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ON THE NUMERICAL SOLUTION FOR DECONVOLUTION PROBLEMS WITH NOISE

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ABSTRACT. In this paper three different stable methods for solving numerically deconvolution problems with noise are studied. The methods examined are the variational regularization method, the dynamical systems method, and the iterative regularized processes. Gravity surveying problem with noise is studied as a model problem. The results obtained by these methods are compared to the exact solution for the model problem. It is found that these three methods are highly stable methods and always converge to the solution even for large size models. The relative higher accuracy is obtained by using the iterative regularized processes.

Key words and phrases: Dynamical systems method, Variational regularization method, Iterative regularized processes.

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1. INTRODUCTION

It is well known that the classical example of an ill-posed problem (see [20]) is encountered in the linear Fredholm integral equation of the first kind with a square integrable kernel:

$$\int_a^b K(s,t)u(t)dt = g(s), \;\; c \le s \le d, \tag{1.1}$$

where the right-hand side g and the kernel K are given functions and u is an unknown function. In this paper, special attentions are given to the deconvolution problem which consists of solving equation of the form

$$Au := \int_0^t K(s-t)u(s)ds := f(t), \;\; 0 \le t \le T, \tag{1.2}$$

where $K(t)$ is given for all $t \ge 0$ and $A : H \to H$ be a linear, injective, bounded operator in a Hilbert space H, $A^{-1} : R(A) \to H$ is unbounded, so that the problem of solving (1.2) is ill-posed (see [17]). Deconvolution problems are important in many engineering applications, in physics and other areas (see [9]). In practice f is measured with some error, so f_δ is known, $\|f_\delta - f\| \le \delta$.

For solving (1.2) numerically one uses regularization methods combined with projection methods to find a stable approximation $u_\delta := u_{m(\delta)} \in H_m \subset H_n$, where H_n is a finite dimensional subspace of H. The problem consists of finding u_δ which solves the following equation

$$P_m A u_{m(\delta)} = P_m f_\delta \quad \text{such that} \quad lim_{\delta\to 0}\|u_{m(\delta)} - u\| = 0,$$

where P_m is the orthoprojection operators on H_m. The above equation is an equation of projection method. The usual approach to solve this projection equation is to use variational or iterative regularization. In this paper we use the Dynamical System Method (DSM) developed in [23]. Convergence rates for the variational, iterative and DSM regularization methods depend on a priori smoothness assumptions on the data (see [17] and the reference cited therein).

Now by using projection methods like Galerkin method with an orthonormal basis or quadrature method ([3] and [7]), equation (1.2) can be written as a linear system

$$A_m u_m = f. \tag{1.3}$$

Problem (1.3) is called discrete ill-posed problem if the matrix A_m is ill-conditioned, that is the condition number

$$\kappa(A_m) = \|A_m\| \, \|A_m^{-1}\| >> 1.$$

Discrete ill-posed problems arise in a variety of applications such as astronomy [4], electrocardiography [5], mathematical physics [26] and other fields. Solving linear algebraic ill-condition system (1.3) in a finite dimensional subspace H_m is also an ill-posed problem.

Our goal in this paper is to compute a stable solution to (1.3), given that noisy data f_δ such that $\|f - f_\delta\| \le \delta$. Three different stable regularization techniques will be considered in this paper. The first one is the variational regularization method (see [10], [25] and [26]) which is most common and well known technique for regularizing ill-posed problems. This method attempts to provide a good estimate of the solution of (1.3) by a solution $u_{\alpha,\delta}$ of the problem

$$\min\{\|Au - f_\delta\|^2 + \alpha\|u\|^2\}, \tag{1.4}$$

where α is the regularization parameter and $u_{\alpha,\delta}$ is the regularization solution. The success of the variational regularization method depends on making a good choice of the regularization parameter which is not easy to find. The reason is that $u_{\alpha,\delta}$ is too sensitive to perturbations in

f, i.e., a small change in f may produce a large change in $u_{\alpha,\delta}$.

The second regularization technique considered here is the dynamical systems method (DSM) which is proposed by A. G. Ramm (see [12]-[24] and the references therein). The DSM is based on an analysis of the solution of Cauchy problem for linear and nonlinear differential equations in Hilbert space. Such an analysis was done for well-posed and some ill-posed problems (see [20], and the references therein), using some integral inequalities. The stopping time is defined by using the generalization of the discrepancy principle [18].

Since iterative regularization methods are important for treating large-scale problems, the third regularization technique considered here is the iterative regularized processes, which is proposed also by A. G. Ramm ([1], [19]) to solve equation (1.3) in case when A is a closed, densely defined in H, unbounded operator. In Section 2 an overview of the variational regularization method, DSM and the iterative regularized processes is presented. In section 3 numerical experiments and comparisons between these methods are presented.

