The Australian Journal of Mathematical Analysis and Applications

AJMAA

Volume 4, Issue 2, Article 18, pp. 1-9, 2007

AN APPROXIMATION OF JORDAN DECOMPOSABLE FUNCTIONS FOR A LIPSCHITZ FUNCTION

IBRAHEEM ALOLYAN

Received 21 March, 2007; accepted 14 November, 2007; published 30 November, 2007.
Communicated by: L. Leindler

MATHEMATICS DEPARTMENT, COLLEGE OF SCIENCE, KING SAUD UNIVERSITY P.O.BOX: 2455, RIYADH 11451, SAUDI ARABIA.
ialolyan05@yahoo.com.

ABSTRACT. The well known Jordan decomposition theorem gives the useful characterization that any function of bounded variation can be written as the difference of two increasing functions. Functions which can be expressed in this way can be used to formulate an exclusion test for the recent Cellular Exclusion Algorithms for numerically computing all zero points or the global minima of functions in a given cellular domain [2, 8, 9]. In this paper we give an algorithm to approximate such increasing functions when only the values of the function of bounded variation can be computed. For this purpose, we are led to introduce the idea of ϵ-increasing functions. It is shown that for any Lipschitz continuous function, we can find two ϵ-increasing functions such that the Lipschitz function can be written as the difference of these functions.

Key words and phrases: Bounded variation, Jordan decomposition, ϵ-increasing.

2000 *Mathematics Subject Classification.* 26A45, 26A48.

ISSN (electronic): 1449-5910

Supported by the research center project # (Math/2007/55).

1. INTRODUCTION

In the theorem of optimization, we have the problem of finding the global minimum of a function defined on a compact domain. In order to solve this problem, Cellular Exclusion Algorithms is used for numerically computing the global minima of functions in a given domain [2, 8, 9]. These algorithms formulate an exclusion test to discard cells that does not contain the global minimum. One of the tests, that are used in Exclusion Algorithms, can be applied to functions that can be written as the difference of two increasing functions [2].

The exclusion test in [2] was applied only to polynomials because they can be easily written as the difference of two increasing polynomials. However, if the function is not a polynomial, we find ourselves in some need of the concept of ϵ-increasing condition which is weaker than increasing condition. As a consequence, we have got a solution to the optimization problem as shown in Theorem 3.1.

The well known Jordan decomposition theorem gives the useful characterization that any function of bounded variation can be written as the difference of two increasing functions. In practice, these functions can not be computed explicit. Therefore, we approximate these two functions and then prove that these functions are ϵ-increasing in Theorem 3.1.

The notion of functions of bounded variation plays a very significant and important role in the theory of real functions [1, 5], numerical analysis [3, 4] and optimization [8]. In the literature, several properties of these functions have been discussed (see for example [1, 6, 7, 8, 10]); nevertheless, we focus our attention to one of these properties known as *Jordan Decomposition Theorem* JDT.

Decomposable functions, which result from JDT, plays an important role in optimization [2, 8]. For example, the Exclusion Algorithm uses decomposable functions as a test function for the minimization condition [2, 9].

Let $f : [a, b] \to \mathbb{R}$ be a function and $\mathscr{P} := \{x_i \in [a, b] : a = x_0 < x_1 < \ldots < x_{m+1} = b\}$ be a partition of $[a, b]$. We recall that the *variation* of f over $\mathscr{P}$ is the nonnegative real number

$$V_{\mathscr{P}}[f; a, b] = \sum_{i=1}^{m+1} |f(x_i) - f(x_{i-1})|.$$

The function f is a function of bounded variation if there exists a number M such that for every partition $\mathscr{P}$ of $[a, b]$, we have

$$\sum_{i=1}^{m+1} |f(x_i) - f(x_{i-1})| \leq M.$$

The total variation of f on $[a, b]$ is defined to be the number

$$V_f[a, b] := \sup_{\mathscr{P} \,|[a,b]} \sum_{i=1}^{m+1} |f(x_i) - f(x_{i-1})|, \tag{1.1}$$

where $\mathscr{P} \,|[a, b]$ means "$\mathscr{P}$ is a partition of $[a, b]$" [7]. For simplicity, we will write $\sup_{\mathscr{P}}$ instead of $\sup_{\mathscr{P} \,|[a,b]}$. The set of all functions of bounded variation on $[a, b]$ is denoted by $\mathbb{BV}[a, b]$ and we have the following proposition which follows immediately from the definition of functions of bounded variation.

