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# ON SANDWICH THEOREMS FOR CERTAIN SUBCLASS OF ANALYTIC FUNCTIONS INVOLVING DZIOK-SRIVASTAVA OPERATOR

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ABSTRACT. The purpose of this present paper is to derive some subordination and superordination results for certain normalized analytic functions in the open unit disk, acted upon by Dziok-Srivastava operator. Relevant connections of the results, which are presented in this paper, with various known results are also considered.

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## 1. INTRODUCTION

Let $\mathcal{H}$ denote the class of functions analytic in the open unit disc $\Delta := \{z : |z| < 1\}$ and $\mathcal{H}[a, n]$ be the subclass of $\mathcal{H}$ consisting of functions of the form $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots$ . Let $\mathcal{A}$ denote the class of all analytic functions of the form $f(z) = z + a_2 z^2 + \cdots$ . If $f, F \in \mathcal{H}$ and $F$ is univalent in $\Delta$ we say that the function $f$ is *subordinate* to $F$, then $F$ is *superordinate* to $f$, written $f(z) \prec F(z)$, if $f(0) = F(0)$ and $f(\Delta) \subseteq F(\Delta)$.

Let $h \in \mathcal{H}$ and let $\phi(r, s, t; z) : \mathbb{C}^3 \times \Delta \to \mathbb{C}$. If $p$ and $\phi(p(z), zp'(z), z^2 p''(z); z)$ are univalent and if $p$ satisfies the second order superordination

$$h(z) \prec \phi(p(z), zp'(z), z^2 p''(z); z), \tag{1.1}$$

then $p$ is the solution of the differential superordination . An analytic function $q$ is called a *subordinant* if $q \prec p$ for all $p$ satisfying (1.1). A univalent subordinant $\widetilde{q}$ that satisfies $q \prec \widetilde{q}$ for all subordinants $q$ of (1.1) is said to be the best subordinant. Recently Miller and Mocanu [10] obtained conditions on $h$, $q$ and $\phi$ for which the following implication holds:

$$h(z) \prec \phi(p(z), zp'(z), z^2 p''(z); z) \Rightarrow q(z) \prec p(z).$$

Using the results of Miller and Mocanu[10], Bulboaca [3] considered certain classes of first order differential superordinations as well as superordination-preserving integral operators [2]. Frasin and Darus [8], Ravichandran [12], Ali et al. [1] and Shanmugam et al. [14] have investigated certain results in terms of $\dfrac{z^2 f'(z)}{f^2(z)}$.

In the present paper, we obtain sufficient condition for a normalized analytic function $f(z)$ to satisfy

$$q_1(z) \prec \frac{z^2 f'(z)}{f^2(z)} \prec q_2(z),$$

where $q_1$ and $q_2$ are given univalent function in $\Delta$. Also, we obtain results for function defined by Dziok - Srivastava operator and Multiplier transformation. Our works generalize the previous results.

For $\alpha_j \in \mathbb{C} (j = 1, 2, \ldots, l)$ and $\beta_j \in \mathbb{C} \setminus \mathbb{Z}_0^- := \{0, -1, -2, \ldots\}, (j = 1, 2, \ldots m)$, the *generalized hypergeometric function* ${}_lF_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z)$ is defined by the infinite series

$${}_lF_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) := \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \ldots (\alpha_l)_n}{(\beta_1)_n \ldots (\beta_m)_n} \frac{z^n}{n!}$$

$$(l \leq m + 1; l, m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}) ,$$

where $(a)_n$ is the Pochhammer symbol defined by

$$(a)_n := \frac{\Gamma(a + n)}{\Gamma(a)} = \begin{cases} 1, & (n = 0); \\ a(a + 1)(a + 2) \ldots (a + n - 1), & (n \in \mathbb{N}). \end{cases}$$

Corresponding to the function $h(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) := z\, {}_lF_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z)$, the Dziok-Srivastava operator [5, 6, 7] (see also [13]) $H^{(l,m)}(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z)$ is defined by the Hadamard product

$$\begin{aligned} H^{(l,m)}(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) f(z) &:= h(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) * f(z) \\ &= z + \sum_{n=2}^{\infty} \frac{(\alpha_1)_{n-1} \ldots (\alpha_l)_{n-1}}{(\beta_1)_{n-1} \ldots (\beta_m)_{n-1}} \frac{a_n z^n}{(n - 1)!}. \end{aligned} \tag{1.2}$$

It is well known, from the work of Srivastava [5, 6, 7], that
$\alpha_1 H^{(l,m)}(\alpha_1+1,\ldots,\alpha_l;\beta_1,\ldots,\beta_m;z)f(z) =$

$$z[H^{(l,m)}(\alpha_1,\ldots,\alpha_l;\beta_1,\ldots,\beta_m;z)f(z)]' \tag{1.3}$$
$$+(\alpha_1-1)H^{(l,m)}(\alpha_1,\ldots,\alpha_l;\beta_1,\ldots,\beta_m;z)f(z).$$

To make the notation simple, we write $H^{l,m}[\alpha_1]f(z) := H^{(l,m)}(\alpha_1,\ldots,\alpha_l;\beta_1,\ldots,\beta_m;z)f(z)$.

We note that $H^{2,1}(a,1;c)f(z) = L(a,c)f(z)$, the familiar Carlson-Shaffer operator and $H^{2,1}(\delta+1,1;1)f(z) = D^\delta f(z)$, the familiar Ruscheweyh derivative operator.

The Multiplier transformation of Srivastava [13] on $\mathcal{A}$, is the operator $I(r,\lambda)$ on $\mathcal{A}$ defined by the following infinite series

$$I(r,\lambda)f(z) := z + \sum_{k=2}^{\infty}\left(\frac{k+\lambda}{1+\lambda}\right)^r a_k z^k. \tag{1.4}$$

A straight forward calculation shows that

$$(1+\lambda)I(r+1,\lambda)f(z) = z[I(r,\lambda)f(z)]' + \lambda I(r,\lambda)f(z). \tag{1.5}$$

The operator $I(r,0)$ is the Sălăgean derivative operators. The operator $I_\lambda^r := I(r,\lambda)$ was studied recently by Cho and Kim[4]. The operator $I_r := I(r,1)$ was studied by Uralegaddi and Somanatha[15].

## 2. PREMILINARIES

We shall need the following definition and results to prove our main results.

