The Australian Journal of Mathematical Analysis and Applications

AJMAA

Volume 4, Issue 2, Article 16, pp. 1-8, 2007

THE CONVERGENCE OF MODIFIED MANN-ISHIKAWA ITERATIONS WHEN APPLIED TO AN ASYMPTOTICALLY PSEUDOCONTRACTIVE MAP

ŞTEFAN M. ŞOLTUZ

Received 26 January, 2007; accepted 5 September, 2007; published 30 November, 2007.
Communicated by: M. Mariani

DEPARTAMENTO DE MATEMATICAS, UNIVERSIDAD DE LOS ANDES, CARRERA 1, NO. 18A-10, BOGOTA, COLOMBIA, AND "T. POPOVICIU" INSTITUTE OF NUMERICAL ANALYSIS, CLUJ-NAPOCA, ROMANIA,
smsoltuz@gmail.com
URL: http://www.uniandes.edu.co/

ABSTRACT. We prove that under minimal conditions the modified Mann and Ishikawa iterations converge when dealing with an asymptotically pseudocontractive map. We give an affirmative answer to the open question from C.E. Chidume and H. Zegeye, Approximate fixed point sequences and convergence theorems for asymptotically pseudocontractive mappings, J. Math. Anal. Appl., 278 (2003), 354–366.

Key words and phrases: Asymptotically hemicontractive map, Modified Mann, Modified Ishikawa iteration.

2000 *Mathematics Subject Classification.* Primary 47H10. Secondary 47H06.

ISSN (electronic): 1449-5910

1. Introduction

Let X be an arbitrary real Banach space and $J : X \to 2^{X^*}$ the *normalized duality mapping* given by

$$Jx := \{f \in X^* : \langle x, f\rangle = \|x\| \, \|f\|, \|f\| = \|x\|\}, \forall x \in X. \tag{1.1}$$

In [13] the following class of maps was introduced:

Definition 1.1. Let X be a normed space and B a subset of X. A map T is said to be *asymptotically pseudocontractive* if there exists a sequence $\{K_n\}$, $K_n \in [1, \infty), \forall n \in \mathbb{N}, \lim_{n\to\infty} K_n = 1$, and there exists $j(x - y) \in J(x - y)$ such that

$$\langle T^n x - T^n y, j(x - y)\rangle \leq K_n \|x - y\|^2, \forall x, y \in B, \forall n \in \mathbb{N}. \tag{1.2}$$

If there exists x^* such that $Tx^* = x^*$, by setting $y := x^*$ in (1.2) we get

$$\langle T^n x - x^*, j(x - x^*)\rangle \leq K_n \|x - x^*\|^2, \forall x, y \in B, \forall n \in \mathbb{N}; \tag{1.3}$$

such a map is called asymptotically hemicontractive.

The modified Mann iteration, (see [8]), is defined by

$$u_{n+1} = (1 - \alpha_n)u_n + \alpha_n T^n u_n. \tag{1.4}$$

The modified Ishikawa iteration is defined, (see [6]), by

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n T^n y_n, \\ y_n &= (1 - \beta_n)x_n + \beta_n T^n x_n. \end{aligned} \tag{1.5}$$

The sequences $\{\alpha_n\} \subset (0, 1)$, $\{\beta_n\} \subset [0, 1)$ satisfy

$$\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0, \ \sum_{n=1}^{\infty} \alpha_n = +\infty. \tag{1.6}$$

We shall give here the most general result concerning the convergence of Mann and Ishikawa iterations dealing with a uniformly Lipschitzian and asymptotically pseudocontractive map. Our result generalizes the main results from [3], [4], [10] and [12]. We also give an affirmative answer to the open question from [4] if the Mann or Ishikawa iteration converges when applied to an asymptotically pseudocontractive (respectively an asymptotically hemicontractive map), in more general spaces than Hilbert spaces.

2. Preliminaries

We recall the following auxiliary results.

Lemma 2.1. [7] *Let X be a Banach space and $x, y \in X$. Then*

$$\|x\| \leq \|x + ry\| \tag{2.1}$$

for all $r > 0$ if and only if there exists $j(x) \in J(x)$ such that $\langle y, j(x)\rangle \geq 0$.

