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ON SEGAL'S QUANTUM OPTION PRICING

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ABSTRACT. We apply the non-commutative extension of classical Itô stochastic calculus, known as quantum stochastic calculus, to the quantum Black-Scholes model in the sense of Segal and Segal [4]. Explicit expressions for the best quantum option price and the associated optimal quantum portfolio are derived.

Key words and phrases: Quantum stochastic calculus, Black-Scholes model, Quantum option pricing.

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1. QUANTUM STOCHASTIC CALCULUS

Let $B_t = \{B_t(\omega)/\, \omega \in \Omega\}$, where $t \geq 0$, be one-dimensional classical Brownian motion. Integration with respect to B_t was defined by Itô. Stochastic integral equations of the form

$$X_t = X_0 + \int_0^t b(s, X_s)\, ds + \int_0^t \sigma(s, X_s)\, dB_s$$

are thought of as stochastic differential equations of the form

$$dX_t = b(t, X_t)\, dt + \sigma(t, X_t)\, dB_t \tag{1.1}$$

where differentials are handled with the use of Itô's formula

$$(dB_t)^2 = dt, \quad dB_t\, dt = dt\, dB_t = (dt)^2 = 0.$$

In [1], Hudson and Parthasarathy defined a non-commutative analogue of classical Itô calculus as follows:

Definition 1.1. The Boson Fock space $\Gamma = \Gamma(L^2(\mathbb{R}_+, \mathcal{C}))$ over $L^2(\mathbb{R}_+, \mathcal{C})$ is the Hilbert space completion of the linear span of the exponential vectors $\psi(f)$ under the inner product

$$< \psi(f), \psi(g) >:= e^{<f,g>}$$

where $f, g \in L^2(\mathbb{R}_+, \mathcal{C})$ and $< f, g >= \int_0^{+\infty} \bar{f}(s)\, g(s)\, ds$. Here and in what follows, $\bar{z}$ denotes the complex conjugate of $z \in \mathbb{C}$.

The annihilation, creation and conservation operators $A(f)$, $A^\dagger(f)$ and $\Lambda(F)$ respectively, are defined on the exponential vectors $\psi(g)$ of Γ as follows.

Definition 1.2.

$$A_t \psi(g) := \int_0^t g(s)\, ds\ \psi(g), \tag{1.2}$$

$$A_t^\dagger \psi(g) := \frac{\partial}{\partial \epsilon}|_{\epsilon=0}\, \psi(g + \epsilon \chi_{[0,t]}), \tag{1.3}$$

$$\Lambda_t \psi(g) := \frac{\partial}{\partial \epsilon}|_{\epsilon=0}\, \psi(e^{\epsilon \chi_{[0,t]}} g). \tag{1.4}$$

Definition 1.3. The basic quantum stochastic differentials dA_t, $dA_t^\dagger$, and $d\Lambda_t$ are defined by

$$dA_t := A_{t+dt} - A_t, \tag{1.5}$$

$$dA_t^\dagger := A_{t+dt}^\dagger - A_t^\dagger, \tag{1.6}$$

$$d\Lambda_t := \Lambda_{t+dt} - \Lambda_t. \tag{1.7}$$

Hudson and Parthasarathy defined stochastic integration with respect to the noise differentials of Definition 1.3 and obtained the quantum stochastic Itô multiplication table:

Table 1.1: Itô table.

$\cdot$	$dA_t^\dagger$	$d\Lambda_t$	dA_t	dt
$dA_t^\dagger$	0	0	0	0
$d\Lambda_t$	$dA_t^\dagger$	$d\Lambda_t$	0	0
dA_t	dt	dA_t	0	0
dt	0	0	0	0

The two fundamental theorems of the Hudson-Parthasarathy quantum stochastic calculus listed below (see Theorems 4.1 and 4.3 of [1]), give formulas for expressing the matrix elements of quantum stochastic integrals in terms of ordinary Riemann-Lebesgue integrals. In what follows, we couple Γ with a "system" Hilbert space $\mathcal{H}$ and we let $\mathcal{E}$:=span $\{u\otimes\psi(f), u \in \mathcal{H}, \psi(f) \in \Gamma\}$ be the "exponential domain" of $\mathcal{H} \otimes \Gamma$.

