theorem 3.5 can be used to establish the uniqueness result for problem (4.7)-(4.10).

5. HIGHER ORDER IMPULSIVE FDIS

Let us start by defining what we mean by a solution of problem (1.9)–(1.12). ∎

Definition 5.1. A function $y \in B_b, k = 0,\dots,m, i = 1,\dots,n-1$, is said to be a solution of (1.9)–(1.12) if y satisfies the equation $y^{(n)}(t) = f(t,y_t)$ a.e. on J, $t \neq t_k$, $k = 1,\dots,m$, and the conditions $y^{(i)}(t_k^+) - y(t_k^-) = I_{k,i}(y(t)),\ t = t_k,\ k = 1,\dots,m,\ \ i = 1,2,\dots,n-1$, $y^{(i)}(0) = y_i, i = 1,\dots,n-1$.

Theorem 5.1. *Assume that* f *is* L^1*-Carathédory and (H1) holds. Then the IVP (1.9)–(1.12) has at least one solution on* $(-\infty,b]$.

Proof. The proof will be given in several steps.
Step 1: Consider the problem,

$$y'(t) = \sum_{i=2}^{n-1} y_i\frac{t^{i-2}}{(i-2)!} + \int_0^t \frac{(t-s)^{n-2}}{(n-2)!}f(s,y_s)ds,\ a.e.\ t\in[0,t_1], \tag{5.1}$$

$$y(t) = \phi(t),\ \ t \in (-\infty,0]. \tag{5.2}$$

Transform the problem (5.1)–(5.2) into a fixed point problem. Consider the operator $G : B \cap C([0,t_1],\mathbb{R}^n) \longrightarrow B\cap C([0,t_1],\mathbb{R}^n)$ defined by,

$$G(y)(t) = \begin{cases} \phi(t), & t\in(-\infty,0] \\ \phi(0) + \sum_{i=1}^{n-1} y_i\frac{t^{i-1}}{(i-1)!} + \int_0^t\int_0^s \frac{(s-u)^{n-2}}{(n-2)!}f(u,y_u)duds, & \in[0,\infty). \end{cases}$$

In analogy to Theorem 3.2, we consider the operator $P_* : C_0 \to C_0$ defined by

$$(P_*z)(t) = \begin{cases} 0 & t\in(-\infty,0], \\ \sum_{i=1}^{n-1} y_i\frac{t^{i-1}}{(i-1)!} + \int_0^t\int_0^s \frac{(s-u)^{n-2}}{(n-2)!}f(u,\bar z_u + x_u)duds & t\in[0,t_1]. \end{cases}$$

As in Theorem 3.2 we can prove that P_* is completely continuous and there exists $M > 0$ such that for every solution of the problem $z = \lambda P_*(z)$ for some $\lambda \in (0,1)$,we have $\|z\|_\infty \le M$. Then by a nonlinear alternative of Leray-Schauder type [12], we deduce that P_* has a fixed point z in U_0. Then G has a fixed point y_0 which is a solution to problem (5.1)–(5.2).
Step 2: Consider now the following problem,

$$y'(t) = \sum_{i=1}^{n-1}[y_0^{(i)}(t_1^-) + I_{1,i}(y_0(t_1^-))]\frac{(t-t_1)^{i-2}}{(i-2)!} + \int_{t_1}^t \frac{(t-s)^{n-2}}{(n-2)!}f(s,y_s)ds,\ t\in[t_1,t_2], \tag{5.3}$$

$$y(t) = y_0(t),\ t \in (-\infty, t_1]. \tag{5.4}$$

Let

$$C_1 = \{y \in C((t_1, t_2], \mathbb{R}^n) : y^{(i)}(t_1^+) \text{ exist}, i = 1, \dots n-1\}.$$