Lemma 2.2. [11] *Let B be a nonempty subset of a Banach space X and let $T : B \to B$ be a map. Then the following conditions are equivalent:*

(i) T is an asymptotically pseudocontractive map,

(ii) for $k_n \in [1, \infty), \forall n \in \mathbb{N}$, we have

$$\|x - y\| \leq \|x - y + r[(k_n I - T^n)x - (k_n I - T^n)y]\|, \forall x, y \in B, \forall r > 0. \tag{2.2}$$

Definition 2.1. Let X be a normed space and B a subset of X, then the map $T : B \to B$ is a uniformly Lipschitzian map if for some $L \geq 1$, we have $\|T^n x - T^n y\| \leq L \|x - y\|, \forall x, y \in B, \forall n \in \mathbb{N}$.

Lemma 2.3. [14] *Let $\{\Psi_n\}$ be a nonnegative sequence satisfying*

$$\Psi_{n+1} \leq (1 - \lambda_n) \Psi_n + \sigma_n, \tag{2.3}$$

where $\lambda_n \in (0, 1)$, $\sum_{n=1}^{\infty} \lambda_n = +\infty$ *and* $\sigma_n = o(\lambda_n)$. *Then* $\lim_{n\to\infty} \Psi_n = 0$.

3. MAIN RESULT

Theorem 3.1. *Let B be a closed convex subset of an arbitrary Banach space X and $(u_n)_n$ defined by (1.4) with $(\alpha_n)_n$ and $(\beta_n)_n$ satisfying (1.6). Let T be an asymptotically pseudo-contractive (or asymptotically hemicontractive) and uniformly Lipschitzian map with $L \geq 1$ self-map of B. If $u_0 \in B$, then the modified Mann iteration (1.4) strongly converges to the nearest x^* fixed point of T.*

Proof. From (1.4) we obtain

$$\begin{aligned} u_n &= u_{n+1} + \alpha_n u_n - \alpha_n T^n u_n \\ &= (1 + \alpha_n^2) u_{n+1} + \alpha_n (\alpha_n k_n I - T^n) u_{n+1} + \\ &\quad - (1 + k_n) \alpha_n^2 u_{n+1} + \alpha_n u_n + \alpha_n (T^n u_{n+1} - T^n u_n) \\ &= (1 + \alpha_n^2) u_{n+1} + \alpha_n (\alpha_n k_n I - T^n) u_{n+1} + \\ &\quad - (1 + k_n) \alpha_n^2 [u_n + \alpha_n (T^n u_n - u_n)] + \alpha_n u_n + \alpha_n (T^n u_{n+1} - T^n u_n) \\ &= (1 + \alpha_n^2) u_{n+1} + \alpha_n (\alpha_n k_n I - T^n) u_{n+1} + (1 + k_n) \alpha_n^3 (u_n - T^n u_n) + \\ &\quad + [1 - (1 + k_n) \alpha_n] \alpha_n u_n + \alpha_n (T^n u_{n+1} - T^n u_n). \end{aligned} \tag{3.1}$$

By using $T^n x^* = x^*$ we observe that

$$x^* = (1 + \alpha_n^2) x^* + \alpha_n (\alpha_n k_n I - T^n) x^* + [1 - (1 + k_n) \alpha_n] \alpha_n x^*. \tag{3.2}$$

From (3.1) and (3.2) we get

$$\begin{aligned} & x^* - u_n \\ &= (1 + \alpha_n^2)(x^* - u_{n+1}) + \\ &+ \alpha_n \left((\alpha_n k_n I - T^n) x^* - (\alpha_n k_n I - T^n) u_{n+1} \right) + \\ &+ [1 - (1 + k_n) \alpha_n] \alpha_n (x^* - u_n) + (1 + k_n) \alpha_n^3 (T^n u_n - u_n) + \\ &+ \alpha_n (T^n u_n - T^n u_{n+1}). \end{aligned} \tag{3.3}$$

The norm of the sum of the first two terms on the right-hand side of (3.3) is equal to

$$(1 + \alpha_n^2) \left\| (x^* - u_{n+1}) + \frac{\alpha_n}{1 + \alpha_n^2} \left((\alpha_n k_n I - T^n) x^* - (\alpha_n k_n I - T^n) u_{n+1} \right) \right\|. \tag{3.4}$$