Proposition 1.1. [7] *If* $f : [a, b] \to \mathbb{R}$ *is a* Lipschitz function *on* $[a, b]$ *(i.e., there is a constant* C *such that*

$$|f(x) - f(y)| \leq C|x - y| \quad \forall x, y \in [a, b]), \tag{1.2}$$

then $f \in \mathbb{BV}[a, b]$.

If we view the sum in (1.1) as a sum of positive and negative parts of the differences $f(x_i) - f(x_{i-1})$, then we can define $P_f[a, b]$ to be the summation of the positive parts of $f(x_i) - f(x_{i-1})$ and $N_f[a, b]$ to be the summation of the negative parts, i.e.,

$$
\begin{aligned}
P_f[a, b] &:= \sup_{\mathscr{P}} \sum_{i=1}^{m+1} (f(x_i) - f(x_{i-1}))^+, \\
N_f[a, b] &:= \sup_{\mathscr{P}} \sum_{i=1}^{m+1} (f(x_i) - f(x_{i-1}))^-,
\end{aligned} \tag{1.3}
$$

where $x^+ := \max\{0, x\}$ and $x^- := \max\{0, -x\}$, then we have

$$
\begin{aligned}
V_f[a, b] &= P_f[a, b] + N_f[a, b], \\
f(b) - f(a) &= P_f[a, b] - N_f[a, b].
\end{aligned}
$$

Varying b in (1.3) we get two functions $P_f[a, .], N_f[a, .] : [a, b] \to \mathbb{R}^+ \cup \{0\}$ defined by

$$
\begin{aligned}
P_f[a, x] &:= \sup_{\mathscr{P}|[a,x]} \sum_{i=1}^{m+1} (f(x_i) - f(x_{i-1}))^+, \\
N_f[a, x] &:= \sup_{\mathscr{P}|[a,x]} \sum_{i=1}^{m+1} (f(x_i) - f(x_{i-1}))^-.
\end{aligned} \tag{1.4}
$$

It can be shown that these two functions are increasing on the interval $[a, b]$. If we take $p_J(x) = P_f[a, x] + f(a)$ and $n_J(x) = N_f[a, x]$ as Jordan functions then we have the following theorem.

Theorem 1.1 (Jordan Decomposition)**.** [7] *If f is a function of bounded variation on $[a, b]$ then f can be written as the difference of two increasing functions*

$$f(x) = p_J(x) - n_J(x).$$

This theorem states that we can write f as the difference of two increasing functions where each function can be computed by finding the supremum among all partitions. However, the supremum sum over all partitions cannot be computed numerically. Therefore, we approximate the functions p_J and n_J by considering the uniform partition $\mathscr{P} := \{a + i(x - a)/(m + 1)\}_{i=0}^{m+1}$ of the interval $[a, x]$, where $m \in \mathbb{N}$ and then study the consequences of this approximation.

In section 2, we explain the need of defining ϵ-increasing functions. In Section 3, we write our algorithm to approximate p_J and n_J; furthermore, we state and prove Theorem 3.1 for the functions p_m and n_m resulting from Algorithm 3.1.

2. NEED OF ϵ-INCREASING DEFINITION

In order to approximate the functions p_J and n_J in Theorem 1.1, we use the uniform partition $\mathscr{P} := \{a + i(x - a)/(m + 1)\}_{i=0}^{m+1}$ of $[a, x]$, where $m \in \mathbb{N}$. Then we define $P_m(x)$ and $N_m(x)$

to be

$$(2.1)\qquad \begin{aligned} P_m(x) &:= \sum_{i=1}^{m+1} (f(x_i) - f(x_{i-1}))^+, \\ N_m(x) &:= \sum_{i=1}^{m+1} (f(x_i) - f(x_{i-1}))^-. \end{aligned}$$

These two functions approximate the functions $P_f[a,x]$ and $N_f[a,x]$, respectively. Moreover, P_m and N_m approach $P_f[a,x]$ and $N_f[a,x]$ as $m \to \infty$. Unfortunately, P_m and N_m are not guaranteed to be increasing if the function f is not monotone. We prove this in the following proposition.