**Definition 1.** [10, Definition 2, p. 817 ] *Let $\mathcal{Q}$ be the set of all functions $f$ that are analytic and injective on $\bar{\Delta} - E(f)$, where*

$$E(f) := \left\{\zeta \in \partial\Delta : \lim_{z\to\zeta} f(z) = \infty\right\}$$

*and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial\Delta - E(f)$.*

**Theorem 2.1.** [9, Theorem 3.4h, p. 132 ] *Let $q(z)$ be univalent in the unit disk $\Delta$ and $\theta$ and $\phi$ be analytic in a domain $D$ containing $q(\Delta)$ with $\phi(\omega) \neq 0$ when $\omega \in q(\Delta)$.*

Set $Q(z) = zq'(z)\phi(q(z))$, $h(z) = \theta(q(z)) + Q(z)$. Suppose that,

(1) $Q(z)$ is starlike univalent in $\Delta$ and
(2) $\Re\left\{\frac{zh'(z)}{Q(z)}\right\} > 0$ for $z \in \Delta$.

If $p(z)$ is analytic in $\Delta$ with $p(\Delta) \subseteq D$, and

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)), \tag{2.1}$$

then $p(z) \prec q(z)$ and $q(z)$ is the best dominant.

**Theorem 2.2.** [3] *Let $q(z)$ be univalent in $\Delta$ , $\vartheta$ and $\varphi$ be analytic in a domain $D$ containing $q(\Delta)$. Suppose that*

(1) $\Re\left[\frac{\vartheta'(q(z))}{\varphi(q(z))}\right] > 0$ *for $z \in \Delta$, and*
(2) $Q(z) = zq'(z)\varphi(q(z))$ *is starlike univalent function in $\Delta$.*

*If $p \in \mathcal{H}[q(0),1] \cap \mathcal{Q}$, with $p(\Delta) \subset D$, and $\vartheta(p(z)) + zp'(z)\varphi(p(z))$ is univalent in $\Delta$, and*

$$\vartheta(q(z)) + zq'(z)\varphi(q(z)) \prec \vartheta(p(z)) + zp'(z)\varphi(p(z)), \tag{2.2}$$

*then $q(z) \prec p(z)$ and $q(z)$ is the best subordinant.*

**Lemma 2.3.** [9, Lemma 1, p. 71 ] *Let $h(z)$ be convex univalent in $\Delta$, with $h(0) = a$ and let $\gamma \in \mathbb{C}$ with $\Re(\gamma) \geq 0$. If $p \in \mathcal{H}$ with $p(0) = \alpha$ and*

$$p(z) + \frac{1}{\gamma} zp'(z) \prec h(z),$$

*then*

$$p(z) \prec q(z) \prec h(z)$$

*where*

$$q(z) = \frac{\gamma}{nz^{\left(\frac{\gamma}{n}\right)-1}} \int_0^z h(t) t^{\left(\frac{\gamma}{n}\right)-1} dt.$$

*The function $q(z)$ is convex and is the best dominant.*

## 3. SANDWICH RESULTS

**Theorem 3.1.** *Let $\alpha, \beta, \gamma$ and $\delta \in \mathbb{C}$, $\delta \neq 0$. Let $0 \neq q(z)$ be univalent in $\Delta$ and $\frac{zq'(z)}{q(z)}$ be starlike univalent in $\Delta$. Further assume that*

$$\Re\left\{\frac{\alpha}{\delta} q(z) + \frac{2\beta}{\delta} q^2(z) - \frac{\gamma}{\delta q(z)} + 1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}\right\} > 0. \tag{3.1}$$

*Let $f \in \mathcal{A}$ and*

$$\Psi(\alpha, \beta, \gamma, \delta; z) = \alpha \frac{z^2 f'(z)}{f^2(z)} + \beta \left(\frac{z^2 f'(z)}{f^2(z)}\right)^2 + \frac{\gamma f^2(z)}{z^2 f'(z)} + \delta \left(\frac{(zf)''(z)}{f'(z)} - \frac{2zf'(z)}{f(z)}\right). \tag{3.2}$$

*If*

$$\Psi(\alpha, \beta, \gamma, \delta; z) \prec \alpha q(z) + \beta q^2(z) + \frac{\gamma}{q(z)} + \frac{\delta zq'(z)}{q(z)}, \tag{3.3}$$

*then*

$$\frac{z^2 f'(z)}{f^2(z)} \prec q(z) \tag{3.4}$$

*and $q$ is the best dominant.*

*Proof.* Let the function $p(z)$ be defined by

$$p(z) := \frac{z^2 f'(z)}{f^2(z)} \quad (z \in \Delta). \tag{3.5}$$

By a straighforward computation, we have

$$\begin{aligned} &\alpha p(z) + \beta p^2(z) + \frac{\gamma}{p(z)} + \frac{\delta zp'(z)}{p(z)} \\ &= \alpha \frac{z^2 f'(z)}{f^2(z)} + \beta \left(\frac{z^2 f'(z)}{f^2(z)}\right)^2 + \frac{\gamma f^2(z)}{z^2 f'(z)} + \delta \left(\frac{(zf)''(z)}{f'(z)} - \frac{2zf'(z)}{f(z)}\right). \end{aligned} \tag{3.6}$$

By setting $\theta(\omega) := \alpha\omega + \beta\omega^2 + \dfrac{\gamma}{\omega}$ and $\phi(\omega) := \dfrac{\delta}{\omega}$, it can be easily verified that $\theta(\omega)$ and $\phi(\omega)$ are analytic in $\mathbb{C} - \{0\}$ . Also, by letting

$$Q(z) = zq'(z)\phi(q(z)) = \frac{\delta zq'(z)}{q(z)} \tag{3.7}$$

and

$$h(z) = \theta(q(z)) + Q(z) = \alpha q(z) + \beta q^2(z) + \frac{\gamma}{q(z)} + \frac{\delta zq'(z)}{q(z)}, \tag{3.8}$$

we find that $Q(z)$ is starlike univalent in $\Delta$ and that

$$\Re\left(\frac{zh'(z)}{Q(z)}\right)=\Re\left(\frac{\alpha}{\delta}q(z)+\frac{2\beta}{\delta}q^2(z)-\frac{\gamma}{\delta q(z)}+1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}\right)>0.$$

The assertion (3.4) of Theorem 3.1 now follows by an application of Theorem 2.1. ∎

For the choice of $q(z)=\dfrac{1+Az}{1+Bz}$ $(-1\le B<A\le 1)$, in Theorem 3.1, we get the following result.

