EXISTENCE OF LARGE SOLUTIONS TO NON-MONOTONE SEMILINEAR ELLIPTIC EQUATIONS

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ABSTRACT. We study the existence of large solutions of the semilinear elliptic equation \( \Delta u = p(x)f(u) \) where \( f \) is not monotonic. We prove existence, on bounded and unbounded domains, under the assumption that \( f \) is Lipschitz continuous, \( f(0) = 0 \), \( f(s) > 0 \) for \( s > 0 \) and there exists a nonnegative, nondecreasing Hölder continuous function \( g \) and a constant \( M \) such that \( g(s) \leq f(s) \leq M g(s) \) for large \( s \). The nonnegative function \( p \) is allowed to be zero on much of the domain.

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1. Introduction

We consider the semilinear elliptic equation

\[ \Delta u = p(x)f(u), \quad x \in \Omega \subseteq \mathbb{R}^n, \quad n \geq 3, \]

where \( \Omega \) is open and connected, the nonnegative function \( p \) can be zero on much of the domain, and \( f \) is a Lipschitz continuous function on \([0, \infty)\) that satisfies \( f(0) = 0, \quad f(s) > 0 \) for \( s > 0 \). In addition, we assume that there exists a nonnegative, nondecreasing Hölder continuous function \( g \) and positive constants \( M \) and \( s_0 \) such that

\[ g(s) \leq f(s) \leq Mg(s) \quad \text{for all} \quad s \geq s_0. \]

We are interested in the existence of large solutions of (1.1) on \( \Omega \); i.e., solutions for which \( u(x) \to \infty \) as \( x \to \partial \Omega \) if \( \Omega \) is bounded, and if \( \Omega \) is unbounded, we also require that \( u(x) \to \infty \) for \( |x| \to \infty \) within \( \Omega \).

Unlike almost all previous work (See, for example, [1, 2, 6, 7, 8, 10, 12, 13], and their references.), we do not require \( f \) to be nondecreasing. The usual requirement that \( f \) be monotonic is necessary, in part, because the proofs depend on the maximum principle. However, where \( f \) is not monotonic, the maximum principle cannot be applied directly to equation (1.1).

The only existence result for non-monotonic \( f \) we are aware of is given by Goncalves and Roncalli [5]. They proved existence under the conditions \( \liminf_{s \to \infty} f(s)/s^b > 0 \) and \( 0 < \sup_{s>0} f(s)/s^a < \infty \), \( 1 < b \leq a < \infty \). These conditions reduce to the existence of position constants \( c_0 \) and \( c_1 \) such that \( c_0 s^a \leq f(s) \leq c_1 s^a \) for \( s \) large, and hence is a special case of our results.

For nondecreasing \( f \), we know that (1.1) has a large solution on a bounded domain if and only if \( f \) satisfies (see [7])

\[ \int_1^{\infty} \left[ \int_0^s f(t) \, dt \right]^{-1/2} \, ds < \infty. \]

We prove here that this remains true for nonmonotone \( f \) (Theorem 2.2). For unbounded domains, we prove results analogous to those for increasing \( f \). In particular, we show that if \( p \) decays rapidly as \( |x| \to \infty \), then, as in the bounded domain case, (1.1) has a large solution if and only if \( f \) satisfies (1.3) (see Corollary 3.4). Our proofs, although comparable to those in [7], require substantial innovations to compensate for the lack of monotonicity.

We note that similar results for systems comparable to (1.1) such as

\[ \begin{align*}
\Delta u &= p(x)f(v) \\
\Delta v &= q(x)h(u)
\end{align*} \]

remain an open problem. Indeed, existence results for large solutions of such systems are known only under the rather restrictive conditions that \( \Omega = \mathbb{R}^n \), \( p \) and \( q \) are spherically symmetric and both \( f \) and \( h \) are nondecreasing (see [3, 11]).

2. Existence of Solutions on Bounded Domains

We first make some preliminary definitions and observations before establishing our existence theorems. In particular, we define the functions \( G \) and \( H \) as follows:

\[ G(s) = \begin{cases}
A \min\{f(t) : s \leq t \leq s_0\}, & 0 \leq s \leq s_0 \\
Af(s_0)g(s)/g(s_0), & s \geq s_0.
\end{cases} \]

\[ H(s) = \begin{cases}
K \max\{f(t) : 0 \leq t \leq s\}, & 0 \leq s \leq s_0, \\
KF_0g(s)/g(s_0), & s \geq s_0,
\end{cases} \]

where \( A \) and \( K \) are positive constants and \( \min\{f(t) : s \leq t \leq s_0\} \geq 0 \) for each \( s \leq s_0 \).
where $0 < A \leq \min \{1, \frac{g(s_0)}{f(s_0)}\}$, $F_0 = \max \{f(t) : 0 \leq t \leq s_0\}$, $K$ is a constant chosen so that $K \geq \max \{1, Mg(s_0)/F_0\}$, and $M$ comes from (1.2). We note without proof that $G$ and $H$ are nondecreasing $C^\infty_{loc}([0, \infty))$ functions which are positive when their argument is positive, and satisfy

$$G(s) \leq f(s) \leq H(s) \text{ for } s \geq 0.$$  

We say that the nonnegative function $p$ is $c$-positive if for any $x_0 \in \Omega$ satisfying $p(x_0) = 0$, there exists a domain $\Omega_0$ such that $x_0 \in \Omega_0$, $\Omega_0 \subset \Omega$, and $p(x) > 0$ for all $x \in \partial\Omega_0$. Thus $p$ can be zero on much of the domain.