2. Overview of the methods and algorithms

2.1. **Variational regularization method.** This method (see [10], [20], [25] and [26]) consists of finding a global minimizer of (1.4), where f_δ is a noisy data and $\|f - f_\delta\| \leq \delta$. The global minimizer of the quadratic functional (1.4) is the unique solution to the linear system $(A^*A + \alpha I)u_{\alpha,\delta} = A^*f_\delta$, where I is the unit matrix. This system has a unique solution $u_{\alpha,\delta} = (A^*A + \alpha I)^{-1}A^*f_\delta$. To determine the suitable α, let $u_{\alpha(\delta),\delta}$ be a solution of (1.4) and consider the equation

$$\|Au_{\alpha,\delta} - f_\delta\| = \tau\delta, \tag{2.1}$$

where $\tau \in]1, 2[$. Equation (2.1) is the usual discrepancy principle. One can prove that equation (1.4) determines $\alpha = \alpha(\delta)$ uniquely, $\alpha(\delta) \to 0$ as $\delta \to 0$, and $u_\delta := u_{\alpha(\delta),\delta} \to y$ where y is the minimal-norm solution to (1.3) (see [20]) as $\delta \to 0$. This justifies the usual discrepancy principle for choosing the regularization parameter ([11]). For more details on the theory of variational regularization method see e.g., [20], Chapter 2.

2.2. **Analysis of the DSM methods.** In the following, we will give a brief description of the analysis of the DSM and for more details on the analysis of DSM see (see [12]-[24] and the references therein). The DSM analysis is bases on a construction of a dynamical system with the trajectory; by using Cauchy problem for nonlinear differential equations in a Hilbert space; starting from an initial approximation point and having a solution to problem (1.3) as a limiting point. It is proved in [20] that if equation (1.3) is solvable and $\|f - f_\delta\| \leq \delta$, the following results hold:

Theorem 2.1. *Assume that $f = Ay$, $y \perp N(A)$, A is a linear operator, closed and densely defined in H. Consider the problem*

$$\frac{du}{dt} = -u + T^{-1}_{\epsilon(t)}A^*f \ , \ u(0) = u_0, \tag{2.2}$$

$N(A) := \{u : Au - f = 0\}$, $u_0 \in H$ is arbitrary, $T_\epsilon = T + \epsilon(t)$, $T = A^*A$, $\epsilon = \epsilon(t)$ is a continuous function monotonically decaying to zero at $t \to \infty$ and $\int_0^\infty \epsilon(s)ds = \infty$. Then problem (2.2) has a unique solution $u(t)$ defined on $[0,\infty)$, and the following limit exists:

$$lim_{t\to\infty}u(t) := u(\infty) \quad and \quad u(\infty) = y.$$

It is pointed out in [20] *that if f_δ is given in place of the exact solution f, calculate its solution $u_\delta(t)$ as $t = t_\delta$, it can be proved that*

$$lim_{\delta\to 0}\|u_\delta(t_\delta) - y\| = 0,$$

if t_δ is suitable chosen. The stopping time t_δ can be uniquely determined, for example by a discrepancy principle, see [24], *for bounded operators A. Also, it is pointed out that the argument in* [20] *remains valid in the case of unbounded A without any essential change.*

2.3. **The dynamical systems algorithm.** The DSM algorithm can be applied as follows:

Step 1. Solve the following ordinary differential equation:

$$\frac{du}{dt} = \Phi(u,t), \quad u(0) = u_0, \tag{2.3}$$

where

$$\Phi(u,t) = -u + (A^*A + \epsilon(t))^{-1}A^*f_\delta, \quad u_0 = 0, \tag{2.4}$$

and the discretization is based on an explicit Runge-Kutta (4,5) formula which is the best formula to apply as a 'first try' for most problems, for more details see [8].

Step 2. The stopping time t_δ is defined by using the following generalization of the discrepancy principle: when $\tau \in]1,2[$, the stopping time is chosen by the formula

$$\|Au_\delta(t_\delta) - f_\delta\| = \tau\delta, \tag{2.5}$$

and we assume that

$$\tau\delta < \|Au_\delta(t) - f_\delta\| \quad \text{for all times} \quad t < t_\delta, \tag{2.6}$$

i.e., t_δ is the first moment t, at which the discrepancy is equal to $\tau\delta$. If

$$\|Au_0 - f_\delta\| > \tau\delta,$$

then formulas (2.5) and (2.6) determine uniquely $t_\delta > 0$, see [24].

2.4. **Iterative regularized processes.** The following iterative regularized processes to solve equation (1.3) with noise is proposed also by A. G. Ramm [19] in case when A is a closed, densely defined in H, unbounded operator. The iterative processes formula:

$$u_{n+1} = Bu_n + T_a^{-1}A^*f, \quad u_0 := u_0^\perp, \quad u_0^\perp \perp N(A); \quad B := aT_a^{-1}, \tag{2.7}$$

where $a = constant > 0$, and the initial element u_0 is arbitrary in the subspace $N^\perp$, $N := N(A) = N(T)$, $T = A^*A$, $T_a = T + aI$, $B \geq 0$, $\|B\| \leq 1$.