Theorem 1.1. *Let*

$$M(t) = \int_0^t E(s)\, d\Lambda(s) + F(s)\, dA(s) + G(s)\, dA^{\dagger}(s) + H(s)\, ds$$

where E, F, G, H are (in general) time dependent adapted processes. Let also $u \otimes \psi(f)$ and $v \otimes \psi(g)$ be in the exponential domain of $\mathcal{H} \otimes \Gamma$. Then

$$< u \otimes \psi(f), M(t)\, v \otimes \psi(g) >=$$

$$\int_0^t < u \otimes \psi(f), \big(\bar{f}(s)\, g(s)\, E(s) + g(s)\, F(s) + \bar{f}(s)\, G(s) + H(s)\big)\, v \otimes \psi(g) > \, ds.$$

Theorem 1.2. *Let*

$$M(t) = \int_0^t E(s)\, d\Lambda(s) + F(s)\, dA(s) + G(s)\, dA^{\dagger}(s) + H(s)\, ds$$

and

$$M'(t) = \int_0^t E'(s)\, d\Lambda(s) + F'(s)\, dA(s) + G'(s)\, dA^{\dagger}(s) + H'(s)\, ds$$

where E, F, G, H, E', F', G', H' are (in general) time dependent adapted processes. Let also $u \otimes \psi(f)$ and $v \otimes \psi(g)$ be in the exponential domain of $\mathcal{H} \otimes \Gamma$. Then

$$< M(t)\, u \otimes \psi(f), M'(t)\, v \otimes \psi(g) >=$$

$$\begin{gathered}\int_0^t \{< M(s)\, u \otimes \psi(f), (\bar{f}(s)\, g(s) E'(s) + g(s)\, F'(s) + \bar{f}(s)\, G'(s) \\ + H'(s)) v \otimes \psi(g) > + < (\bar{g}(s)\, f(s)\, E(s) + f(s)\, F(s) + \bar{g}(s)\, G(s) \\ + H(s)) u \otimes \psi(f), M'(s)\, v \otimes \psi(g) > + < (f(s) E(s) + G(s)) u \otimes \psi(f), \\ (g(s) E'(s) + G'(s)) v \otimes \psi(g) >\}\, ds.\end{gathered}$$

The fundamental result which connects classical with quantum stochastics is that the processes B_t and P_t defined by

$$B_t = A_t + A_t^{\dagger}$$

and

$$P_t = \Lambda_t + \sqrt{\lambda}(A_t + A_t^{\dagger}) + \lambda t$$

are identified through their vacuum characteristic functionals

$$< \psi(0), e^{i\, s\, B_t}\, \psi(0) >= e^{-\frac{s^2}{2}\, t}$$

and

$$< \psi(0), e^{i\, s\, P_t}\, \psi(0) >= e^{\lambda\, \left(e^{i\, s} - 1\right)\, t}$$

with classical Brownian motion and the Poisson process of intensity $\lambda > 0$ respectively.

Within the framework of Hudson-Parthasarathy Quantum Stochastic Calculus, classical quantum mechanical evolution equations take the form

$$\begin{aligned} dU_t &= -((iH + \frac{1}{2}\, L^* L)\, dt + L^*\, W\, dA_t - L\, dA_t^{\dagger} \\ &+ (1 - W)\, d\Lambda_t) U_t \end{aligned} \tag{1.8}$$

with $U_0 = 1$, where, for each $t \geq 0$, U_t is a unitary operator defined on the tensor product $\mathcal{H} \otimes \Gamma(L^2(\mathbb{R}_+, \mathcal{C}))$ of the system Hilbert space $\mathcal{H}$ and the noise (or reservoir) Fock space Γ. Here