Proposition 2.1. *If the function $f : [a,b] \to \mathbb{R}$ is not monotone on $[a,b]$, then no m can be chosen so that the functions P_m and N_m in (2.1) are increasing with respect to the uniform partition $\mathscr{P} := \{a + i(x-a)/(m+1)\}_{i=0}^{m+1}$.*

Proof. Suppose first that f is increasing on the interval $[a,c]$ and decreasing on the interval $[c,d]$ for $c, d \in (a,b]$ and $c < d$. In this case, the function P_m is increasing on $[a,c]$; nevertheless, if we let $e = \min\{c + (c-a)/m, d\}$ and consider the interval $\mathscr{I} = (c,e)$, then $P_m(x)$ is less than $P_m(c)$ for all $x \in \mathscr{I}$. In order to prove that, let x be any point in (c,e), and $\mathscr{P}$ be the uniform partition of $[a,x]$, then $f(x) < f(c)$ and $x_m = a + m(x-a)/(m+1) \in (a + m(c-a)/(m+1), c)$ and we have

$$\begin{aligned} P_m(x) &= \sum_{i=1}^{m+1} (f(x_i) - f(x_{i-1}))^+ \\ &= f(x_m) - f(x_0) + (f(x) - f(x_m))^+ \\ &< f(x_m) - f(x_0) + f(c) - f(x_m) \\ &= f(c) - f(x_0) = P_m(c). \end{aligned}$$

Therefore, the function P_m is not increasing on (c,e). If the function f is decreasing on the interval $[a,j]$ and increasing on the interval $[j,k]$ for $j, k \in (a,b]$ and $j < k$ then the function N_m is not increasing on the interval (j,k) by the same argument that was done for P_m in the first case. ∎

We define next the concept of oscillation of a function.

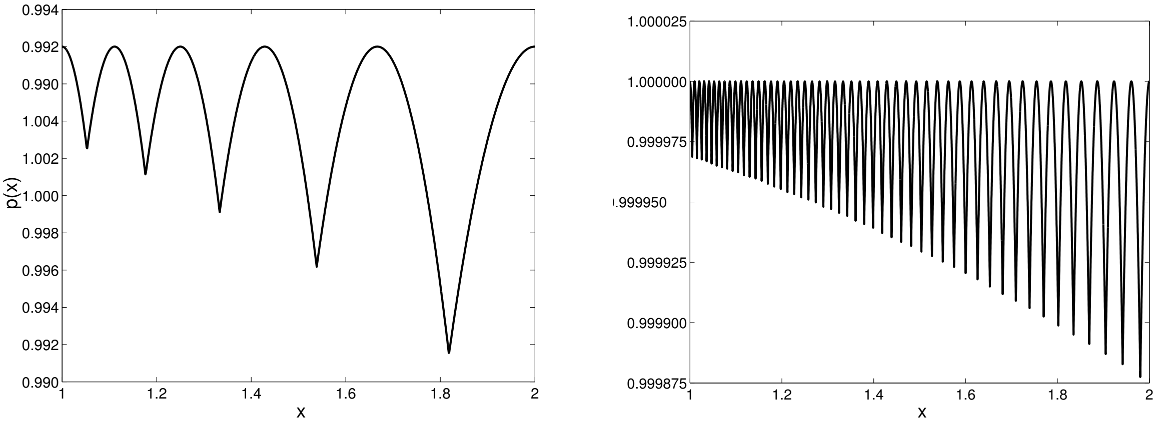
Definition 2.1 (Oscillation)**.** We say that a continuous function $f : [a,b] \to \mathbb{R}$ *oscillates* k times on the interval $[a,b]$ if there are exactly k points $s_1, s_2, \ldots, s_k \in (a,b)$ such that for all $i = 1, \ldots, k$, the value $f(s_i)$ is either a strict local maximum or a strict local minimum. If the function f has an infinite number of maximum and minimum points on (a,b), we say that f oscillates infinitely often.

For example, the function $f : [-4,4] \to \mathbb{R}$, defined by $f(x) = x(x-1)(x-2)(x-3)$, oscillates 3 times and the function $g : (0,1) \to \mathbb{R}$, defined by $g(x) = x\sin(1/x)$, oscillates infinitely often. In the following example, the function f is smooth and oscillates only one time; nevertheless, the function P_m in this case is not increasing for any choice of m.

Example 2.1. Let $f : [0,2] \to \mathbb{R}$ be defined by $f(x) := 1 - (1-x)^2$. In theory, this function can be written as the difference of two increasing functions, e.g., $p_J(x) = 2x$ and $n_J(x) = x^2$. In order to compute p_m and n_m numerically, we notice that this function is increasing on the interval $[0,1]$; therefore, the value of $P_m(x)$ will increase on this interval for any choice of m.