Set $C_* = B \cap C([0, t_1], \mathbb{R}^n) \cap C_1$. Transform the problem (5.3)–(5.4) into a fixed point problem. Consider the operator

$$G_1(y)(t) = \begin{cases} y_0(t), & t \in (-\infty, t_1], \\ \sum_{i=1}^{n-1} [y_0^{(i)}(t_1^-) + I_{1,i}(y_0(t_1^-))] \dfrac{(t-t_1)^{i-1}}{(i-1)!} & \\ + \displaystyle\int_{t_1}^{t} \int_{t_1}^{s} \frac{(s-u)^{n-2}}{(n-2)!} f(u, y_u) du ds, & t \in (t_1, t_2] \end{cases}$$

Set

$$C_{t_1} = \{z \in C_* : z_{t_1} = 0\}.$$

Let the operator $P_{**} : C_{t_1} \to C_{t_1}$ defined by:

$$(P_{**}z)(t) = \begin{cases} 0, & t \in (-\infty, t_1], \\ \sum_{i=1}^{n-1} [y_0^{(i)}(t_1^-) + I_{1,i}(y_0(t_1^-))] \dfrac{(t-t_1)^{i-1}}{(i-1)!} & \\ + \displaystyle\int_{t_1}^{t} \int_{0}^{s} \frac{(s-u)^{n-2}}{(n-2)!} f(u, \bar{z}_u + x_u) du ds, & t \in [t_1, t_2]. \end{cases}$$

As in Theorem 3.2 we can show that P_{**} is continuous and completely continuous, and if z is a possible solution of the equations $z = \lambda P_{**}(z)$ and $z_0 = \lambda y_0$, for some $\lambda \in (0, 1)$, there exists $K_{*_1} > 0$ such that

$$\|z\|_\infty \leq K_{*_1}.$$

Set

$$U_1 = \{z \in C_{t_1} : \sup\{\|z(t)\| : t_1 \leq t \leq t_2\} \leq K_{*_1} + 1\}.$$

As a consequence of the nonlinear alternative of Leray-Schauder type [12], we deduce that P_{**} has a fixed point z in U_1. Thus G_1 has a fixed point y which is an solution to problem (5.3)–(5.4). Denote this solution by y_1.

Step 3: We continue this process and taking into account that $y_m := y\Big|_{[t_m, b]}$ is a solution to the problem, for a.e. $t \in (t_m, b]$,

$$y'(t) = \sum_{i=2}^{n-1} [y_{m-1}^{(i)}(t_m^-) + I_{1,i}(y_{m-1}(t_m))] \frac{(t-t_m)^{i-2}}{(i-2)!} + \int_{t_m}^{t} \frac{(t-s)^{n-2}}{(n-2)!} f(s, y_s) ds, \tag{5.5}$$

$$y(t) = y_{m-1}(t),\ t \in (-\infty, t_{m-1}]. \tag{5.6}$$

The solution y of the problem (1.9)-(1.12) is then defined by

$$y(t) = \begin{cases} y_0(t), & \text{if } t \in (-\infty, t_1], \\ y_1(t), & \text{if } t \in (t_1, t_2], \\ \dots & \\ y_m(t), & \text{if } t \in (t_m, b]. \end{cases}$$

∎

In this next part, we give sufficient conditions for local existence and uniqueness of solutions of the problem (1.9)–(1.12).

Theorem 5.2. *Assume (A1), and the condition:*

(A^*1) *There exist a constants* $0 < c_{k,i} < 1,\ k = 1, \dots, m,\ i = 1, \dots, n-1,$ *such that*

$$\|I_{k,i}(y) - I_{k,i}(x)\| \le c_{k,i}|y - x|, \quad \text{for each} \quad y, x \in \mathbb{R}^n.$$

are satisfied. Then the IVP (1.9)-(1.12) has a unique solution.

Proof. For the prove see Theorem 3.3.

