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A NONLINEAR PROXIMAL ALTERNATING DIRECTIONS METHOD FOR STRUCTURED VARIATIONAL INEQUALITIES

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ABSTRACT. In this paper, we present a nonlinear proximal alternating directions method (NPADM) for solving a class of structured variational inequalities (SVI). By choosing suitable Bregman functions, we generalize the proximal alternating directions method proposed by He, et al. [13]. The convergence of the method is proved under quite mild assumptions and flexible parameter conditions.

Key words and phrases: Alternating directions method, Bregman function, Proximal point algorithm, Structured variational inequalities.

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1. INTRODUCTION

A variational inequality (VI) problem is to find a vector $u^* \in \Omega$ such that

$$(u' - u^*)^T F(u^*) \geq 0, \qquad \forall u' \in \Omega, \tag{1.1}$$

where Ω is a nonempty closed convex subset of $\mathbb{R}^n$, and F is a mapping from Ω into $\mathbb{R}^n$. In this paper, we consider the VI problem with the following structure:

$$u = \begin{pmatrix} x \\ y \end{pmatrix}, \qquad F(u) = \begin{pmatrix} f(x) \\ g(y) \end{pmatrix}, \tag{1.2}$$

$$\Omega = \{(x, y) \mid x \in \mathbb{X}, \ \ y \in \mathbb{Y}, \ \ Ax + By = b\}, \tag{1.3}$$

where $\mathbb{X}$ and $\mathbb{Y}$ are given nonempty closed convex subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, $A \in \mathbb{R}^{l\times n}$ and $B \in \mathbb{R}^{l\times m}$ are given matrices, $b \in \mathbb{R}^l$ is a given vector, $f : \mathbb{X} \to \mathbb{R}^n$ and $g : \mathbb{Y} \to \mathbb{R}^m$ are given monotone operators. Such problems have many important applications, especially in economics and transportation equilibrium problems, which can be found in Bertsekas and Gafin [1], Dafermos [7], Eckstein and Fukushima [9], Fukushima [10] and Nagurney and Ramanujam [14].

By attaching a Lagrange multiplier vector $\lambda \in \mathbb{R}^l$ to the linear constraint $Ax + By = b$, one obtains an equivalent form of problem (1.1)-(1.3):

$$w^* \in \mathbb{W}, \qquad (w' - w^*)^T Q(w^*) \geq 0, \qquad \forall w' \in \mathbb{W}, \tag{1.4}$$

where

$$w = \begin{pmatrix} x \\ y \\ \lambda \end{pmatrix}, \qquad Q(w) = \begin{pmatrix} f(x) - A^T\lambda \\ g(y) - B^T\lambda \\ Ax + By - b \end{pmatrix}, \qquad \mathbb{W} = \mathbb{X} \times \mathbb{Y} \times \mathbb{R}^l. \tag{1.5}$$

In the following, we denote VI problem (1.4)-(1.5) by SVI$(\mathbb{W}, Q)$.

The classical proximal point algorithm generates a sequence $\{w^k\}$ via the following scheme:

$$w^{k+1} \in \mathbb{W}, \qquad \left(w' - w^{k+1}\right)^T \left[Q(w^{k+1}) + c\left(w^{k+1} - w^k\right)\right] \geq 0, \qquad \forall w' \in \mathbb{W}. \tag{1.6}$$

Much recent research work has centered on nonlinear generalizations of (1.6) based on Bregman functions [4, 6, 8]. Suppose ϕ is a Bregman function (defined in Section 3). Then an alternative to (1.6) is the nonlinear proximal point algorithm (NPPA):

$$w^{k+1} \in \mathbb{W}, \quad \left(w' - w^{k+1}\right)^T \left\{Q(w^{k+1}) + c\left[\nabla\phi\left(w^{k+1}\right) - \nabla\phi\left(w^k\right)\right]\right\} \geq 0, \quad \forall w' \in \mathbb{W}.$$