Nevertheless, on the interval $(1, (m+1)/m)$; P_m will take values less than $P_m(1) = 1$. To explain this, let x be any point in $(1, (m+1)/m)$, and $\mathscr{P} := \{ix/(m+1)\}_{i=0}^{m+1}$ be the uniform partition of $[0, x]$, then $f(x) < f(1)$ and $mx/(m+1) \in (m/(m+1), 1)$ and we have

$$\begin{aligned}
P_m(x) =& \sum_{i=1}^{m+1} (f(x_i) - f(x_{i-1}))^+ \\
=& f(mx/(m+1)) - f(0) + (f(x) - f(mx/(m+1)))^+ \\
<& f(mx/(m+1)) - f(0) + (f(1) - f(mx/(m+1))) \\
=& f(1) - f(0) = 1 = P_m(1).
\end{aligned}$$



(a) m = 10 (b) m = 100

Figure 1: The function $P_m(x)$ in Example 2.1

Therefore, the function P_m is not increasing on $(1, (m+1)/m)$. In fact, $P_m(x)$ equals the value $1 = P_m(1)$ if there is i, $1 \le i \le m+1$, such that $ix/(m+1) = 1$, i.e., the point 1 is one of mesh points. Solving for x, we get $x = (m+1)/i$; therefore, $P_m(x)$ equals one if $x = (m+1)/(m+1) = 1$, $x = (m+1)/m$, $x = (m+1)/(m-1), \ldots$ or $x = m+1$. In Figure 1, we plot the function p_m on the interval $[1, 2]$ for $m = 10$ and 100. We can see from the plots that for $m = 10$, we have $P_m(x) = 1$ on the interval $[1, 2]$ for the values $x = 1, 10/9, 10/8, 10/7, 10/6,$ and $10/5$; however, for all other values; P_m is strictly less than one. For $m = 100$, the function P_m oscillates more because it equals one at $1, 100/99, 100/98, \ldots, 100/50$ and it is strictly less than one for all other values on $[1, 2]$. Although the functions in Figure 1 increase in oscillations, the amplitude decreases as m increases.

Proposition 2.1 and Example 2.1 show that m cannot be chosen in such a way that P_m and N_m are always increasing. In fact,the functions P_m and N_m will oscillate in a small ϵ-neighborhood of $f(c)$. As m increases, the number ϵ becomes smaller and smaller. This idea led us to introduce a new definition that we call ϵ-increasing and we will use it in the Decomposition Algorithm.

Definition 2.2 (ϵ-Increasing)**.** We say that the function $f : [a, b] \to \mathbb{R}$ is ϵ-increasing, where ϵ is a nonnegative number, if $f(x) \le f(y) + \epsilon$ for all $x < y$ and $x, y \in [a, b]$.

If a function f is increasing, then $f(x) \le f(y)$ for all $x \le y$. This implies that $f(x) \le f(y) + \epsilon$ for all $\epsilon \ge 0$; therefore, the function f is ϵ-increasing.

If a function f is ϵ_1-increasing, then f is ϵ-increasing for all $\epsilon \geq \epsilon_1$. For example, the function $f : [-10^{-2}, \infty) \to \mathbb{R}$, defined by $f(x) = x^2$ is 10^{-4}-increasing because $f(x) \leq f(y) + 10^{-4}$ for all $x \leq y$. In fact, f is ϵ-increasing for all $\epsilon \geq 10^{-4}$. However, this function is not increasing because $f(-10^{-2}) > f(0)$.

Similarly, we can define the concept of what we call ϵ-decreasing for decreasing function of some perturbation.

3. DECOMPOSITION ALGORITHM

In this section, we introduce the *Decomposition Algorithm* to compute the functions p_m and n_m that approximate p_J and n_J, respectively. These functions are computed with respect to the uniform partition $\pi := \{a + i(x-a)/(m+1)\}_{i=0}^{m+1}$ of the interval $[a, x]$ where m is the number of points between a and x. Consequently, the functions p_m and n_m will not be always increasing. We showed in Example 2.1 that p_m and n_m cannot be increasing on $[a, b]$ even for smooth functions and for any value of m. However, we prove in Theorem 3.1 that the resulting functions, p_m and n_m, are ϵ-increasing under some assumptions. In this algorithm we define p_m and n_m to be

$$\begin{aligned} p_m(x) &:= f(a) + \sum_{i=1}^{m+1} (f(x_i) - f(x_{i-1}))^+, \\ n_m(x) &:= \sum_{i=1}^{m+1} (f(x_i) - f(x_{i-1}))^-. \end{aligned} \tag{3.1}$$