Using (2.1) with

$$\begin{aligned} x &:= x^* - u_{n+1}, \\ y &:= (\alpha_n k_n I - T^n) x^* - (\alpha_n k_n I - T^n) u_{n+1}, \\ r &:= \frac{\alpha_n}{1 + \alpha_n^2}, \end{aligned} \tag{3.5}$$

we obtain

$$\left\|(1+\alpha_n^2)(x^*-u_{n+1})+\alpha_n\left((\alpha_n k_n I-T^n)x^*-(\alpha_n k_n I-T^n)u_{n+1}\right)\right\| \geq (1+\alpha_n^2)\left\|x^*-u_{n+1}\right\|. \tag{3.6}$$

From (3.3) and (3.6) it follows that

$$\begin{aligned}&\|x^*-u_n\| \\ &\overset{?}{\geq} \left\|(1+\alpha_n^2)(x^*-u_{n+1})+\alpha_n\left((\alpha_n k_n I-T^n)x^*-(\alpha_n k_n I-T^n)u_{n+1}\right)\right\|+ \\ &+[1-(1+k_n)\alpha_n]\alpha_n\left\|x^*-u_n\right\|-(1+k_n)\alpha_n^3\left\|T^n u_n-u_n\right\|+ \\ &-\alpha_n\left\|T^n u_n-T^n u_{n+1}\right\| \\ &\geq (1+\alpha_n^2)\left\|x^*-u_{n+1}\right\|+[1-(1+k_n)\alpha_n]\alpha_n\left\|x^*-u_n\right\|+ \\ &-(1+k_n)\alpha_n^3\left\|T^n u_n-u_n\right\|-\alpha_n\left\|T^n u_n-T^n u_{n+1}\right\|.\end{aligned} \tag{3.7}$$

We shall prove later the first inequality from (3.7). Supposing that (3.7) holds, we obtain

$$\begin{aligned}&(1+\alpha_n^2)\left\|x^*-u_{n+1}\right\| \\ &\leq \{1-[1-(1+k_n)\alpha_n]\alpha_n\}\left\|x^*-u_n\right\|+(1+k_n)\alpha_n^3\left\|T^n u_n-u_n\right\|+ \\ &+\alpha_n\left\|T^n u_n-T^n u_{n+1}\right\|.\end{aligned} \tag{3.8}$$

Also, we know that

$$\begin{aligned}\|u_n-T^n u_n\| &\leq \|T^n u_n-T^n x^*\|+\|x^*-u_n\| \\ &\leq L\left\|x^*-u_n\right\|+\left\|x^*-u_n\right\| \\ &=(L+1)\left\|x^*-u_n\right\|.\end{aligned} \tag{3.9}$$

Using (1.4), (3.9) and the fact that T is a uniformly Lipschitzian map, we obtain

$$\begin{aligned}\|T^n u_{n+1}-T^n u_n\| &\leq L\left\|u_{n+1}-u_n\right\| \\ &=\alpha_n L\left\|u_n-T^n u_n\right\| \\ &\leq \alpha_n L\,(L+1)\left\|x^*-u_n\right\|.\end{aligned} \tag{3.10}$$

From (3.8), (3.9) and (3.10), by using $(1+\alpha_n^2)^{-1}\leq 1, \forall n\in\mathbb{N}$, we get

$$\begin{aligned}&\|x^*-u_{n+1}\| \\ &\leq \{1-[1-(1+k_n)\alpha_n]\alpha_n\}\left\|x^*-u_n\right\|+ \\ &+(1+k_n)\alpha_n^3\,(L+1)\left\|x^*-u_n\right\|+\alpha_n^2 L\,(L+1)\left\|x^*-u_n\right\|.\end{aligned} \tag{3.11}$$

The condition $\lim_{n\to\infty}\alpha_n=0$ implies the existence of $n_0\in\mathbb{N}$, such that

$$\alpha_n^2\leq\frac{1}{18\ (1+L)}, \forall n\geq n_0 \tag{3.12}$$

Condition (3.12) assures the following inequalities, $\forall n\geq n_0$,

$$\begin{aligned}\alpha_n &\leq \frac{1}{3\left((1+k_n)+L\,(1+L)\right)}, \\ \alpha_n^2 &\leq \frac{1}{3\,(1+L)\,(1+k_n)}, \\ \alpha_n &\leq \frac{1}{3}.\end{aligned} \tag{3.13}$$