H, L, W are in $\mathcal{B}(\mathcal{H})$, the space of bounded linear operators on $\mathcal{H}$, with W unitary and H self-adjoint. Notice that for $L = W = -1$ equation (1.8) reduces to a classical SDE of the form (1.1). Here and in what follows we identify time-independent, bounded, system space operators X with their ampliation $X \otimes 1$ to $\mathcal{H} \otimes \Gamma(L^2(\mathbb{R}_+, \mathcal{C}))$. The quantum stochastic differential equation (analogue of the Heisenberg equation for quantum mechanical observables) satisfied by the flow

$$j_t(X) = U_t^* \, X \, U_t$$

where X is a bounded system space operator, is (see [3])

$$\begin{aligned} \text{(1.9)} \quad dj_t(X) &= j_t \left(i \, [H, X] - \frac{1}{2} \left(L^* L X + X L^* L - 2 L^* X L \right) \right) dt \\ &+ j_t \left([L^*, X] \; W \right) dA_t + j_t \left(W^* \; [X, L] \right) dA_t^\dagger + j_t \left(W^* \, X \, W - X \right) d\Lambda_t \end{aligned}$$

with $j_0(X) = X, \quad t \in [0, T]$ where $[x, y] := xy - yx$ is the usual commutator.

2. SEGAL'S OPTION PRICING MODEL

In recent years the fields of Quantum Economics and Quantum Finance have appeared in order to interpret erratic stock market behavior with the use of quantum mechanical concepts as in [4]. Within the framework of Hudson-Parthasarathy quantum stochastic calculus, the stock process $\{X_t \, / \, t \geq 0\}$ of the classical Black-Scholes theory is replaced by the quantum mechanical process $j_t(X) = U_t^* \, X \otimes 1 \, U_t$ where, for each $t \geq 0$, U_t is a unitary operator defined on the tensor product $\mathcal{H} \otimes \Gamma(L^2(\mathbb{R}_+, \mathcal{C}))$ of a system Hilbert space $\mathcal{H}$ and the noise Boson Fock space $\Gamma = \Gamma(L^2(\mathbb{R}_+, \mathcal{C}))$. We assume that U_t satisfies the quantum stochastic differential equation

$$dU_t = -((iH + \frac{1}{2} \, L^* L) \, dt + L^* \, dA_t - L \, dA_t^\dagger) \, U_t, \quad U_0 = 1$$

where $X > 0$, H, L, are in $\mathcal{B}(\mathcal{H})$, the space of bounded linear operators on $\mathcal{H}$, with X and H self-adjoint. The value process V_t is defined for $t \in [0, T]$ by

$$V_t = a_t \, j_t(X) + b_t \, \beta_t$$

with terminal condition

$$V_T = (j_T(X) - K)^+ = \max(0, j_T(X) - K)$$

where $K > 0$ is a bounded self-adjoint system space operator corresponding to the strike price of the quantum option, a_t is a real-valued function, b_t is in general an observable quantum stochastic processes (i.e b_t is a self-adjoint operator for each $t \geq 0$) and

$$\beta_t = \beta_0 \, e^{t \, r}$$

where β_0 and r are positive real numbers. Therefore

$$b_t = (V_t - a_t \, j_t(X)) \, \beta_t^{-1}.$$

We interpret the above in the sense of expectation i.e given $u \otimes \psi(f)$ in the exponential domain of $\mathcal{H} \otimes \Gamma$, where we will always assume $u \neq 0$ so that $\| u \otimes \psi(f) \| \neq 0$,

$$< u \otimes \psi(f), V_t \, u \otimes \psi(f) >= a_t \; < u \otimes \psi(f), j_t(X) \, u \otimes \psi(f) >$$

$$+ < u \otimes \psi(f), b_t \, u \otimes \psi(f) > \; \beta_t$$

i.e the value process is always in reference to a particular quantum mechanical state and

$$< u \otimes \psi(f), V_T \, u \otimes \psi(f) >= \max(0, < u \otimes \psi(f), (j_T(X) - K) \, u \otimes \psi(f) >).$$

As in the classical case we assume that the portfolio $(a_t, b_t), t \in [0,T]$ is self -financing i.e

$$dV_t = a_t\, dj_t(X) + b_t\, d\beta_t$$

which implies

$$da_t \cdot j_t(X) + da_t \cdot dj_t(X) + db_t \cdot \beta_t + db_t \cdot d\beta_t = 0.$$