Algorithm 3.1. INPUT: a, f and x.
OUTPUT: p and n evaluated at x.
choose m (number of points between a and x).

```
d := (x − a)/(m + 1).
for i = 0 : m + 1
    x_i := a + id.
p = f(a).
n = 0.
for i = 1 : m + 1
    if s = f(x_i) − f(x_{i−1}) ≥ 0
        p = p + s.
    else
        n = n + s.
```

Note that we can choose $m + 1 = 2^k$ to decrease the number of computations when we increase m (i.e., if we choose $m + 1 = 2^{k+1}$ then we already have computed the the values $f(x_i)$ for i is even). The main question about this algorithm is how to compute m such that p and n are increasing. Unfortunately, m cannot always be computed even for smooth functions.

We have discussed in Proposition 1.1 that each Lipschitz function on $[a, b]$ is a function of bounded variation in this interval. In our next discussion, we choose the space of Lipschitz functions because in this space, the value $|f(x_i) - f(x_{i-1})|$ can be controlled by $|x_i - x_{i-1}|$. We come now to the main theorem in this paper which shows that p_m and n_m in Algorithm 3.1 have, in fact, a special property that can be exploited numerically.

Theorem 3.1. *If $f : [a, b] \to \mathbb{R}$ is a Lipschitz function with a Lipschitz constant C, then for all $\epsilon > 0$ there exists an $m := m_\epsilon \in \mathbb{N}$ such that p_m and n_m in Algorithm 3.1 are ϵ-increasing.*

In order to prove this theorem, we need to prove the following propositions. We will always use the partitions $\{x_i\}_{i=0}^{m+1}$ and $\{y_i\}_{i=0}^{m+1}$ to be the uniform partitions of $[a, x]$ and $[a, y]$, respectively.

Lemma 3.1. (1) *If f is increasing on $[a, b]$ then p_m is increasing and we have $p_m(x) = f(x)$ and $n_m(x) = 0$ for all $x \in [a, b]$.*
(2) *If f is decreasing on $[a, b]$ then n_m is increasing and we have $p_m(x) = f(a)$ and $n_m(x) = f(a) - f(x)$ for all $x \in [a, b]$.*

Proof. This follows immediately from the way p_m and n_m are constructed in Algorithm 3.1 ∎

In the following discussion, we will consider the function p_m and prove that this function is ϵ-increasing. A similar argument can be made to prove that the function n_m is also ϵ-increasing.

Proposition 3.1. *For fixed $x \in [a, b]$, the sequences $(p_m(x))$, and $(n_m(x))$ converge to $p_J(x)$ and $n_J(x)$, respectively. In other words, the sequences (p_m), and (n_m) converge pointwise to p_J and n_J, respectively.*

Proof. The proof of this proposition is straight forward. It suffices to note that

$$\lim_{m\to\infty} \sum_{i=1}^{m+1} (f(x_i) - f(x_{i-1}))^+ = \sup_{\mathscr{P}\,|[a,b]} \sum_{i=1}^{m+1} (f(x_i) - f(x_{i-1}))^+,$$

and

$$\lim_{m\to\infty} \sum_{i=1}^{m+1} (f(x_i) - f(x_{i-1}))^- = \sup_{\mathscr{P}\,|[a,b]} \sum_{i=1}^{m+1} (f(x_i) - f(x_{i-1}))^-.$$

∎

Proposition 3.2. *If $x < y$, then for any $\epsilon > 0$, there exists an $m \in \mathbb{N}$ such that*

$$|p_m(x) - p_m(y)| < 4C(m + 2)(y - x), \tag{3.2}$$

$$|n_m(x) - n_m(y)| < 4C(m + 2)(y - x). \tag{3.3}$$

Proof. We prove (3.2) and notice that a similar argument holds for (3.3).

$$\begin{aligned}
\big|p_m(x) - p_m(y)\big| &= \sum_{i=1}^{m+1} \big[(f(x_i) - f(x_{i-1}))^+ - (f(y_i) - f(y_{i-1}))^+\big] \\
&\le \sum_{i=1}^{m+1} \big|f(x_i) - f(x_{i-1})\big| + \big|f(y_i) - f(y_{i-1})\big| \\
&\le C \sum_{i=1}^{m+1} \big[(x_i - x_{i-1}) + (y_i - y_{i-1})\big] \\
&\le C \sum_{i=1}^{m+1} \big[|y_i - x_{i-1}| + |y_{i-1} - x_i|\big] \\
&\le 4C \sum_{i=1}^{m+1} \big(y_i - x_i\big) \\
&\le 4C(m + 2)(y - x).
\end{aligned}$$