**Example 1.** *Assume that (3.1) holds. If* $f\in\mathcal{A}$ *, and*

$$\Psi(\alpha,\beta,\gamma,\delta;z)\prec\alpha\left(\frac{1+Az}{1+Bz}\right)+\beta\left(\frac{1+Az}{1+Bz}\right)^2+\frac{\gamma(1+Bz)}{(1+Az)}+\frac{\delta(A-B)z}{(1+Az)(1+Bz)}$$

*where* $\Psi(\alpha,\beta,\gamma,\delta;z)$ *as defined by (3.2), then*

$$\frac{z^2f'(z)}{f^2(z)}\prec\frac{1+Az}{1+Bz}$$

*and* $\dfrac{1+Az}{1+Bz}$ *is the best dominant.*

*In particular, we have*

$$\Psi(\alpha,\beta,\gamma,\delta;z)\prec\alpha\left(\frac{1+z}{1-z}\right)+\beta\left(\frac{1+z}{1-z}\right)^2+\frac{\gamma(1-z)}{(1+z)}+\frac{2\delta z}{1-z^2},$$

*implies*

$$\Re\left(\frac{z^2f'(z)}{f^2(z)}\right)>0.$$

**Theorem 3.2.** *Let* $\alpha,\beta,\gamma\in\mathbb{C}$, $\gamma\neq 0$. *Let* $q(z)$ *be univalent in* $\Delta$. *Let* $f\in\mathcal{A}$ *and*

$$\Re\left(\frac{\alpha}{\gamma}+\frac{2\beta}{\gamma}q(z)+1+\frac{zq''(z)}{q'(z)}\right)>0. \tag{3.9}$$

*Let*

$$\xi(\alpha,\beta,\gamma;z)=\alpha\frac{z^2f'(z)}{f^2(z)}+\beta\left(\frac{z^2f'(z)}{f^2(z)}\right)^2-\gamma z^2\left(\frac{z}{f(z)}\right)''. \tag{3.10}$$

*If*

$$\xi(\alpha,\beta,\gamma;z)\prec\alpha q(z)+\beta q^2(z)+\gamma zq'(z), \tag{3.11}$$

*then*

$$\frac{z^2f'(z)}{f^2(z)}\prec q(z)$$

*and* $q$ *is the best dominant.*

*Proof.* Define the function $p(z)$ by

$$p(z):=\frac{z^2f'(z)}{f^2(z)}.$$

Now,

$$\xi(\alpha,\beta,\gamma;z)=\alpha\left(\frac{z^2f'(z)}{f^2(z)}\right)+\beta\left(\frac{z^2f'(z)}{f^2(z)}\right)^2-\gamma z^2\left(\frac{z}{f(z)}\right)''.$$

$$=\alpha p(z)+\beta p^2(z)+\gamma zp'(z). \tag{3.12}$$

By using (3.12) in subordination (3.11), we have

$$\alpha p(z) + \beta p^2(z) + \gamma z p'(z) \prec \alpha q(z) + \beta q^2(z) + \gamma z q'(z). \tag{3.13}$$

The subordination (3.13) is same as (2.1) with $\theta(\omega) := \alpha\omega + \beta\omega^2$ and $\phi(\omega) := \gamma$. Now our result follows as an application of Theorem 2.1. ∎

By taking $\alpha, \beta, \gamma, \delta$ as real and $\alpha = \beta = \gamma = 0$ and $\delta = 1$ in Theorem 3.1, then we have the following result obtained by Ravichandran et al. [12].

**Corollary 3.3.** *If $f \in \mathcal{A}$ and*

$$\frac{(zf)''(z)}{f'(z)} - \frac{2zf'(z)}{f(z)} \prec \frac{zq'(z)}{q(z)},$$

*then*

$$\frac{z^2 f'(z)}{f^2(z)} \prec q(z)$$

*and $q$ is the best dominant.*

**Remark 1.** *Taking $\alpha = 1, \beta = 0$ in Theorem 3.2, we get the result obtained by Shanmugam et al.* [14].

**Theorem 3.4.** *Let $\alpha, \beta, \gamma$ and $\delta \in \mathbb{C}$, $\delta \neq 0$. Let $0 \neq q(z)$ be convex univalent in $\Delta$ with $q(0) = 1$ and $\frac{zq'(z)}{q(z)}$ be starlike univalent in $\Delta$. Let $f \in \mathcal{A}$, $0 \neq \dfrac{z^2 f'(z)}{f^2(z)} \in \mathcal{H}[1,1] \cap \mathcal{Q}$, with*

$$\Re\left(\frac{\alpha}{\delta} q(z) + \frac{2\beta}{\delta} q^2(z) - \frac{\gamma}{\delta q(z)}\right) > 0. \tag{3.14}$$

*If $\Psi(\alpha, \beta, \gamma, \delta; z)$ as defined by (3.2) is univalent in $\Delta$ and*

$$\alpha q(z) + \beta q^2(z) + \frac{\gamma}{q(z)} + \frac{\delta z q'(z)}{q(z)} \prec \Psi(\alpha, \beta, \gamma, \delta; z), \tag{3.15}$$

*then*

$$q(z) \prec \frac{z^2 f'(z)}{f^2(z)} \tag{3.16}$$

*and $q$ is the best subordinant.*

*Proof.* By setting $\vartheta(\omega) := \alpha\omega + \beta\omega^2 + \dfrac{\gamma}{\omega}$ and $\phi(\omega) := \dfrac{\delta}{\omega}$, it is easily observed that $\vartheta(\omega)$ and $\phi(\omega)$ is analytic in $\mathbb{C} - \{0\}$ . Since $q$ is convex univalent it follows that,

$$\Re\left[\frac{\vartheta'(q(z))}{\phi(q(z))}\right] = \Re\left[\frac{\alpha}{\delta} q(z) + \frac{2\beta}{\delta} q^2(z) - \frac{\gamma}{\delta q(z)}\right] > 0, \quad (z \in \Delta; \alpha, \beta, \gamma, \delta \in \mathbb{C}, \delta \neq 0).$$