He et. al. [13] proposed the proximal alternating directions method (PADM), which generates the new triplet $w^{k+1} = \left(x^{k+1}, y^{k+1}, \lambda^{k+1}\right) \in \mathbb{W}$ from $w^k = \left(x^k, y^k, \lambda^k\right) \in \mathbb{W}$ by the following scheme, for all $x' \in \mathbb{X}$ and $y' \in \mathbb{Y}$ we have

$$\left(x' - x^{k+1}\right)^T \left\{f(x^{k+1}) - A^T\left[\lambda^k - \beta\left(Ax^{k+1} + By^k - b\right)\right] + r\left(x^{k+1} - x^k\right)\right\} \geq 0, \tag{1.7}$$

$$\left(y' - y^{k+1}\right)^T \left\{g(y^{k+1}) - B^T\left[\lambda^k - \beta\left(Ax^{k+1} + By^{k+1} - b\right)\right] + s\left(y^{k+1} - y^k\right)\right\} \geq 0, \tag{1.8}$$

and

$$\lambda^{k+1} = \lambda^k - \beta\left(Ax^{k+1} + By^{k+1} - b\right), \tag{1.9}$$

where $\beta, \ r, \ s$ are given positive constants. It is clear that the proximal alternating directions method (ADM) (1.7)-(1.9) adopts the new information in the iteration whenever possible. The sub-VI problems (1.7) and (1.8) are easier to be solved than the original alternating directions method [11, 12]. Combining the advantages of the NPPA and the PADM, we propose a nonlinear proximal alternating directions method (NPADM).

The rest of this paper is organized as follows. In the next section, we summarize some preliminaries. Section 3 presents a nonlinear proximal ADM method and gives some preparations concerning the convergence analysis. Then we prove the convergence of the proposed method in Section 4. Finally, in Section 5, some concluding remarks are made.

2. PRELIMINARIES

In this section, we state the definitions of the D-function and Bregman function, and summarize some basic properties of them.

Definition 2.1. [2] D-Function. Let $\mathbb{S} \subseteq \mathbb{R}^n$ be a nonempty open convex set and let $\bar{\mathbb{S}}$ denote its closure. Let $\phi : \bar{\mathbb{S}} \to \mathbb{R}$ be a strictly convex function that is continuously differentiable on $\mathbb{S}$. The function $D_\phi : \bar{\mathbb{S}} \times \mathbb{S} \to \mathbb{R}$, defined by

$$D_\phi(u, v) = \phi(u) - \phi(v) - (\nabla\phi(v))^T (u - v)\,, \tag{2.1}$$

is called the D-function.

Remark 2.1. From the strict convexity of ϕ, one can prove that $D_\phi(u, v) \geq 0$, and $D_\phi(u, v) = 0$ if and only if $u = v$.

D-functions do not in general behave like Euclidean distances. Generally, they are not symmetric and do not obey the triangle inequality. They have simple properties like that of squares of Euclidean distances, as captured in the following lemma:

Lemma 2.1. *(*[5]*, Lemma 3.1) Given $\phi : \mathbb{R}^n \to (-\infty, +\infty]$, D_ϕ defined in Eq. (2.1), and $u, v, t \in \mathbb{R}^n$ such that $\phi(u), \phi(v), \phi(t)$ are finite and ϕ is differentiable at v and t, then we have*

$$D_\phi(u, v) = D_\phi(u, t) + D_\phi(t, v) + [\nabla\phi(t) - \nabla\phi(v)]^T (u - t)\,. \tag{2.2}$$

Proof. The result can be conformed by a straightforward substitution of the definition (2.1) into (2.2). ∎

To apply the D-function D_ϕ effectively in the proposed method, we need to add some additional conditions on the function ϕ.

Definition 2.2. [3] Bregman Function. Let $\mathbb{S} \subseteq \mathbb{R}^n$ be a nonempty open convex set, and let $\bar{\mathbb{S}}$ be its closure. A function $\phi : \bar{\mathbb{S}} \to \mathbb{R}$ is called a Bregman function with zone $\mathbb{S}$ if it satisfies the following conditions:

(i) ϕ is strictly convex and continuous on $\bar{\mathbb{S}}$.
(ii) ϕ is continuously differentiable on $\mathbb{S}$.
(iii) Given any $u \in \bar{\mathbb{S}}$ and $\alpha \in \mathbb{R}$, the right partial level set $\mathrm{L}(u, \alpha) = \{v \in \mathbb{S} | D_\phi(u, v) \leq \alpha\}$ is bounded.
(iv) If $\{v^k\} \subset \mathbb{S}$ is a convergent sequence with limit v^∞, then $D_\phi(v^\infty, v^k) \to 0$.
(v) If $\{u^k\} \subset \bar{\mathbb{S}}$ and $\{v^k\} \subset \mathbb{S}$ are sequences such that $v^k \to v^\infty$ and $\{u^k\}$ is bounded, and furthermore $D_\phi(u^k, v^k) \to 0$, then one has $u^k \to v^\infty$.