∎

Proposition 3.3. *For any $\epsilon > 0$, and $x \in [a, b]$, there exists an $m \in \mathbb{N}$ and an open neighborhood $I = (x - \delta, x + \delta)$ of x, where $\delta > 0$ such that*

$$|p_m(y) - p_J(y)| \leq \epsilon \quad \forall y \in I, \tag{3.4}$$

$$|n_m(y) - n_J(y)| \leq \epsilon \quad \forall y \in I. \tag{3.5}$$

Proof. Since the sequence of functions (p_m) converges pointwise to p_J, we choose an m such that

$$|p_m(x) - p_J(x)| < \epsilon/3. \tag{3.6}$$

Choose δ such that $\delta < \epsilon/12C(m + 2)$, then we have from Proposition 3.2 that

$$|p_m(x) - p_m(y)| < \epsilon/3. \tag{3.7}$$

Furthermore, we have

$$|p_J(x) - p_J(y)| < C\delta < \epsilon/3. \tag{3.8}$$

Now we have

$$\begin{aligned} |p_m(y) - p_J(y)| \leq & |p_m(y) - p_m(x)| + |p_m(x) - p_J(x)| \\ & + |p_J(x) - p_J(y)| \\ < & \epsilon. \end{aligned}$$

The inequality (3.5) can be proved similarly.
∎

Theorem 3.2. *The sequences (p_m), and (n_m) converge uniformly to p_J and n_J, respectively.*

Proof. Since the interval $[a, b]$ is compact, it follows immediately from Proposition 3.3 that (p_m), and (n_m) converge uniformly to p_J and n_J, respectively ∎

After proving the previous propositions, and Theorem 3.2, we now have the tools to prove Theorem 3.1.

Proof of Theorem 3.1. Both functions p_J and n_J are increasing as we mentioned. For any $\epsilon > 0$ choose an m such that $|p_m(x) - p_J(x)| \leq \epsilon/2$ for any $x \in [a, b]$. Thus, for any $x < y$,

$$p_m(x) \leq p_J(x) + \epsilon/2 \leq p_J(y) + \epsilon/2 \leq p_m(y) + \epsilon.$$

This proves that the function p_m is ϵ-increasing. □

If we choose ϵ to be the machine epsilon ϵ_{mach} of some computer, then the functions p_m and n_m will be increasing in this computer because if $x < y$ then $p_m(x) \leq p_m(y) + \epsilon_{\text{mach}} = p_m(y)$ and $n_m(x) \leq n_m(y) + \epsilon_{\text{mach}} = n_m(y)$.

REFERENCES

[1] J. BANAŚ and W. EL-SAYED, Functions of generalized bounded variation. *Zeszyty Nauk. Politech. Rzeszowskiej Mat. Fiz.*, no. 12, 1991, pp. 91–110.

[2] M. ERDMANN, On the Implementation and Analysis of Cellular Exclusion Algorithms. PhD thesis, Colorado State University, 2001.

[3] E.GODLEWSKI and P. RAVIART, *Hyperbolic Systems of Conservation Laws*, Ellipses, Paris, 1991.

[4] E. GODLEWSKI and P. RAVIART, *Numerical Approximation of Hyperbolic Systems of Conservation Laws*, Springer, New York, 1996.

[5] E. KREYSZIG, *Introductory Functional Analysis with Applications*, Wiley Classics Library, John Wiley and Sons, Inc., New York, 1989.

[6] F. MORICZ, Pointwise behavior of Fourier integrals of functions of bounded variation over $\mathbb{R}$. *J. of Math. Anal. and Appl.*, **297**, Issue 2, 2004, pp. 527-539

[7] W. WADE, *An Introduction to Analysis*, Prentice-Hall Inc, New Jersey, 1995.

[8] X. XIAO PING and Z. BO, Properties of set-valued functions of bounded variation in a Banach space. *J. Harbin Inst. Tech.*, no. 3(107), 1991, pp. 102–105.

[9] Z. XU, J. ZHANG and Y. LEUNG, A cell exclusion algorithm for determining all the solutions of a nonlinear system of equations, *Appl. Math. Compu.,* 86, 1997, pp. 235–259.

[10] P. ZINGANO and S. STEINBERG, On the Hardy-Littlewood theorem for functions of bounded variation. *SIAM J. Math. Anal.* no. 5 (33), 2002, pp. 1199–1210.