Using (3.11) and (3.13) we observe that

$$\begin{aligned}(3.14)\quad &\{1-[1-(1+k_n)\alpha_n]\alpha_n\}+(1+k_n)\alpha_n^3\,(L+1)+\alpha_n^2L\,(L+1)\\ &=1-\alpha_n+\alpha_n\left[((1+k_n)+L\,(1+L))\,\alpha_n+(1+k_n)\,(L+1)\,\alpha_n^2\right]\\ &\leq 1-\alpha_n+\alpha_n\left(\frac{1}{3}+\frac{1}{3}\right)=1-\frac{1}{3}\alpha_n.\end{aligned}$$

Relations $(3.11)\,,(3.14)\,,$ lead us to

$$(3.15)\quad \|x^*-u_{n+1}\|\leq\left(1-\frac{1}{3}\alpha_n\right)\|x^*-u_n\|\,,\forall n\geq n_0.$$

Setting in (2.3) from lemma 2.3

$$\begin{aligned}(3.16)\quad &\Psi_n:=\|x^*-u_n\|\,,\forall n\geq n_0,\\ &\lambda_n:=\frac{1}{3}\alpha_n,\forall n\geq n_0,\\ &\sigma_n:=0,\forall n\in\mathbb{N},\end{aligned}$$

we get

$$(3.17)\quad \lim_{n\to\infty}\Psi_n=\lim_{n\to\infty}\|x^*-u_n\|=0.$$

We prove now the first inequality from (3.7) . Set in (3.7)

$$\begin{aligned}(3.18)\quad a&=\left(1+\alpha_n^2\right)(x^*-u_{n+1})\,,\\ a'&=(1+\alpha_n^2)(x^*-u_{n+1})+\\ &\quad+\alpha_n\left((\alpha_nk_nI-T^n)x^*-(\alpha_nk_nI-T^n)u_{n+1}\right),\\ b&=[1-(1+k_n)\alpha_n]\alpha_n\,(x^*-u_n)\,,\\ c&=(1+k_n)\alpha_n^3\,(T^nu_n-u_n)\,,\\ d&=\alpha_n\,(T^nu_{n+1}-T^nu_n)\,,\end{aligned}$$

to obtain

$$\begin{aligned}(3.19)\quad &\|a'+b+c+d\|\\ &=\|x^*-u_n\|=||(1+\alpha_n^2)(x^*-u_{n+1})+\\ &+\alpha_n\left((\alpha_nk_nI-T^n)x^*-(\alpha_nk_nI-T^n)u_{n+1}\right)+\\ &+[1-(1+k_n)\alpha_n]\alpha_n\,(x^*-u_n)+(1+k_n)\alpha_n^3\,(T^nu_n-u_n)+\\ &+\alpha_n\,(T^nu_n-T^nu_{n+1})\,||\\ &\overset{?}{\geq}\left\|(1+\alpha_n^2)(x^*-u_{n+1})+\alpha_n\left((\alpha_nk_nI-T^n)x^*-(\alpha_nk_nI-T^n)u_{n+1}\right)\right\|+\\ &+[1-(1+k_n)\alpha_n]\alpha_n\,\|x^*-u_n\|-(1+k_n)\alpha_n^3\,\|T^nu_n-u_n\|+\\ &-\alpha_n\,\|T^nu_n-T^nu_{n+1}\|\\ &=\|a'\|+\|b\|-\|c\|-\|d\|\,.\end{aligned}$$

We shall now prove (3.19) using the following relations

$$
\begin{aligned}
&\|a'+b+c+d\|+\|c\|+\|d\| \\
&\geq \|a'+b+c+d\|+\|c+d\| \\
&\overset{?}{\geq} \|a'\|+\|b\| \\
&\geq \|a\|+\|b\| .
\end{aligned} \tag{3.20}
$$