The following lemma due to Solodov and Svaiter [15] will be useful in proving the convergence of the proposed method.

Lemma 2.2. *Let ϕ be a convex function that satisfies conditions (i) and (ii) in Definition 2.2. Suppose that, for $\{u^k\} \subset \bar{\mathbb{S}}$ and $\{v^k\} \subset \mathbb{S}$, the following limiting condition is satisfied:*

$$\lim_{k\to\infty} D_\phi(u^k, v^k) = 0.$$

If one of the sequences $\{u^k\}$ and $\{v^k\}$ is convergent, then the other also converges to the same limit.

Proof. See Theorem 2.4 in [15]. ∎

In the proposed method, we use D-functions to replace the quadratic terms in (1.7)-(1.8). And the related Bregman functions φ and ψ need to satisfy Condition A, which is stated as follows:
Condition A: *Let φ and ψ be Bregman functions with zone $\mathbb{X}_0$ and zone $\mathbb{Y}_0$, respectively, where $\mathbb{X} \subset \mathbb{X}_0$ and $\mathbb{Y} \subset \mathbb{Y}_0$. And $\nabla\varphi$ and $\nabla\psi$ are Lipschitz continuous, i.e., there exist constants $\nu > 0$ and $\tau > 0$, such that*

$$\|\nabla\varphi(x_1) - \nabla\varphi(x_2)\| \leq \nu \|x_1 - x_2\|, \qquad \forall\, x_1, x_2 \in \mathbb{X}_0 \tag{2.3}$$

and

$$\|\nabla\psi(y_1) - \nabla\psi(y_2)\| \leq \tau \|y_1 - y_2\|, \qquad \forall\, y_1, y_2 \in \mathbb{Y}_0. \tag{2.4}$$

For convenience, we make some basic assumptions to guarantee that the problem under consideration is solvable and the NPADM is well defined.
Assumption A: $f(x)$ and $g(y)$ are continuous on $\mathbb{X}$ and $\mathbb{Y}$, respectively.
Assumption B: The solution set of SVI$(\mathbb{W}, Q)$, denoted by $\mathbb{W}^*$, is nonempty.

3. A NONLINEAR PROXIMAL ADM METHOD AND ITS MAIN PROPERTIES

We now formally present a nonlinear proximal ADM method for monotone variational inequalities. Starting with an initial arbitrary triplet $w^0 = \left(x^0, y^0, \lambda^0\right) \in \mathbb{X} \times \mathbb{Y} \times \mathbb{R}^l$ and three positive constants r, s and β. A sequence $\left\{w^k\right\} = \left\{\left(x^k, y^k, \lambda^k\right)\right\} \subset \mathbb{X} \times \mathbb{Y} \times \mathbb{R}^l$, $k \geq 0$ is successively generated by the following steps:
Step 1. *Find $x^{k+1} \in \mathbb{X}$ such that*

$$\left(x' - x^{k+1}\right)^T \left[f_{k+1}(x^{k+1}) + r\left(\nabla\varphi(x^{k+1}) - \nabla\varphi(x^k)\right)\right] \geq 0, \qquad \forall x' \in \mathbb{X}. \tag{3.1}$$

Step 2. *Find $y^{k+1} \in \mathbb{Y}$ such that*

$$\left(y' - y^{k+1}\right)^T \left[g_{k+1}(y^{k+1}) + s\left(\nabla\psi(y^{k+1}) - \nabla\psi(y^k)\right)\right] \geq 0, \qquad \forall y' \in \mathbb{Y}. \tag{3.2}$$

Step 3. *Update λ^{k+1} via*

$$\lambda^{k+1} = \lambda^k - \beta\left(Ax^{k+1} + By^{k+1} - b\right). \tag{3.3}$$

Here

$$f_{k+1}(x) = f(x) - A^T\left[\lambda^k - \beta\left(Ax + By^k - b\right)\right], \tag{3.4}$$

$$g_{k+1}(y) = g(y) - B^T\left[\lambda^k - \beta\left(Ax^{k+1} + By - b\right)\right]. \tag{3.5}$$

Throughout the rest of the paper, we denote the function $\phi : \mathbb{W} \to \mathbb{R}$ by

$$\phi(w) = r\varphi(x) + s\psi(y) + \frac{1}{2\beta}\|\lambda\|^2 + \frac{\beta}{2}\|By\|^2, \tag{3.6}$$

which is also a Bregman function.
First, we have the following lemma.