Theorem 2.2. *Assume that $f = Ay$, $y \perp N(A)$, A is a linear operator, closed and densely defined in H. Under the above assumptions it can be proved that*

$$\lim_{n\to\infty} \|u_n - y\| = 0.$$

Proof. (see [19]).

It's pointed out in [19] that the iterative process (2.7) yields stable solutions of equation (1.3), when f_δ in place of f, where f_δ is given such that $\|f - f_\delta\| \leq \delta$, the stopping rule, i.e., the number $n(\delta)$ such that $\lim_{\delta\to 0}\|u_{n,\delta} - y\| = 0$, is found for any fixed small δ as a minimizer for the problem

$$c\,(n+1)\delta + \|B^n(u_0 - y)\| = min, \quad \text{where} \quad c = \frac{1}{2\sqrt{a}}\,. \tag{2.8}$$

∎

2.5. **Iterative regularized algorithm.** This algorithm can be applied by using the following steps:
Step 1. Choose $u_0 = u_0^{\perp}, \quad u_0^{\perp} \perp N(A)$
Step 2. $u_{n+1} = Bu_n + T_a^{-1} A^* f_\delta$
Step 3. Check the stopping rule: find $n(\delta)$ for any fixed small δ as the minimizer for the problem

$$c\,(n+1)\,\delta + E(n) = min., \quad c := \frac{1}{2\sqrt{a}}$$

where $E(n) := \|B^n w\|, \quad w := u_0^{\perp} - y, \quad w \perp N, \quad B^n w = Bw_n = B(u_n - y).$

3. Application

In this section we will apply the above stable methods to solve numerically gravity surveying problem as a model problem ([27]). This model has a convolution type kernel and can be expressed in the form of equation (1.2). To explain the model, assume that a one dimensional horizontal mass distribution $u(t)$ lies at depth d below the surface from 0 to 1 on t axis; for the geometry and the location of the s and t axes see Figure 1. From measurements of the vertical component of the gravitational field, denoted $g(s)$, at the surface from 0 to 1 on s axis, it is required to compute the mass distribution, denoted $u(t)$, along the t axis, i.e., an inverse problem. The contribution to g from an infinitesimal part dt of the mass distribution at t is given by

$$dg = \frac{\sin(\theta)}{r^2} u(t)dt,$$

where $r = \sqrt{(d^2 + (s-t)^2)}$ is the distance between the two points on the s and t. Using that $sin\theta = \frac{d}{r}$, one get

$$\frac{\sin(\theta)}{r^2} u(t)dt = \frac{d}{(d^2 + (s-t)^2)^{\frac{3}{2}}} u(t)dt.$$

The total value of $g(s)$ for any s is therefore

$$g(s) = \int_0^1 \frac{d}{(d^2 + (s-t)^2)^{\frac{3}{2}}} u(t)dt.$$

Thus, we arrive at a deconvolution problem for computing the desired quantity f with kernel given by $K(s-t) = d(d^2 + (s-t)^2)^{\frac{-3}{2}}$.:

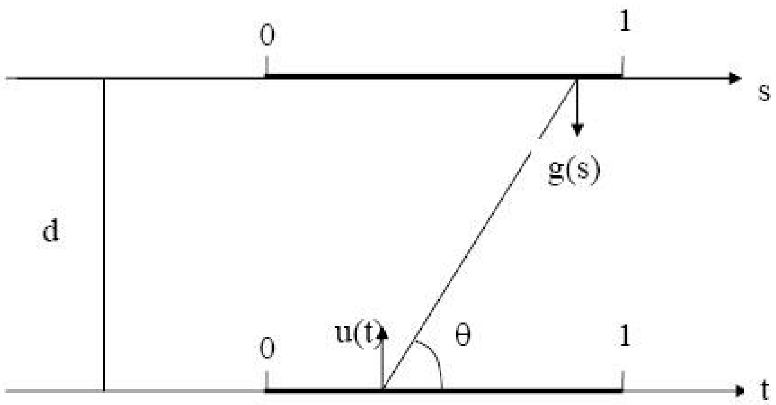

Figure 1: The geometry of gravity surveying.