By the Quantum Itô table of Section 1, and the homomorphism property $j_t(x\,y) = j_t(x)\, j_t(y)$, we obtain

$$dj_t(X) = j_t(\alpha^\dagger)\, dA_t^\dagger + j_t(\alpha)\, dA_t + j_t(\theta)\, dt,$$
$$(dj_t(X))^2 = j_t(\alpha\, \alpha^\dagger)\, dt$$

while for $k \geq 2$, $(dj_t(X))^k = 0$. Here, and in what follows,

$$\begin{aligned} \alpha &= [L^*, X],\ \alpha^\dagger = [X, L] \\ \theta &= i\,[H, X] - \frac{1}{2}\,\{L^*\,L\,X + X\,L^*\,L - 2\,L^*\,X\,L\}. \end{aligned}$$

In the above framework, let $V_t := F(t, j_t(X))$ where $F : [0,T] \times \mathcal{B}(\mathcal{H} \otimes \Gamma) \longrightarrow \mathcal{B}(\mathcal{H} \otimes \Gamma)$ is the extension to self-adjoint operators $x = j_t(X)$ of the analytic function

$$F(t,x) = \sum_{n,k=0}^{+\infty} a_{n,k}(t_0, x_0)\, (t - t_0)^n\, (x - x_0)^k$$

where x and $a_{n,k}(t_0, x_0)$ are in $\mathbb{C}$, and for $\lambda, \mu \in \{0, 1, ...\}$

$$F_{\lambda\,\mu}(t,x) := \frac{\partial^{\lambda+\mu} F}{\partial t^\lambda\, \partial x^\mu}(t,x).$$

Notice that if 1 denotes the identity operator then

$$a_{n,k}(t_0, x_0) = a_{n,k}(t_0, x_0)\, 1 = \frac{1}{n!\,k!}\, F_{n\,k}(t_0, x_0).$$

Moreover for $(t_0, x_0) = (0,0)$ we have

$$V_t = \sum_{n,k=0}^{+\infty} a_{n,k}(0,0)\, t^n\, j_t(X)^k = \sum_{n,k=0}^{+\infty} a_{n,k}(0,0)\, t^n\, j_t(X^k).$$

By the Quantum Itô table

$$\begin{gathered} dV_t = (a_{1,0}(t, j_t(X)) + a_{0,1}(t, j_t(X))\, j_t(\theta) + a_{0,2}(t, j_t(X))\, j_t(\alpha\,\alpha^\dagger))\, dt \\ + a_{0,1}(t, j_t(X))\, j_t(\alpha^\dagger)\, dA_t^\dagger + a_{0,1}(t, j_t(X))\, j_t(\alpha)\, dA_t \end{gathered}$$

while using the self-financing property we obtain

$$dV_t = (a_t\, j_t(\theta) + V_t\, r - a_t\, j_t(X)\, r)\, dt + a_t\, j_t(\alpha^\dagger)\, dA_t^\dagger + a_t\, j_t(\alpha)\, dA_t.$$

Equating the coefficients of dt and the quantum stochastic differentials in the two expressions for dV_t we obtain

$$\begin{gathered} a_{1,0}(t, j_t(X)) + a_{0,1}(t, j_t(X))\, j_t(\theta) + a_{0,2}(t, j_t(X))\, j_t(\alpha\,\alpha^\dagger) \\ = a_t\, j_t(\theta) + V_t\, r - a_t\, j_t(X)\, r \end{gathered}$$

and

$$a_{0,1}(t, j_t(X)) = a_t$$

By combining the above two equations and simplifying we obtain

$$a_{1,0}(t, j_t(X)) + a_{0,2}(t, j_t(X))\, j_t([L^*, X]\,[X, L]) + a_{0,1}(t, j_t(X))\, j_t(X)\, r - V_t\, r = 0$$

which can be written as

$$F_{1\,0}(t, j_t(X)) + \frac{1}{2}\, F_{0\,2}(t, j_t(X))\, j_t([L^*, X]\,[X, L]) + F_{0\,1}(t, j_t(X))\, j_t(X)\, r$$
$$= F(t, j_t(X))\, r$$

with $F(T, j_T(X)) = (j_T(X) - K)^+$. Letting $x = j_t(X)$, $y = j_t(L)$ be arbitrary elements in $\mathcal{B}(\mathcal{H} \otimes \Gamma)$ and $g(x) = [y^*, x]\,[x, y]$, $h(x) = x\, r$, we obtain

$$F_{1\,0}(t, x) + \frac{1}{2}\, F_{0\,2}(t, x)\, g(x) + F_{0\,1}(t, x)\, h(x) = F(t, x)\, r.$$