5.1. **Global Existence and Uniqueness Result.** In this subsection, we present an existence and uniqueness result for higher order impulsive functional differential equations with infinite delay. More precisely, we consider the problem,

$$y^{(n)} = f(t, y_t),\ t \in [0, \infty),\ t \ne t_k,\ k = 1, ..., \tag{5.7}$$

$$y^{(i)}(t_k^+) - y^{(i)}(t_k^-) = I_{k,i}(y(t_k)),\ \ t = t_k,\ k = 1, \dots,\ i = 1, \dots, n-1, \tag{5.8}$$

$$y(0) = y_i, \quad i = 1, \dots, n-1, \tag{5.9}$$

$$y_0 = \phi \in B, \tag{5.10}$$

where $I_{k,i}, f$ are as in problem (1.9)-(1.12), and $g : J \times B \to \mathbb{R}^n$ is a given function. ∎

Theorem 5.3. *Assume:*

(H^*2) *There exist constants* $d_{k,i} > 0,\ k = 1, \dots, i = 1, \dots,$ *such that, for all* $x, \overline{x} \in \mathbb{R}^n$*,*

$$\|I_{k,i}(x) - I_{k,i}(\overline{x})\| \le d_{k,i}\|x - \overline{x}\|, \ \text{for each},\ x, y \in \mathbb{R}^n$$

(H^*3) *For all* $R > 0$ *there exist* $l_R \in L^1_{loc}([0, \infty), \mathbb{R}_+)$ *such that*

$$\|f(t, x) - f(t, \overline{x})\| \le l_R(t)\|x - \overline{x}\|_B,$$

for each $x, \overline{x} \in B$ *with* $\|x\|, \|\overline{x}\| \le R$; *for a.e.* $t \in [0, \infty)$;

(H^*4) *There exist a continuous non-decreasing function* $\psi : [0, \infty) \longrightarrow (0, \infty)$, $p \in L^1([0, \infty), \mathbb{R}_+)$ *such that*

$$\|f(t, u)\| \le p(t)\psi(\|u\|_B) \ \text{for each}\ (t, u) \in [0, \infty) \times B,$$

$$\int_0^\infty \frac{du}{\psi(u)} = \infty,$$

where

$$K_k = \sup\{|K(t)| : \ t \in [0, t_k]\},\ \ k = 1, \dots .$$

If $\sum_{k=1}^{\infty} d_{k,1} < 1,$ *and* $\sum_{k=1}^{\infty} d_{k,i} < 1,\ i = 2, 3, \dots$*, then the initial value problem (5.7)-(5.10) has unique solution.*

Proof. By analogies of Theorem 3.5, it can be shown that problem (5.7)–(5.10) has unique solution. The details are left to the reader. ∎

6. EXAMPLE

In this, section we give an example to illustrate our main results.

(6.1) $$y''(t) = \frac{e^{-\gamma t}\|y_t\|_{B_\gamma}}{(t+1)(t+2)},\ a.e.\ t \in J := [0,\infty)] \setminus \{1,2,\ldots\},$$

(6.2) $$y(t_k^+) - y(t_k^-) = b_k y(t_k^-), \quad k = 1,\ldots,$$

(6.3) $$y'(t_k^+) - y'(t_k^-) = \bar{b}_k y(t_k^-), \quad k = 1,\ldots,m,$$

(6.4) $$y(t) = \phi(t),\ \ t \in (-\infty, 0].$$

Let $\mathcal{D} = \{\psi : (-\infty, 0] \to \mathbb{R}^n \mid \psi$ is continuous everywhere except for a countable number of points $\bar{t}$ at which $\psi(\bar{t}^-)$ and $\psi(\bar{t}^+)$ exist, $\psi(\bar{t}^-) = \psi(\bar{t})\}$, b_k and $\bar{b}_k, k = 1,\ldots,$ are real sequences. Let γ be a positive real constant and

$$B_\gamma = \{y \in \mathcal{D} \cap PC_* : \lim_{\theta\to-\infty} e^{\gamma\theta}y(\theta),\ \text{exists in } \mathbb{R}^n\}.$$

The norm of B_γ is given by

$$\|y\|_\gamma = \sup_{-\infty<\theta\leq 0} e^{\gamma\theta}\|y(\theta)\|.$$

Let $y : (-\infty,\infty) \to \mathbb{R}^n$ such that $y_0 \in B_\gamma$.