The assertion (3.16) of Theorem 3.4 follows by an application of Theorem 2.2. ∎

**Theorem 3.5.** *Let $\alpha, \beta, \gamma \in \mathbb{C}$, $\gamma \neq 0$. Let $q(z)$ be convex univalent in $\Delta$ with $q(0) = 1$. Let $f \in \mathcal{A}$, $0 \neq \dfrac{z^2 f'(z)}{f^2(z)} \in \mathcal{H}[1,1] \cap \mathcal{Q}$, with*

$$\Re\left(\frac{\alpha}{\gamma} + \frac{2\beta}{\gamma} q(z)\right) > 0. \tag{3.17}$$

*If $\xi(\alpha, \beta, \gamma; z)$ as defined by (3.10) is univalent in $\Delta$ and*

$$\alpha q(z) + \beta q^2(z) + \gamma z q'(z) \prec \xi(\alpha, \beta, \gamma; z),$$

*then*

$$q(z) \prec \frac{z^2 f'(z)}{f^2(z)}$$

*and $q$ is the best subordinant.*

*Proof.* Theorem 3.5 follows from Theorem 2.2 by taking $p(z) := \frac{z^2 f'(z)}{f^2(z)}$. ∎

We remark here that Theorem 3.4 and Theorem 3.5 can be easily restated, for different choices of the function $q(z)$.

Combining Theorem 3.1 and Theorem 3.4 we get the following sandwich theorem.

**Theorem 3.6.** *Let $0 \neq q_1(z)$ and $0 \neq q_2(z)$ be convex univalent satisfying (3.14) and (3.1) respectively and $\frac{zq_i'(z)}{q_i(z)}$ be starlike univalent in $\Delta$ for $i = 1, 2$. If $0 \neq \frac{z^2 f'(z)}{f^2(z)} \in \mathcal{H}[1,1] \cap \mathcal{Q}$, $\Psi(\alpha, \beta, \gamma, \delta; z)$ as defined by (3.2) is univalent in $\Delta$ and*

$$\alpha q_1(z) + \beta q_1^2(z) + \frac{\gamma}{q_1(z)} + \frac{\delta z q_1'(z)}{q_1(z)} \prec \Psi(\alpha, \beta, \gamma, \delta; z) \prec \alpha q_2(z) + \beta q_2^2(z) + \frac{\gamma}{q_2(z)} + \frac{\delta z q_2'(z)}{q_2(z)},$$

*then*

$$q_1(z) \prec \frac{z^2 f'(z)}{f^2(z)} \prec q_2(z)$$

*and $q_1(z)$ and $q_2(z)$ are respectively the best subordinant and best dominant.*

By taking $\alpha = \beta = \gamma = 0$ and $\delta = 1$ in Theorem 3.6, then we have the following corollary of Ali et al. [1].

**Theorem 3.7.** *Let $q_i(z) \neq 0$ be univalent in $\Delta$ and $\frac{zq_i'(z)}{q_i(z)}$ is starlike univalent in $\Delta$ for $i = 1, 2$. If $f \in \mathcal{A}$, $0 \neq \frac{z^2 f'(z)}{f^2(z)} \in \mathcal{H}[1,1] \cap \mathcal{Q}$, $\frac{(zf)''(z)}{f'(z)} - 2\frac{zf'(z)}{f(z)}$ is univalent in $\Delta$, then*

$$\frac{zq_1'(z)}{q_1(z)} \prec \frac{(zf)''(z)}{f'(z)} - 2\frac{zf'(z)}{f(z)} \prec \frac{zq_2'(z)}{q_2(z)},$$

*implies*

$$q_1(z) \prec \frac{z^2 f'(z)}{f^2(z)} \prec q_2(z)$$

*and $q_1(z)$ and $q_2(z)$ are respectively the best subordinant and best dominant.*

**Theorem 3.8.** *Let $q_1(z)$ and $q_2(z)$ be convex univalent satisfying (3.17) and (3.9) respectively. If $0 \neq \frac{z^2 f'(z)}{f^2(z)} \in \mathcal{H}[1,1] \cap \mathcal{Q}$, $\xi(\alpha, \beta, \gamma; z)$ as defined by (3.10) is univalent in $\Delta$ and*

$$\alpha q_1(z) + \beta q_1^2(z) + \gamma z q_1'(z) \prec \xi(\alpha, \beta, \gamma; z) \prec \alpha q_2(z) + \beta q_2^2(z) + \gamma z q_2'(z),$$

*then*

$$q_1(z) \prec \frac{z^2 f'(z)}{f^2(z)} \prec q_2(z)$$

*and $q_1(z)$ and $q_2(z)$ are respectively the best subordinant and best dominant.*

For $q_1(z) = \frac{1 + A_1 z}{1 + B_1 z}$, $q_2(z) = \frac{1 + A_2 z}{1 + B_2 z}$ , $(-1 \leq B_2 \leq B_1 < A_1 \leq A_2 \leq 1)$, in Theorem 3.6, we have the following:

**Corollary 3.9.** *If* $f \in \mathcal{A}$, $\dfrac{z^2 f'(z)}{f^2(z)} \in \mathcal{H}[1,1] \cap \mathcal{Q}$, *and*

$$\Psi_1(\alpha,\beta,\gamma,\delta;z) \prec \Psi(\alpha,\beta,\gamma,\delta;z) \prec \Psi_2(\alpha,\beta,\gamma,\delta;z),$$

*then*

$$\frac{1+A_1z}{1+B_1z} \prec \frac{z^2 f'(z)}{f^2(z)} \prec \frac{1+A_2z}{1+B_2z}$$

*where* $\Psi(\alpha,\beta,\gamma,\delta;z)$ *is as defined by (3.2) and*

$$\Psi_1(\alpha,\beta,\gamma,\delta;z) := \alpha\left(\frac{1+A_1z}{1+B_1z}\right) + \beta\left(\frac{1+A_1z}{1+B_1z}\right)^2 + \gamma\left(\frac{1+B_1z}{1+A_1z}\right) + \frac{\delta(A_1-B_1)z}{(1+A_1z)(1+B_1z)}.$$

$$\Psi_2(\alpha,\beta,\gamma,\delta;z) := \alpha\left(\frac{1+A_2z}{1+B_2z}\right) + \beta\left(\frac{1+A_2z}{1+B_2z}\right)^2 + \gamma\left(\frac{1+B_2z}{1+A_2z}\right) + \frac{\delta(A_2-B_2)z}{(1+A_2z)(1+B_2z)}.$$