Lemma 3.1. *Let* $w^{k+1} = \left(x^{k+1}, y^{k+1}, \lambda^{k+1}\right)$ *be generated by (3.1)-(3.5) from given* $w^k = \left(x^k, y^k, \lambda^k\right)$*. Then for any* $w^* = (x^*, y^*, \lambda^*) \in \mathbb{W}^*$*, it holds that*

$$\begin{aligned}(3.7)\qquad &\left(\lambda^k - \lambda^*\right)^T \left(Ax^{k+1} + By^{k+1} - b\right) \\ &\geq \beta \left\|Ax^{k+1} + By^{k+1} - b\right\|^2 + \beta \left(Ax^{k+1} - Ax^*\right)^T \left(By^k - By^{k+1}\right) \\ &\quad + \ r \left(x^{k+1} - x^*\right)^T \left(\nabla\varphi(x^{k+1}) - \nabla\varphi(x^k)\right) \\ &\quad\quad + \ s \left(y^{k+1} - y^*\right)^T \left(\nabla\psi(y^{k+1}) - \nabla\psi(y^k)\right).\end{aligned}$$

Proof. Since w^* is a solution of $\mathrm{SVI}(\mathbb{W}, Q)$ and $x^{k+1} \in \mathbb{X}$, $y^{k+1} \in \mathbb{Y}$, we have

$$(3.8)\qquad \left(x^{k+1} - x^*\right)^T \left[f(x^*) - A^T\lambda^*\right] \geq 0,$$

and

$$(3.9)\qquad \left(y^{k+1} - y^*\right)^T \left[g(x^*) - B^T\lambda^*\right] \geq 0.$$

On the other hand, from (3.1)-(3.2), $x^* \in \mathbb{X}$ and $y^* \in \mathbb{Y}$, it follows that

$$\begin{aligned}(3.10)\qquad &\left(x^* - x^{k+1}\right)^T \\ &\left\{f(x^{k+1}) - A^T\left[\lambda^k - \beta\left(Ax^{k+1} + By^k - b\right)\right] + r\left(\nabla\varphi(x^{k+1}) - \nabla\varphi(x^k)\right)\right\} \geq 0,\end{aligned}$$

and

$$\begin{aligned}(3.11)\qquad &\left(y^* - y^{k+1}\right)^T \\ &\left\{g(y^{k+1}) - B^T\left[\lambda^k - \beta\left(Ax^{k+1} + By^{k+1} - b\right)\right] + s\left(\nabla\psi(y^{k+1}) - \nabla\psi(y^k)\right)\right\} \geq 0.\end{aligned}$$

Adding Eqs. (3.8) and (3.10) and using the monotonicity of f, we have

$$\begin{aligned}(3.12)\qquad &\left(x^{k+1} - x^*\right)^T A^T \left[\left(\lambda^k - \lambda^*\right) - \beta\left(Ax^{k+1} + By^k - b\right)\right] \\ &\geq \ r\left(x^{k+1} - x^*\right)^T \left(\nabla\varphi(x^{k+1}) - \nabla\varphi(x^k)\right).\end{aligned}$$

In a similar way, adding Eqs. (3.9) and (3.11) and using the monotonicity of g, we have

$$\begin{aligned}(3.13)\qquad &\left(y^{k+1} - y^*\right)^T B^T \left[\left(\lambda^k - \lambda^*\right) - \beta\left(Ax^{k+1} + By^{k+1} - b\right)\right] \\ &\geq \ s\left(y^{k+1} - y^*\right)^T \left(\nabla\psi(y^{k+1}) - \nabla\psi(y^k)\right).\end{aligned}$$

Adding (3.12) and (3.13), and using $Ax^* + By^* = b$, it follows the assertion. ∎