Then

$$\lim_{\theta\to\infty} e^{\gamma\theta}y_t(\theta) = \lim_{\theta\to\infty} e^{\gamma\theta}y(t+\theta) = e^{\gamma t}\lim_{\theta\to\infty} e^{\gamma\theta}y(\theta)e^{\gamma t}\lim_{\theta\to\infty} e^{\gamma\theta}y_0(\theta) < \infty.$$

Hence $y_t \in B_\gamma$.

Finally we prove that

$$\|y_t\| \leq K(t)\sup\{\|y(s)\| : 0 \leq s \leq t\} + M(t)\|y_0\|_\gamma,$$

where $K = M = 1,\ H = 1$, and

$\|y_t(\theta)\| = \|y(t+\theta)\|$.

If $\theta + t \leq 0$, we get

$$\|y_t(\theta)\| \leq \sup\{\|y(s)\| : -\infty < s \leq 0\}.$$

Yet, for $t + \theta \geq 0$, we have

$$\|y_t(\theta)\| \leq \sup\{\|y(s)\| : 0 < s \leq t\}.$$

Thus for all $t + \theta \in \mathbb{R}$, we get

$$\|y_t(\theta)\| \leq \sup\{\|y(s)\| : -\infty < s \leq 0\} + \sup\{\|y(s)\| : 0 \leq s \leq t\}.$$

Then

$$\|y_t\|_\gamma \leq \|y_0\|_\gamma + \sup\{\|y(s)\| : 0 \leq s \leq t\}.$$

Finally, we prove that

$$\|y(t)\| \leq H\|y_t\|_\gamma.$$

Let $y \in B_\gamma$, then

$$\|y(t)\| = e^{\gamma 0}e^{\gamma 0}\|y(t)\| \leq \sup\{\|e^{\gamma\theta}y(t+\theta)\| : \theta \in (-\infty, 0]\}.$$

Hence,

$$\|y(t)\| \leq \|y_t\|_{B_\gamma}.$$

Then, $(B_\gamma, \|\cdot\|)$ is a Banach space. We can conclude that B_γ is a phase space.

With $f(t,u) = \frac{e^{-\gamma t}\|u\|_{B_\gamma}}{(t+1)(t+2)},\ (t,u) \in [0,\infty) \times B,$

$$\|f(t,u)\| \leq p(t)\psi(\|u\|_{B_\gamma}),$$

where $\psi(x) = 1 + x$ and $p(t) = \frac{e^{-\gamma t}}{(1+t)(2+t)}$.

$$\|f(t,u)\| = \frac{e^{-\gamma t}\|u\|_{B_\gamma}}{(t+1)(t+2)} \leq p(t)[e^{-\gamma t}\|u\|_{B_\gamma} + 1] \Longrightarrow |f(t,u)| \leq p(t)[\|u|_{B_\gamma} + 1],$$

and

$$\int_0^\infty p(t)dt \leq \int_0^\infty \frac{dt}{(1+t)(2+t)} = \log 2 < \int_0^\infty \frac{dx}{\psi(x)} = \infty.$$

Let $x, y \in B_\gamma$, then we have

$$\|f(t,x) - f(t,y)\| = \frac{e^{-\gamma t}}{(t+1)(t+2)}\|\|x\|_{B_\gamma} - \|y\|_{B_\gamma}\| \leq \frac{1}{(t+1)(t+2)}\|x-y\|_{B_\gamma}.$$

Hence the conditions $(H2)$ and $(H3)$ of Theorem 3.5 are satisfied. Assume that $\sum_{k=1}^{\infty} b_k < 1,$ and $\sum_{k=1}^{\infty} \bar{b}_k < \infty$. Then by Theorem 3.5, the problem (6.1)-(6.4) has a unique solution.

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