*The functions* $\dfrac{1+A_1z}{1+B_1z}$ *and* $\dfrac{1+A_2z}{1+B_2z}$ *are respectively the best subordinant and best dominant.*

## 4. APPLICATION TO DZIOK-SRIVASTAVA OPERATOR

**Theorem 4.1.** *Let* $0 \neq q(z)$ *be univalent in* $\Delta$ *and* $\frac{zq'(z)}{q(z)}$ *be starlike univalent in* $\Delta$ *and satisfies (3.1). Let* $f \in \mathcal{A}$ *and*

(4.1)
$$\begin{aligned}&\phi(\alpha,\beta,\gamma,\delta,l,m;z)\\ =\ & \alpha\left(\frac{zH^{l,m}[\alpha_1+1]f(z)}{(H^{l,m}[\alpha_1]f(z))^2}\right) + \beta\left(\frac{zH^{l,m}[\alpha_1+1]f(z)}{(H^{l,m}[\alpha_1]f(z))^2}\right)^2 + \gamma\frac{\left(H^{l,m}[\alpha_1]f(z)\right)^2}{zH^{l,m}[\alpha_1+1]f(z)}\\ +\ & \delta\left\{(\alpha_1-1) + \frac{(\alpha_1+1)H^{l,m}[\alpha+2]f(z)}{H^{l,m}[\alpha_1+1]f(z)} - \frac{2\alpha_1 H^{l,m}[\alpha_1+1]f(z)}{H^{l,m}[\alpha_1]f(z)}\right\}.\end{aligned}$$

*If* $f \in \mathcal{A}$ *and*

$$\phi(\alpha,\beta,\gamma,\delta,l,m;z) \prec \alpha q(z) + \beta q^2(z) + \frac{\gamma}{q(z)} + \frac{\delta zq'(z)}{q(z)},$$

*then*

$$\left(\frac{zH^{l,m}[\alpha_1+1]f(z)}{(H^{l,m}[\alpha_1]f(z))^2}\right) \prec q(z) \tag{4.2}$$

*and* $q$ *is the best dominant.*

*Proof.* Define the function $p(z)$ by

$$p(z) := \left(\frac{zH^{l,m}[\alpha_1+1]f(z)}{(H^{l,m}[\alpha_1]f(z))^2}\right). \tag{4.3}$$

By taking logarithmic derivative of $p(z)$ given by (4.3), we have

$$\frac{zp'(z)}{p(z)} = 1 + \frac{z\left(H^{l,m}[\alpha_1+1]f(z)\right)'}{H^{l,m}[\alpha_1+1]f(z)} - \frac{2z(H^{l,m}[\alpha_1]f(z))'}{H^{l,m}[\alpha_1]f(z)}. \tag{4.4}$$

By using the identity

$$z(H^{l,m}[\alpha_1]f(z))' = \alpha_1 H^{l,m}[\alpha_1+1]f(z) - (\alpha_1-1)H^{l,m}[\alpha_1]f(z)$$

and (4.3) in (4.4) we obtain

$$\frac{zp'(z)}{p(z)} = (\alpha_1-1) + (\alpha_1+1)\frac{H^{l,m}[\alpha_1+2]f(z)}{H^{l,m}[\alpha_1+1]f(z)} - \frac{2\alpha_1 H^{l,m}[\alpha_1+1]f(z)}{H^{l,m}[\alpha_1]f(z)}.$$

The assertion (4.2) of Theorem 4.1 now follows from Theorem 2.1. ∎

Taking $l = 2, m = 1,$ and $\alpha_2 = 1$ in Theorem 4.1, we get

**Corollary 4.2.** *Let $0 \neq q(z)$ be univalent in $\Delta$ with $q(0) = 1$. If $f \in \mathcal{A}$ and*

$$\begin{aligned}(4.5)\qquad &\phi_1(\alpha,\beta,\gamma,\delta,a,c;z)\\ := \;& \alpha\left(\frac{zL(a+1,c)f(z)}{(L(a,c)f(z))^2}\right) + \beta\left(\frac{zL(a+1,c)f(z)}{(L(a,c)f(z))^2}\right)^2 + \gamma\frac{(L(a,c)f(z))^2}{zL(a+1,c)f(z)}\\ +\;& \delta\left\{(a-1) + \frac{(a+1)L(a+2,c)f(z)}{L(a+1,c)f(z)} - \frac{2aL(a+1,c)f(z)}{L(a,c)f(z)}\right\}.\end{aligned}$$

*If*

$$\phi_1(\alpha,\beta,\gamma,\delta,a,c;z) \prec \alpha q(z) + \beta q^2(z) + \frac{\gamma}{q(z)} + \frac{\delta z q'(z)}{q(z)},$$

*then*

$$\left(\frac{zL(a+1,c)f(z)}{(L(a,c)f(z))^2}\right) \prec q(z)$$

*and $q$ is the best dominant, where $L(a,c)$ is the familiar Carlson-Shaffer operator.*

By taking $\alpha = \beta = \gamma = 0, \delta = 1, a = c = 1$ and $q(z) = 1 + (1-b)z$ in corollary 4.2, then we have the following:

**Corollary 4.3.** *If $f \in \mathcal{A}$ and*

$$2 + \frac{zf''(z)}{f'(z)} - \frac{2zf'(z)}{f(z)} \prec \frac{(1-b)z}{1+(1-b)z}, \quad (0 \leq b < 1),$$

*then*

$$\frac{z^2 f'(z)}{f^2(z)} \prec 1 + (1-b)z.$$

*By restating the above Corollary 4.3, we get the following result of Frasin and Darus* [8].