Letting

$$\begin{aligned} u(t, x) &= F(T - t, x), \\ u_{1\,0}(t, x) &= -F_{1\,0}(T - t, x), \\ u_{0\,2}(t, x) &= F_{0\,2}(T - t, x), \\ u_{0\,1}(t, x) &= F_{0\,1}(T - t, x) \end{aligned}$$

we obtain the Quantum Black-Scholes Equation

$$u_{1\,0}(t, x) = \frac{1}{2}\, u_{0\,2}(t, x)\, g(x) + u_{0\,1}(t, x)\, h(x) - u(t, x)\, r \tag{2.1}$$

with

$$u(0, j_T(X)) = (j_T(X) - K)^+.$$

To solve the Quantum Black-Scholes Equation we assume that

$$j_t(X^2) = j_t([L^*, X]\,[X, L])$$

which implies that $[X, L] = W\, X$ and $[L^*, X] = X\, W^*$ where W is an arbitrary unitary operator acting on the system space. The Quantum Black-Scholes Equation (2.1) now takes the form

$$u_{1\,0}(t, x) = \frac{1}{2}\, u_{0\,2}(t, x)\, x^2 + u_{0\,1}(t, x)\, x\, r - u(t, x)\, r \tag{2.2}$$

where we may assume that x is a bounded self-adjoint operator. At $(0, 0)$

$$u(t, x) = F(T - t, x) = \sum_{n,k=0}^{+\infty} a_{n,k}(0, 0)\, (T - t)^n\, x^k$$

and, since $x = j_t(X) > 0$ and K are invertible, we may let $x = K\, e^z$ where z is a bounded self-adjoint operator commuting with K. Letting

$$\omega(t, z) := u(t, K\, e^z) = \sum_{n,k=0}^{+\infty} a_{n,k}(0, 0)\, (T - t)^n\, (K\, e^z)^k,$$

and using

$$\omega_{0\,2}(t, z) - \omega_{0\,1}(t, z) = u_{0\,2}(t, x)\ x^2$$

we obtain

$$\omega_{1\,0}(t, z) = \frac{1}{2}\, \omega_{0\,2}(t, z) + \omega_{0\,1}(t, z)\, (r - \frac{1}{2}) - \omega(t, z)\, r \tag{2.3}$$

with

$$\omega(0, z_T) = (j_T(X) - K)^+$$

where z_T is defined by $K\, e^{z_T} = j_T(X)$. The quantum analogue of the classical Black-Scholes option pricing model is as follows:

Theorem 2.1. *The solution of (2.3) is given by*

$$\omega(t,z) = K\,e^z\;\Phi(g(t,K\,e^z)) - K\,\Phi(h(t,K\,e^z))\,e^{-r\,t}$$

where

$$\begin{aligned} g(t,K\,e^z) &= z\,t^{-1/2} + (r+0.5)\,t^{1/2} \\ h(t,K\,e^z) &= z\,t^{-1/2} + (r-0.5)\,t^{1/2} \\ \Phi(x) &= \frac{1}{2} + \frac{1}{\sqrt{2\,\pi}}\sum_{n=0}^{+\infty}\frac{(-1)^n}{2^n\,n!}\,\frac{x^{2\,n+1}}{2\,n+1}. \end{aligned}$$

Moreover, a reasonable price for a quantum option is $\omega(T,z_0)$ *where* z_0 *is defined by* $X = K\,e^{z_0}$. *The associated quantum portfolio* (a_t,b_t) *is given by*

$$\begin{aligned} a_t &= \omega_{0\,1}(t-T,z_t) \\ b_t &= (\omega(T-t,z_t) - a_t\,j_t(X))\,e^{-t\,r}\,{\beta_0}^{-1} \end{aligned}$$

where z_t *is defined by* $j_t(X) = K\,e^{z_t}$.

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