Notice that

$$\begin{aligned}(3.14)\qquad \frac{1}{2\beta}\left\|\lambda^k - \lambda^*\right\|^2 = \ &\frac{1}{2\beta}\left\|\lambda^{k+1} - \lambda^*\right\|^2 \\ &- \frac{1}{2\beta}\left\|\lambda^k - \lambda^{k+1}\right\|^2 + \frac{1}{\beta}\left(\lambda^k - \lambda^*\right)^T\left(\lambda^k - \lambda^{k+1}\right).\end{aligned}$$

Then from (3.3) and Lemma 3.1, we get

$$\begin{aligned}(3.15)\qquad \frac{1}{2\beta}\left\|\lambda^k - \lambda^*\right\|^2 \geq &\frac{1}{2\beta}\left\|\lambda^{k+1} - \lambda^*\right\|^2 + \frac{\beta}{2}\left\|Ax^{k+1} + By^{k+1} - b\right\|^2 \\ &+ \ \beta\left(Ax^{k+1} - Ax^*\right)^T\left(By^k - By^{k+1}\right) \\ &\quad + \ r\left(x^{k+1} - x^*\right)^T\left(\nabla\varphi(x^{k+1}) - \nabla\varphi(x^k)\right) \\ &\quad\quad + \ s\left(y^{k+1} - y^*\right)^T\left(\nabla\psi(y^{k+1}) - \nabla\psi(y^k)\right).\end{aligned}$$

Similarly, we have

$$\frac{\beta}{2}\left\|By^k - By^*\right\|^2 = \frac{\beta}{2}\left\|By^{k+1} - By^*\right\|^2 - \frac{\beta}{2}\left\|By^k - By^{k+1}\right\|^2 + \beta\left(By^k - By^*\right)^T\left(By^k - By^{k+1}\right). \tag{3.16}$$

Adding (3.15) and (3.16), and using $Ax^* + By^* = b$, we obtain

$$\begin{aligned}&\frac{1}{2\beta}\left\|\lambda^k - \lambda^*\right\|^2 + \frac{\beta}{2}\left\|B(y^k - y^*)\right\|^2 \\ &\geq \frac{1}{2\beta}\left\|\lambda^{k+1} - \lambda^*\right\|^2 + \frac{\beta}{2}\left\|B\left(y^{k+1} - y^*\right)\right\|^2 + \frac{\beta}{2}\left\|Ax^{k+1} + By^k - b\right\|^2 \\ &\quad + r\left(x^{k+1} - x^*\right)^T\left(\nabla\varphi(x^{k+1}) - \nabla\varphi(x^k)\right) \\ &\quad + s\left(y^{k+1} - y^*\right)^T\left(\nabla\psi(y^{k+1}) - \nabla\psi(y^k)\right).\end{aligned} \tag{3.17}$$

4. CONVERGENCE OF THE PROPOSED METHOD

Now, we are in the stage to prove the main theorem of this paper, which shows that the generated sequence $\{w^k\}$ is monotone under the D-functions.

Theorem 4.1. *Let $\{w^k\}$ be a sequence generated by the proposed method and let $w^* = (x^*, y^*, \lambda^*)$ be a solution of SVI$(\mathbb{W}, Q)$. Then for all k, we have*

$$\begin{aligned}D_\phi(w^*, w^{k+1}) &\leq D_\phi(w^*, w^k) \\ &\quad - \left[rD_\varphi(x^{k+1}, x^k) + sD_\psi(y^{k+1}, y^k) + \frac{\beta}{2}\left\|Ax^{k+1} + By^k - b\right\|^2\right],\end{aligned} \tag{4.1}$$

where $w^* = (x^*, y^*, \lambda^*)$ and ϕ is defined in (3.6). In particular, for all k, we have

$$D_\phi(w^*, w^{k+1}) \leq D_\phi(w^*, w^k). \tag{4.2}$$

Proof. From Lemma 2.1, we know that

$$\begin{aligned}rD_\varphi(x^*, x^k) &= rD_\varphi(x^*, x^{k+1}) + rD_\varphi(x^{k+1}, x^k) \\ &\quad + r\left(x^* - x^{k+1}\right)^T(\nabla\varphi(x^{k+1}) - \nabla\varphi(x^k)).\end{aligned} \tag{4.3}$$

Similarly, we have

$$\begin{aligned}sD_\psi(y^*, y^k) &= sD_\psi(y^*, y^{k+1}) + sD_\psi(y^{k+1}, y^k) \\ &\quad + s\left(y^* - y^{k+1}\right)^T(\nabla\psi(y^{k+1}) - \nabla\psi(y^k)).\end{aligned} \tag{4.4}$$