**Corollary 4.4.** *If $f \in \mathcal{A}$ and*

$$\left|\frac{(zf(z))''}{f'(z)} - \frac{2zf'(z)}{f(z)}\right| < \frac{1-b}{2-b}, \quad (0 \leq b < 1),$$

*then*

$$\left|\frac{z^2 f'(z)}{f^2(z)} - 1\right| \leq 1 - b.$$

**Theorem 4.5.** *Let $0 \neq q(z)$ be convex univalent in $\Delta$ and $\frac{zq'(z)}{q(z)}$ be starlike univalent in $\Delta$. Let $q(z)$ satisfies (3.14). Let $f \in \mathcal{A}$, $0 \neq \left(\frac{zH^{l,m}[\alpha_1+1]f(z)}{(H^{l,m}[\alpha_1]f(z))^2}\right) \in \mathcal{H}[1,1] \cap \mathcal{Q}$. If $\phi(\alpha,\beta,\gamma,\delta,l,m;z)$ as defined by (4.1) is univalent in $\Delta$ and*

$$\alpha q(z) + \beta q^2(z) + \frac{\gamma}{q(z)} + \frac{\delta z q'(z)}{q(z)} \prec \phi(\alpha,\beta,\gamma,\delta,l,m;z),$$

*then*

$$q(z) \prec \left(\frac{zH^{l,m}[\alpha_1+1]f(z)}{(H^{l,m}[\alpha_1]f(z))^2}\right)$$

*and $q$ is the best subordinant.*

*Proof.* Theorem 4.5 follows from Theorem 2.2 by taking

$$p(z) := \left(\frac{zH^{l,m}[\alpha_1+1]f(z)}{(H^{l,m}[\alpha_1]f(z))^2}\right).$$

By combining Theorem 4.1 and Theorem 4.5 we get the following sandwich theorem. ∎

**Theorem 4.6.** *Let $0 \neq q_1(z)$ and $0 \neq q_2(z)$ be convex univalent satisfying (3.14) and (3.1) respectively and $\frac{zq_i'(z)}{q_i(z)}$ be starlike univalent in $\Delta$ for $i = 1, 2$. If $f \in \mathcal{A}$,*

$$0 \neq \left(\frac{zH^{l,m}[\alpha_1+1]f(z)}{(H^{l,m}[\alpha_1]f(z))^2}\right) \in \mathcal{H}[1,1] \cap \mathcal{Q}$$

*and $\phi(\alpha, \beta, \gamma, \delta, l, m; z)$ as defined by (4.1) is univalent in $\Delta$. Further if*

$$\alpha q_1(z)+\beta q_1^2(z)+\frac{\gamma}{q_1(z)}+\frac{\delta zq_1'(z)}{q_1(z)} \prec \phi(\alpha,\beta,\gamma,\delta,l,m;z) \prec \alpha q_2(z)+\beta q_2^2(z)+\frac{\gamma}{q_2(z)}+\frac{\delta zq_2'(z)}{q_2(z)},$$

*then*

$$q_1(z) \prec \left(\frac{zH^{l,m}[\alpha_1+1]f(z)}{(H^{l,m}[\alpha_1]f(z))^2}\right) \prec q_2(z)$$

*and $q_1$ and $q_2$ are respectively the best subordinant and best dominant.*

## 5. APPLICATION TO MULTIPLIER TRANSFORMATION

**Theorem 5.1.** *Let $h \in \mathcal{H}, h(0) = 1, h'(0) \neq 0, \lambda \neq -1$, which satisfies the inequality*

$$\Re\left[1 + \frac{zh''(z)}{h'(z)}\right] > \frac{-1}{2}, \quad (z \in \Delta).$$

*If $f \in \mathcal{A}_m$ satisfies the differential subordination.*

$$\frac{2z\,(I(r+1,\lambda)f(z))^2}{(I(r,\lambda)f(z))^3} - \frac{z(I(r+2,\lambda)f(z))}{I(r+1,\lambda)f(z)} \prec h(z),$$

*then*

$$\frac{zI(r+1,\lambda)f(z)}{(I(r,\lambda)f(z))^2} \prec q(z), \tag{5.1}$$

*where*

$$q(z) = \frac{-(1+\lambda)}{nz^{\left(\frac{-(1+\lambda)}{n}\right)-1}} \int_0^z h(t)t^{\left(-\frac{(1+\lambda)}{n}\right)-1}dt.$$

*The function $q$ is the convex and is the best dominant.*

*Proof.* Define the function $p(z)$ by

$$p(z) := \frac{zI(r+1,\lambda)f(z)}{(I(r,\lambda)f(z))^2}. \tag{5.2}$$

By taking logarithmic derivative of $p(z)$ given by (5.2), we get

$$\frac{zp'(z)}{p(z)} = 1 + \frac{z(I(r+1,\lambda)f(z))'}{I(r+1,\lambda)f(z)} - \frac{2z(I(r,\lambda)f(z)'}{I(r,\lambda)f(z)}. \tag{5.3}$$

By using the identity

$$z(I(r,\lambda)f(z))' = (1+\lambda)I(r+1,\lambda)f(z) - \lambda\,(I(r,\lambda)f(z))$$

and (5.2) in (5.3), we obtain

$$\frac{zp'(z)}{p(z)} = (1+\lambda) + (1+\lambda)\frac{I(r+2,\lambda)f(z)}{I(r+1,\lambda)f(z)} - 2(1+\lambda)\frac{I(r+1,\lambda)f(z)}{I(r,\lambda)f(z)}.$$

Hence,

$$p(z) + \frac{zp'(z)}{-(1+\lambda)} = \frac{2z\left(I(r+1,\lambda)f(z)\right)^2}{\left(I(r,\lambda)f(z)\right)^3} - \frac{z(I(r+2,\lambda)f(z))}{I(r+1,\lambda)f(z)}.$$

The assertion (5.1) of Theorem 5.1 follows from Lemma 2.3. ∎

**Theorem 5.2.** *Let* $0 \neq q(z)$ *be univalent in* $\Delta$ *with* $q(0) = 1$. *Suppose* $\frac{zq'(z)}{q(z)}$ *be starlike univalent in* $\Delta$ *and satisfies (3.1). Let* $f \in \mathcal{A}$ *and*

(5.4)
$$\begin{aligned} &\Lambda(\alpha,\beta,\gamma,\delta,r,\lambda;z) \\ := \ & \alpha\left(\frac{zI(r+1,\lambda)f(z)}{(I(r,\lambda)f(z))^2}\right) + \beta\left(\frac{zI(r+1,\lambda)f(z)}{(I(r,\lambda)f(z))^2}\right)^2 + \frac{\gamma(I(r,\lambda)f(z))^2}{zI(r+1,\lambda)f(z)} \\ + \ & \delta\left\{(1+\lambda) + (1+\lambda)\frac{I(r+2,\lambda)f(z)}{I(r+1,\lambda)f(z)} - 2(1+\lambda)\frac{I(r+1,\lambda)f(z)}{I(r,\lambda)f(z)}\right\}. \end{aligned}$$