The last inequality from (3.20) is given by (2.1) , that is $\|a'\| \geq \|a\|$. We further prove that

$$\|a'+b+c+d\|+\|c+d\| \overset{?}{\geq} \|a'\|+\|b\| . \tag{3.21}$$

By using

$$x^*-u_{n+1}=x^*-u_n+\alpha_n\left(u_n-T^nu_n\right), \tag{3.22}$$

we obtain

$$
\begin{aligned}
\|a'\| = &||(1+\alpha_n^2)(x^*-u_{n+1})+ \\
&+\alpha_n\left((\alpha_nk_nI-T^n)x^*-(\alpha_nk_nI-T^n)u_{n+1}\right)|| \\
&=||(1+\alpha_n^2)(x^*-u_{n+1})+k_n\alpha_n^2(x^*-u_{n+1}) \\
&-\alpha_n\left(T^nx^*-T^nu_{n+1}\right)|| \\
&=||\left(1+k_n\right)\alpha_n^2\left(\alpha_nu_n-\alpha_nT^nu_n\right)+ \\
&+\left(1+k_n\right)\alpha_n^2\left(x^*-u_n\right)+\left(x^*-u_{n+1}\right)+ \\
&-\alpha_n\left(T^nx^*-T^nu_{n+1}\right)|| \\
&=||\left(1+k_n\right)\alpha_n^3\left(u_n-T^nu_n\right)+ \\
&-\alpha_n\left(T^nx^*-T^nu_{n+1}\right)+\left(1+k_n\right)\alpha_n^2\left(x^*-u_n\right)+ \\
&+\left(x^*-u_n\right)+\alpha_n\left(u_n-T^nu_n\right)|| \\
&=||\left(1+k_n\right)\alpha_n^3\left(u_n-T^nu_n\right)+ \\
&-\alpha_n\left(T^nx^*-T^nu_{n+1}-T^nx^*+T^nu_n\right)+\left(1+k_n\right)\alpha_n^2\left(x^*-u_n\right)+ \\
&+\left(x^*-u_n\right)+\alpha_n\left(u_n-T^nu_n\right)+\alpha_n\left(-T^nx^*+T^nu_n\right)|| \\
&=||\left(1+k_n\right)\alpha_n^3\left(u_n-T^nu_n\right)+ \\
&-\alpha_n\left(-T^nu_{n+1}+T^nu_n\right)+\left(1+k_n\right)\alpha_n^2\left(x^*-u_n\right)+ \\
&+\left(x^*-u_n\right)+\alpha_n\left(u_n-x^*\right)||.
\end{aligned} \tag{3.23}
$$

The last equality is true because $T^nx^*=x^*$. Finally, we have

$$
\begin{aligned}
\|a'\| = &||\left(1+k_n\right)\alpha_n^3\left(u_n-T^nu_n\right)+ \\
&-\alpha_n\left(T^nu_n-T^nu_{n+1}\right)+ \\
&+\left(1-\alpha_n+\left(1+k_n\right)\alpha_n^2\right)\left(x^*-u_n\right)||
\end{aligned} \tag{3.24}
$$

$$
\begin{aligned}
&\leq \left\| -\left((1+k_n)\,\alpha_n^3 \,(u_n - T^n u_n) + \alpha_n \,(T^n u_n - T^n u_{n+1}) \right) \right\| + \\
&+ \left(1 - \alpha_n + (1+k_n)\,\alpha_n^2\right) \|x^* - u_n\| \\
&= \|x^* - u_n\| + \\
&+ \left\| -\left((1+k_n)\,\alpha_n^3 \,(u_n - T^n u_n) + \alpha_n \,(T^n u_n - T^n u_{n+1}) \right) \right\| + \\
&- \left(1 - (1+k_n)\,\alpha_n^2\right) \|x^* - u_n\| \\
&= \|a' + b + c + d\| + \|c + d\| - \|b\| .
\end{aligned}
$$

The last equality is true because we already know that

$$\|x^* - u_n\| = \|a' + b + c + d\| . \tag{3.25}$$

∎

Remark 3.1. If $\lim_{n\to\infty} \alpha_n \neq 0$, then our Theorem *3.1* holds supposing condition (3.12) is satisfied.

The modified Ishikawa iteration also converges, being equivalent to the modified Mann iteration.

Theorem 3.2. [11] *Let B be a closed convex subset of an arbitrary Banach space X, $(x_n)_n$ and $(u_n)_n$ defined by (1.5) and (1.4) with $(\alpha_n)_n$ and $(\beta_n)_n$ satisfying (1.6). Let T be an asymptotically pseudocontractive and uniformly Lipschitzian with $L \geq 1$ self-map of B. Let x^* be a fixed point of T. If $u_0 = x_0 \in B$, then the following two assertions are equivalent:*
(i) the modified Mann iteration (1.4) strongly converges to x^*,
(ii) the modified Ishikawa iteration (1.5) strongly converges to x^*.