Using the definitions of D-Function and ϕ in (3.6), we obtain

$$D_\phi(w, \bar{w}) = rD_\varphi(x, \bar{x}) + sD_\psi(y, \bar{y}) + \frac{1}{2\beta}\left\|\lambda - \bar{\lambda}\right\|^2 + \frac{\beta}{2}\|By - B\bar{y}\|^2.$$

From the above notation of D_ϕ, and then adding (3.17), (4.3) and (4.4), we further get (4.1). Inequality (4.2) follows directly from (4.1). ∎

The following lemma plays an important role in the convergence analysis of the proposed method.

Lemma 4.2. *Let* $w^{k+1} = \left(x^{k+1}, y^{k+1}, \lambda^{k+1}\right)$ *be generated by (3.1)-(3.5) from given* $w^k = \left(x^k, y^k, \lambda^k\right)$. *Then for any* $w' = (x', y', \lambda') \in \mathbb{W}$*, we have*

$$(x' - x^{k+1})^T f_{k+1}(x^{k+1}) \geq (x' - x^{k+1})^T r\left(\nabla\varphi(x^k) - \nabla\varphi(x^{k+1})\right) \tag{4.5}$$

and

$$(y' - y^{k+1})^T g_{k+1}(y^{k+1}) \geq (y' - y^{k+1})^T s\left(\nabla\psi(y^k) - \nabla\psi(y^{k+1})\right). \tag{4.6}$$

Proof. The desired result follows from (3.1) and (3.2), immediately. ∎

The two lemmas given below are devoted to prove the convergence of the NPADM.

Lemma 4.3. *Let* $\{w^k\}$ *be a sequence generated by the proposed method. Then,* $\{w^k\}$ *is bounded and every accumulation point of* $\{w^k\}$ *is a solution point of SVI*$(\mathbb{W}, Q)$.

Proof. From (4.2) in Theorem 4.1, we know that

$$w^k \in \mathrm{L}(w^*, \alpha) = \{w \mid D_\phi(w^*, w) \leq \alpha\} \qquad \forall\, k \geq 0,$$

where $\alpha = D_\phi(w^*, w^0)$. This implies that $\{w^k\}$ is bounded because of condition (iii) in Definition 2.2. The first assertion is proved.

It follows from (4.1) that

$$\lim_{k\to\infty} rD_\varphi(x^{k+1}, x^k) + sD_\psi(y^{k+1}, y^k) + \frac{\beta}{2}\left\|Ax^{k+1} + By^k - b\right\|^2 = 0.$$

Consequently,

$$\lim_{k\to\infty} D_\varphi(x^{k+1}, x^k) = 0, \qquad \lim_{k\to\infty} D_\psi(y^{k+1}, y^k) = 0, \tag{4.7}$$

and

$$\lim_{k\to\infty} \left\|Ax^{k+1} + By^k - b\right\| = 0. \tag{4.8}$$

Because $\{w^k\}$ is bounded, it has at least one accumulation point. Let w^∞ be an accumulation point of $\{w^k\}$ and the subsequence $\{w^{k_j}\}$ converges to $w^\infty = (x^\infty, y^\infty, \lambda^\infty)$, i.e.,

$$\lim_{j\to\infty} \left\|x^{k_j} - x^\infty\right\| = 0, \qquad \lim_{j\to\infty} \left\|y^{k_j} - y^\infty\right\| = 0, \tag{4.9}$$

and

$$\lim_{j\to\infty} \left\|\lambda^{k_j} - \lambda^\infty\right\| = 0. \tag{4.10}$$

From (4.7), we know that

$$\lim_{j\to\infty} D_\psi(y^{k_j+1}, y^{k_j}) = 0. \tag{4.11}$$

It follows from the boundedness of $\{y^{k_j+1}\}$, (4.9) and (4.11) that $\{y^{k_j+1}\}$ also converges to y^∞ since the condition (v) in Definition 2.2 is satisfied. Then we can easily obtain

$$\lim_{j\to\infty} \left\|y^{k_j+1} - y^{k_j}\right\| = 0. \tag{4.12}$$

Because $\nabla\psi$ is Lipschitz continuous (Condition A),

$$\lim_{j\to\infty} \left\|\nabla\psi(y^{k_j+1}) - \nabla\psi(y^{k_j})\right\| = 0. \tag{4.13}$$