*If*

$$\Lambda(\alpha,\beta,\gamma,\delta,r,\lambda;z) \prec \alpha q(z) + \beta q^2(z) + \frac{\gamma}{q(z)} + \frac{\delta zq'(z)}{q(z)},$$

*then*

$$\frac{zI(r+1,\lambda)f(z)}{(I(r,\lambda)f(z))^2} \prec q(z)$$

*and* $q$ *is the best dominant.*

*Proof.* Define the function $p(z)$ by

$$p(z) := \frac{zI(r+1,\lambda)f(z)}{(I(r,\lambda)f(z))^2}.$$

By a straightforward computation we have

$$\frac{zp'(z)}{p(z)} = (1+\lambda) + (1+\lambda)\frac{I(r+2,\lambda)f(z)}{I(r+1,\lambda)f(z)} - 2(1+\lambda)\frac{I(r+1,\lambda)f(z)}{I(r,\lambda)f(z)}.$$

Now our result follows as an application of Theorem 2.1. ∎

Since the superordination results are dual of the subordination, we state the results pertaining to the superordination, using the duality.

**Theorem 5.3.** *Let* $0 \neq q(z)$ *be convex univalent in* $\Delta$ *with* $q(0) = 1$. *Suppose* $\frac{zq'(z)}{q(z)}$ *be starlike univalent in* $\Delta$ *and satisfies (3.14). Let* $f \in \mathcal{A}$, $0 \neq \left(\frac{zI(r+1,\lambda)f(z)}{(I(r,\lambda)f(z))^2}\right) \in \mathcal{H}[1,1] \cap \mathcal{Q}$. *If* $\Lambda(\alpha,\beta,\gamma,\delta,r,\lambda;z)$ *as defined by (5.4) is univalent in* $\Delta$ *and*

$$\alpha q(z) + \beta q^2(z) + \frac{\gamma}{q(z)} + \frac{\delta zq'(z)}{q(z)} \prec \Lambda(\alpha,\beta,\gamma,\delta,r,\lambda;z),$$

*then*

$$q(z) \prec \left(\frac{zI(r+1,\lambda)f(z)}{(I(r,\lambda)f(z))^2}\right)$$

*and* $q$ *is the best subordinant.*

Combining Theorem 5.2 and Theorem 5.3, we state the following sandwich theorem.

**Theorem 5.4.** *Let $0 \neq q_1(z)$ and $0 \neq q_2(z)$ be convex univalent in $\Delta$ satisfying (3.14) and (3.1) respectively and $\frac{zq_i'(z)}{q_i(z)}$ be starlike univalent in $\Delta$ for $i = 1, 2$. If $f \in \mathcal{A}$,*

$$0 \neq \left( \frac{zI(r+1,\lambda)f(z)}{(I(r,\lambda)f(z))^2} \right) \in \mathcal{H}[1,1] \cap \mathcal{Q}$$

*and $\Lambda(\alpha,\beta,\gamma,\delta,r,\lambda;z)$ as defined by (5.4) is univalent in $\Delta$. Further if*

$$\alpha q_1(z) + \beta q_1^2(z) + \frac{\gamma}{q_1(z)} + \frac{\delta z q_1'(z)}{q_1(z)} \prec \Lambda(\alpha,\beta,\gamma,\delta,r,\lambda;z) \prec \alpha q_2(z) + \beta q_2^2(z) + \frac{\gamma}{q_2(z)} + \frac{\delta z q_2'(z)}{q_2(z)},$$

*then*

$$q_1(z) \prec \left( \frac{zI(r+1,\lambda)f(z)}{(I(r,\lambda)f(z))^2} \right) \prec q_2(z)$$

*and $q_1$ and $q_2$ are respectively the best subordinant and best dominant.*

By taking $\lambda = 0$ in Theorem 5.2, we get the following:

**Corollary 5.5.** *Let $0 \neq q(z)$ be univalent in $\Delta$ with $q(0) = 1$, satisfying (3.1). If $f \in \mathcal{A}$ and*

$$\eta(\alpha,\beta,\gamma,\delta,m;z) := \alpha \left( \frac{z\mathcal{D}^{m+1}f(z)}{(\mathcal{D}^m f(z))^2} \right) + \beta \left( \frac{z\mathcal{D}^{m+1}f(z)}{(\mathcal{D}^m f(z))^2} \right)^2 + \frac{\gamma(\mathcal{D}^m f(z))^2}{z\mathcal{D}^{m+1}f(z)} + \delta \left\{ 1 + \frac{\mathcal{D}^{m+2}f(z)}{\mathcal{D}^{m+1}f(z)} - \frac{2\mathcal{D}^{m+1}f(z)}{\mathcal{D}^m f(z)} \right\}. \tag{5.5}$$

*If*

$$\eta(\alpha,\beta,\gamma,\delta,m;z) \prec \alpha q(z) + \beta q^2(z) + \frac{\gamma}{q(z)} + \frac{\delta z q'(z)}{q(z)},$$

*then*

$$\frac{z\mathcal{D}^{m+1}f(z)}{(\mathcal{D}^m f(z))^2} \prec q(z)$$

*and $q(z)$ is the best dominant, where $\mathcal{D}^m f(z)$ is the Sălăgean operator.*

For the choice of $q_1(z) = \dfrac{1+A_1z}{1+B_1z}$ , $q_2(z) = \dfrac{1+A_2z}{1+B_2z}$ $(-1 \leq B_2 \leq B_1 < A_1 \leq A_2 \leq 1)$ and $\lambda = 0$ in Theorem 5.4, we have the following:

**Example 2.** *Let $f \in \mathcal{A}$ , $\left( \dfrac{z\mathcal{D}^{m+1}f(z)}{(\mathcal{D}^m f(z))^2} \right) \in \mathcal{H}[1,1] \cap \mathcal{Q}$ and $\eta(\alpha,\beta,\gamma,\delta,m;z)$ as defined by (5.5) is univalent in $\Delta$. Further if*

$$\eta_1(\alpha,\beta,\gamma,\delta,m;z) \prec \eta(\alpha,\beta,\gamma,\delta,m;z) \prec \eta_2(\alpha,\beta,\gamma,\delta,m;z),$$

*then*

$$\frac{1+A_1z}{1+B_1z} \prec \left( \frac{z\mathcal{D}^{m+1}f(z)}{(\mathcal{D}^m f(z))^2} \right) \prec \frac{1+A_2z}{1+B_2z}$$