Remark 3.2. Each fixed point has its own basin of attraction. The map T has no unique fixed point. The starting point is crucial for the convergence of Mann or Ishikawa iteration. For example, take $T = I$, the identity map on B, with $k_n = 1, \forall n \in \mathbb{N}$. Each point of B becomes a fixed point and the starting point is directly a fixed point.

Theorem 3.1 generalizes the Theorem from [3] because in [3] the set B is bounded, the space X is uniformly convex and $(\alpha_n)_n$ and $(\beta_n)_n$ satisfy some additional conditions. We also generalize Theorem 1 from [12], because the space is smooth and the following conditions are required: $\sum (k_n - 1) < +\infty$, $\sum \alpha_n^2 < +\infty$ and $\sum \beta_n < +\infty$. Our Theorem 3.1 generalizes the main results from [2] and [10] because the map T satisfies the following restrictive condition:

$$\langle T^n x_{n+1} - x^*, j\,(x_{n+1} - x^*)\rangle \leq k_n \|x_{n+1} - x^*\|^2 - \phi\left(\|x_{n+1} - x^*\|\right), \tag{3.26}$$

where $(x_n)_n$ is the modified Mann (respectively modified Ishikawa) iterations, x^* is a fixed point and $\phi : [0, \infty) \to [0, \infty)$ is a strictly increasing function with $\phi\,(0) = 0$.

In [1] and [5] the convergence of (1.4) and (1.5) is shown, dealing with an asymptotically pseudocontractive map without being uniformly Lipschitzian. However, in [1] and [5] the assumptions are more restrictive than those from our Theorem 3.1: the Banach space is uniformly smooth, the set B is bounded, respectively $T(B)$ is bounded and the map T satisfies condition (3.26).

REFERENCES

[1] S.S. CHANG, Some results for asymptotically pseudo-contractive mappings and asymptotically nonexpansive mappings, *Proc. Amer. Math. Soc.*, **129** (2000), 845–853.

[2] S.S. CHANG, J.Y. PARK and Y.J. CHO, Iterative approximations of fixed points for asymptotically nonexpansive mappings in Banach spaces, *Bull. Korean Math. Soc.*, **37** (2000), 109–119.

[3] C.E. CHIDUME, Convergence theorems for asymptotically pseudocontractive mappings, *Nonlinear Analysis*, **49** (2002), 1–11.

[4] C.E. CHIDUME and H. ZEGEYE, Approximate fixed point sequences and convergence theorems for asymptotically pseudocontractive mappings, *J. Math. Anal. Appl.*, **278** (2003), 354–366.

[5] D.I. IGBOKWE, Iterative construction of fixed points of asymptotically pseudocontractive maps, *Panamer. Math. J.*, **13** (2003), 83–97.

[6] S. ISHIKAWA, Fixed points by a new iteration method, *Proc. Amer. Math. Soc.*, **44** (1974), 147-150.

[7] T. KATO, Nonlinear semigroup and evolution equations, *J. Math. Soc. Japan*, **19**(1967), 508–520.

[8] W.R. MANN, Mean value in iteration, *Proc. Amer. Math. Soc.*, **4** (1953), 506–510.

[9] M.O. OSILIKE, Iterative approximation of fiexd points of asymptotically demicontractive mappings, *Indian J. Pure Appl. Math.*, **29** (1998), 1291–1300.

[10] M.O. OSILIKE and D.I. IGBBOKWE, Convergence theorems for asymptotically pseudocontractive maps, *Bull. Korean Math. Soc.*, **39** (2002), 389–399.

[11] B.E. RHOADES and ŞTEFAN M. ŞOLTUZ, The equivalence between the convergences of Ishikawa and Mann iterations for asymptotically pseudocontractive map, *J. Math. Anal. Appl.*, **283** (2003), 681–688.

[12] B.K. SHARMA and D.R. SAHU, Existence and approximation results for asymptotically pseudocontractive mappings, *Indian J. Pure Appl. Math.*, **31** (2000), 185–196.

[13] J. SCHU, Iterative construction of fixed points of asymptotically nonexpansive mappings, *J. Math. Anal. Appl.*, **158** (1991), 407–413.

[14] H. ZHOU and J. YUTING, *Approximation of fixed points of strongly pseudocontractive maps without Lipschitz assumption*, *Proc. Amer. Math. Soc.*, **125** (1997), 1705–1709.