Similarly, we get

$$\lim_{j\to\infty} \left\|x^{k_j+1} - x^\infty\right\| = 0, \qquad \lim_{j\to\infty} \left\|\nabla\varphi(x^{k_j+1}) - \nabla\varphi(x^{k_j})\right\| = 0. \tag{4.14}$$

(4.8) and (4.12) yield that

$$\lim_{j\to\infty}\left\|Ax^{k_j+1}+By^{k_j+1}-b\right\| \overset{(3.3)}{=} \lim_{j\to\infty}\frac{1}{\beta}\left\|\lambda^{k_j}-\lambda^{k_j+1}\right\| = 0. \tag{4.15}$$

From this and (4.10), we can see that $\left\{\lambda^{k_j+1}\right\}$ also converges to λ^∞.

Since (see (3.4) and (3.5))

$$f_{k+1}(x^{k+1}) = f(x^{k+1}) - A^T\lambda^{k+1} + A^THB\left(y^k - y^{k+1}\right),$$

and

$$g_{k+1}(y^{k+1}) = g(y^{k+1}) - B^T\lambda^{k+1},$$

it follows from (4.5)-(4.6) and (4.12)-(4.15) that

$$\begin{cases} \lim_{j\to\infty}\left(x'-x^{k_j+1}\right)^T\left\{f(x^{k_j+1})-A^T\lambda^{k_j+1}\right\}\geq 0, & \forall x'\in\mathbb{X},\\ \lim_{j\to\infty}\left(y'-y^{k_j+1}\right)^T\left\{g(y^{k_j+1})-B^T\lambda^{k_j+1}\right\}\geq 0, & \forall y'\in\mathbb{Y},\\ \lim_{j\to\infty}Ax^{k_j+1}+By^{k_j+1}-b=0, \end{cases}$$

and consequently

$$\begin{cases} (x'-x^\infty)^T\left\{f(x^\infty)-A^T\lambda^\infty\right\}\geq 0, & \forall x'\in\mathbb{X},\\ (y'-y^\infty)^T\left\{g(y^\infty)-B^T\lambda^\infty\right\}\geq 0, & \forall y'\in\mathbb{Y},\\ Ax^\infty+By^\infty-b=0, \end{cases}$$

because $\left\{w^{k_j+1}\right\}$ also converges to w^∞. This means that w^∞ is a solution point of SVI$(\mathbb{W},Q)$. ∎

Lemma 4.4. *Let* $\left\{w^k\right\}$ *be a sequence generated by the proposed method. Then,* $\left\{w^k\right\}$ *is convergent.*

Proof. Since $\left\{w^k\right\}$ is bounded, it has at least one accumulation point w^∞. Let $\left\{w^k\right\}_{k\in\mathbb{K}}$ be a subsequence which converges to w^∞. Then from condition (iv) in Definition 2.2, we have

$$\lim_{k\to\infty,k\in\mathbb{K}} D_\phi(w^\infty,w^k)=0.$$

From Lemma 4.3 and Theorem 4.1, we know that w^∞ is a solution point of SVI$(\mathbb{W},Q)$ and

$$0\leq D_\phi(w^\infty,w^{k+1})\leq D_\phi(w^\infty,w^k).$$

Obviously, the sequence $\{D_\phi(w^\infty,w^k)\}$ converges and the limit is equal to zero. Moreover, applying Lemma 2.2 with u^k and v^k being replaced by w^∞ and w^k, respectively, we see that $\left\{w^k\right\}$ converges to w^∞. ∎

From the above analysis, we give the convergence theorem for the NPADM.

Theorem 4.5. *Let* $\left\{w^k\right\}$ *be a sequence generated by the proposed method. Then,* $\left\{w^k\right\}$ *converges to a solution of SVI*$(\mathbb{W},Q)$.

Proof. It is an immediate consequence of Lemmas 4.3 and 4.4. ∎

5. CONCLUDING REMARKS

In this paper, we suggest a nonlinear proximal alternating directions method. It is shown that the method has global convergence under proper conditions. Our main work is using D-functions to substitute the quadratic terms in the proximal alternating directions method. It is our belief that the research on the choices of φ and ψ is important in application and we hope this paper may stimulate further investigation in this direction.

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