3.1. **Numerical treatment.** Using a quadrature rule [2],[6] we can approximate the integral in our Fredholm integral equation as follows

$$\int_0^1 K(s,t)u(t)dt \simeq \sum_{j=1}^{n} \omega_j K(s_i,t_j)\tilde{u}(t_j) = f(s_i), \quad i,j = 1,...,n. \tag{3.1}$$

where $\tilde{u}(t_j)$ and $f(s_i)$ are the approximation of $u(t)$ and $g(s)$ respectively. When these equations (3.1) are rewritten in matrix notation, one obtains the system $A_n u_n = f$, where A_n is an $n \times n$ matrix. The elements of A_n, f and u_n are given by

$$a_{ij} = \omega_j K(s_i,t_j), \quad f_i = f(s_i), \quad u_j = \tilde{u}(t_j), \quad i,j = 1,...,n.$$

Now the midpoint rule is used to discretize the gravity surveying problem, with quadrature and collocations points equidistantly distributed in the interval $[0,1]$ as $s_i = t_i = (i-0.5)/n, \quad i = 1,...,n$. Thus, the matrix elements are given by

$$a_{ij} = \frac{\frac{d}{n}}{(d^2 + (s_i - t_j)^2)^{\frac{3}{2}}} = \frac{n^2 d}{(n^2 d^2 + (i-j)^2)^{\frac{3}{2}}}, \quad i,j = 1,...,n.$$

The exact solution of the model problem, is chosen such as $u_{orig}(t) = sin(\pi t) + 0.5 sin(2\pi t)$, and the elements of the exact solution u thus consists of the sampled values of u_n at the abscissas $t_j = (j-0.5)/n$ for $j = 1,...,n$. Finally, the right-hand side f is computed as $f = A_n u_n$. Due to our choice of quadrature and collocation points we obtain a symmetric matrix, and the depth is chosen such that $t_i \le d \le s_i$ and $d \le r$, then in the following computational work the depth d is chosen to be $d = 0.25$.

At this stage, we emphasize that in practice the right-hand side is usually a perturbed version of this f. That is, we solve the system $A_n u_{n,\delta} = f_\delta$, where $f_\delta = f + \delta$, and the vector δ represents the perturbation of the exact data.

For the numerical computations, the quadrature rule with $n = 20$, leads to a 20×20, linear ill-posed system of equations: $A_n u_n = f$, where the condition number of the matrix A_n is equal to $3.71137471\, e^5$. Perturbed the right-hand side vector f; by adding a noise term δ to the last row in f; in order to have f_δ. Let us take $\tau = 1.9$, $\delta = 0.02$, $c_0 = 0.1$, $c_1 = 0.1$, $\epsilon(t) := \frac{c_0}{c_1 + t}$.

For the iterative processes, let $a = 4, \ \delta = 0.02$,

$$u_0 = [0.001 \quad 0.002 \quad 0.003 \quad 0.004 \quad 0.005 \quad 0 \quad 0 \quad 0 \ldots 0 \quad 0]^t. \tag{3.2}$$

Table 1 shows the results obtained when the iterative regularized processes is applied to solve the gravity model problem, where the third column gives the relative error $:= \frac{\|u^{exact} - u^{approx}\|}{\|u^{exact}\|}$. Table 2 shows the results obtained by using variational regularization method with stopping rule (2.1), the DSM method with stopping rule (2.5) and the iterative regularized processes with stopping rule (2.8). The results show that the DSM is superior to variational regularization terms of accuracy. Moreover, The higher accuracy is obtained by using the iterative regularized processes with stopping rule (2.8). Also the numerical computations show that the relative error in all methods are not decaying further as the the dimension of the matrix A_n increases, because the major component in these errors come from the noise level and not from the error of the computational method. The above methods are tested also with different values of $d, d = 0.5, 0.75, 1.0$, and 1.5, it is found that although the condition number become bigger, the methods always converge to the solution even for large size 7 model problem and the iterative regularized processes, DSM is superior to variational regularization method.

Finally, it is important to mention that CPU computer time for both DSM and variational regularization, is approximately the same and it is very small, it is nearly 0.0013 minute for this problem. Also the CPU computer time for the iterative regularized processes is very small: about 0.00208 minute for 50 iteration, 0.00338 minute for 150 iteration and 0.0052 for 270 iteration, Table 1. Finally, we pointed out that the approximate solutions are obtained by using Matlab version7.

n	$c\,(n+1)\delta + E(n)$	$\frac{c(n+1)\delta+E(n)}{n}$	Relative error
5	0.255	0.051	0.0636
10	0.227	0.022	0.0487
50	0.356	0.0071	0.0297
100	0.587	0.0058	0.0256
150	0.828	0.0055	0.0235
200	1.072	0.0053	0.0224
250	1.318	0.00527	0.0221
270	1.417	0.00524	0.0222

Table1

Method	Relative error	$\alpha, t_\delta, , n(\delta)$
Variational regularization	$2.85e-2$	$\alpha = 0.03984$
DSM	$2.36e-2$	$t_\delta \;=\; 6.7$
Iterative Process	$2.22e-2$	$n(\delta) \;=\; 270$

Table 2

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