*where*

$$\eta_1(\alpha,\beta,\gamma,\delta,m;z) := \alpha \left( \frac{1+A_1z}{1+B_1z} \right) + \beta \left( \frac{1+A_1z}{1+B_1z} \right)^2 + \gamma \left( \frac{1+B_1z}{1+A_1z} \right) + \frac{\delta(A_1-B_1)z}{(1+A_1z)(1+B_1z)}.$$

$$\eta_2(\alpha,\beta,\gamma,\delta,m;z) := \alpha \left( \frac{1+A_2z}{1+B_2z} \right) + \beta \left( \frac{1+A_2z}{1+B_2z} \right)^2 + \gamma \left( \frac{1+B_2z}{1+A_2z} \right) + \frac{\delta(A_2-B_2)z}{(1+A_2z)(1+B_2z)}.$$

*The functions $\dfrac{1+A_1z}{1+B_1z}$ and $\dfrac{1+A_2z}{1+B_2z}$ are respectively best subordinant and best dominant.*

**Theorem 5.6.** *Let $q(z)$ be univalent in $\Delta$ with $q(0) = 1$ and satisfies (3.9). Let $f \in \mathcal{A}$ and*

$$\Omega(\alpha,\beta,\gamma,r,\lambda;z) := \left(\frac{zI(r+1,\lambda)f(z)}{(I(r,\lambda)f(z))^2}\right) \times \left[\alpha + \beta\left(\frac{zI(r+1,\lambda)f(z)}{(I(r,\lambda)f(z))^2}\right) + \right.$$

$$\left. \gamma\left\{(1+\lambda) + (1+\lambda)\frac{I(r+2,\lambda)f(z)}{I(r+1,\lambda)f(z)} - 2(1+\lambda)\frac{I(r+1,\lambda)f(z)}{I(r,\lambda)f(z)}\right\}\right]. \tag{5.6}$$

*If*

$$\Omega(\alpha,\beta,\gamma,r,\lambda;z) \prec \alpha q(z) + \beta q^2(z) + \gamma z q'(z),$$

*then*

$$\frac{zI(r+1,\lambda)f(z)}{(I(r,\lambda)f(z))^2} \prec q(z)$$

*and $q$ is the best dominant.*

*Proof.* Theorem 5.6 follows by taking $p(z) := \dfrac{zI(r+1,\lambda)f(z)}{(I(r,\lambda)f(z))^2}$. ∎

Since the superordination results are dual of the subordination, we state the results pertaining to the superordination, using the duality.

**Theorem 5.7.** *Let $q(z)$ be convex univalent in $\Delta$ with $q(0) = 1$ and satisfies (3.17). Let $f \in \mathcal{A}$, $0 \neq \left(\dfrac{zI(r+1,\lambda)f(z)}{(I(r,\lambda)f(z))^2}\right) \in \mathcal{H}[1,1] \cap \mathcal{Q}$. If $\Omega(\alpha,\beta,\gamma,r,\lambda;z)$ as defined by (5.6) is univalent in $\Delta$ and*

$$\alpha q(z) + \beta q^2(z) + \gamma z q'(z) \prec \Omega(\alpha,\beta,\gamma,r,\lambda;z),$$

*then*

$$q(z) \prec \left(\frac{zI(r+1,\lambda)f(z)}{(I(r,\lambda)f(z))^2}\right)$$

*and $q$ is the best subordinant.*

Combining Theorem (5.6) and Theorem (5.7), we state the following sandwich theorems.

**Theorem 5.8.** *Let $q_1(z)$ and $q_2(z)$ be convex univalent in $\Delta$ satisfying (3.17) and (3.9) respectively. If $f \in \mathcal{A}$, $0 \neq \left(\dfrac{zI(r+1,\lambda)f(z)}{(I(r,\lambda)f(z))^2}\right) \in \mathcal{H}[1,1] \cap \mathcal{Q}$ and $\Omega(\alpha,\beta,\gamma,r,\lambda;z)$ as defined by (5.6) is univalent in $\Delta$. Further if*

$$\alpha q_1(z) + \beta q_1^2(z) + \gamma z q_1'(z) \prec \Omega(\alpha,\beta,\gamma,r,\lambda;z) \prec \alpha q_2(z) + \beta q_2^2(z) + \gamma z q_2'(z),$$

*then*

$$q_1(z) \prec \left(\frac{zI(r+1,\lambda)f(z)}{(I(r,\lambda)f(z))^2}\right) \prec q_2(z)$$

*and $q_1$ and $q_2$ are respectively the best subordinant and best dominant.*

By taking $\alpha = 1, \beta = 0, \lambda = 0$ and $q_1(z) = \dfrac{1+A_1 z}{1+B_1 z}$, $q_2(z) = \dfrac{1+A_2 z}{1+B_2 z}$, $(-1 \leq B_2 \leq B_1 < A_1 \leq A_2 \leq 1)$ in Theorem 5.8, we get the following corollary of Shanmugam et al. [14]

**Corollary 5.9.** *Let $q_1(z)$ and $q_2(z)$ be convex univalent in $\Delta$ and satisfies (3.17) and (3.9) respectively. Let $f \in \mathcal{A}$ and*

$$\phi_1 \prec (1+\gamma)\frac{z\mathcal{D}^{m+1}f(z)}{(\mathcal{D}^m f(z))^2} + \gamma z\frac{\mathcal{D}^{m+2}f(z)}{(\mathcal{D}^m f(z))^2} - 2\gamma\frac{(\mathcal{D}^{m+1}f(z))^2}{(\mathcal{D}^m f(z))^3} \prec \phi_2,$$

*where*

$$\phi_1 = \frac{1+A_1 z}{1+B_1 z} + \gamma \frac{(A_1 - B_1)z}{(1+B_1 z)^2}$$
$$\phi_2 = \frac{1+A_2 z}{1+B_2 z} + \gamma \frac{(A_2 - B_2)z}{(1+B_2 z)^2}$$

*then*

$$\frac{1+A_1 z}{1+B_1 z} \prec \frac{z\mathcal{D}^{m+1} f(z)}{(\mathcal{D}^m f(z))^2} \prec \frac{1+A_2 z}{1+B_2 z}.$$

*The functions* $\frac{1+A_1 z}{1+B_1 z}$ *and* $\frac{1+A_2 z}{1+B_2 z}$ *are repectively best subordinant and best dominant.*

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