**Lemma 2.1.** Suppose $\Omega$ is a bounded domain in $\mathbb{R}^n$ with a $C^{2, \gamma}$ boundary, and $p$ is a nonnegative $C^\infty(\overline{\Omega})$ function that is $c$-positive on $\Omega$. Suppose $f$ is Lipschitz continuous on $[0, \infty)$, $f(0) = 0$, $f(s) > 0$ for $s > 0$ and satisfies (1.2). Then for any nonnegative constant $c$, the boundary value problem

$$\Delta v = p(x)f(v), \quad x \in \Omega,$$

$$v(x) = c, \quad x \in \partial\Omega$$

has a nonnegative classical solution $v$ on $\Omega$.

**Proof.** From [4] (See Theorem 14.10) we have that for any nonnegative constant $c$ there exist nonnegative classical solutions $v_1$ and $v_2$ to the following boundary value problems

$$\Delta v_1 = p(x)G(v_1), \quad x \in \Omega,$$

$$v_1(x) = c, \quad x \in \partial\Omega,$$

$$\Delta v_2 = p(x)H(v_2), \quad x \in \Omega,$$

$$v_2(x) = c, \quad x \in \partial\Omega.$$  

We claim that $v_1 \geq v_2$ on $\overline{\Omega}$. Indeed, suppose $v_1 < v_2$ at some point in $\overline{\Omega}$. Let $\varepsilon > 0$ be small enough such that $\max_{\overline{\Omega}}[v_2(x) - v_1(x) - \varepsilon h(r)] > 0$, where $h(r) = (1 + r^2)^{-1/2}$, $r = |x|$. Then $0 < v_2(x_0) - v_1(x_0) - \varepsilon h(r) \equiv \max_{\overline{\Omega}}[v_2(x) - v_1(x) - \varepsilon h(r)]$ and hence at $x_0$ we have $0 \geq \Delta(v_2 - v_1 - \varepsilon h(r)) = p(x_0)[H(v_2(x_0)) - G(v_1(x_0))] - \varepsilon \Delta h(r) \geq -\varepsilon \Delta h(r) > 0$, a contradiction. The last inequality holds because $n \geq 3$. Thus, $v_2 \leq v_1$ in $\overline{\Omega}$.

Now, letting $\overline{v} = v_1$ and $\underline{v} = v_2$ we have that $\underline{v} \leq \overline{v}$ in $\Omega$ and

$$\Delta \overline{v} = p(x)G(\overline{v}) \leq p(x)f(\overline{v}), \quad x \in \Omega,$$

$$\Delta \underline{v} = p(x)H(\underline{v}) \geq p(x)f(\underline{v}), \quad x \in \Omega.$$  

Thus, $\overline{v}$ and $\underline{v}$ are upper and lower solutions, respectively, of $\Delta v = p(x)f(v)$ on $\Omega$, and hence the monotone iteration scheme (see [14]) gives the existence of a classical solution, $v$, to equation (2.4) on $\Omega$ with $\underline{v} \leq v \leq \overline{v}$.

The following is our main result for this section.

**Theorem 2.2.** Suppose $\Omega$, $p$ and $f$ satisfy the hypothesis of the lemma above. Then equation (1.1) has a nonnegative large solution in $\Omega$ if and only if $f$ satisfies (1.3).

**Proof.** Suppose $f$ satisfies (1.3). Let $v_k$ and $w_k$ be the nonnegative solutions of (see 7)

$$\Delta v_k = p(x)G(v_k), \quad x \in \Omega,$$

$$v_k(x) = k, \quad x \in \partial\Omega,$$

$$\Delta w_k = p(x)H(w_k), \quad x \in \Omega,$$

$$w_k(x) = k, \quad x \in \partial\Omega.$$
Then $v_k$ and $w_k$ are monotonically increasing. We shall construct a monotone sequence of functions $\{u_k\}$ which satisfies, for each $k$,
\[
\Delta u_k = p(x)f(u_k), \quad x \in \Omega,
\]
\[
u_k(x) = k, \quad x \in \partial \Omega.
\]
We start with $k = 1$. Letting $\overline{\alpha_1} = v_1$ and $\underline{u_1} = w_1$ we have that there exists a nonnegative classical solution $u_1$ of
\[
\Delta u_1 = p(x)f(u_1), \quad x \in \Omega,
\]
\[
u_1(x) = 1, \quad x \in \partial \Omega,
\]
with $w_1 = \underline{u_1} \leq u_1 \leq \overline{\alpha_1} = v_1$. We then consider the following system of equations
\[
\Delta v_2 = p(x)G(v_2), \quad x \in \Omega,
\]
\[
v_2(x) = 2, \quad x \in \partial \Omega,
\]
\[
\Delta u_1 = p(x)f(u_1), \quad x \in \Omega,
\]
\[
u_1(x) = 1, \quad x \in \partial \Omega.
\]
Letting $\overline{\alpha_2} = v_2$ and $\underline{u_2} = u_1$ we have that there exists a nonnegative classical solution $u_2$ of
\[
\Delta u_2 = p(x)f(u_2), \quad x \in \Omega,
\]
\[
u_2(x) = 2, \quad x \in \partial \Omega,
\]
with $w_1 \leq u_1 \leq u_2 \leq \overline{\alpha_2} = v_2$. Continuing this line of reasoning we have that there exists a nonnegative classical solution $u_k$ to
\[
\Delta u_k = p(x)f(u_k), \quad x \in \Omega,
\]
\[
u_k(x) = k, \quad x \in \partial \Omega,
\]
with $w_1 \leq u_{k-1} \leq u_k \leq v_k$, $k \geq 2$. Clearly the sequence $\{u_k\}$ is monotone. We note that since $f$ satisfies (1.3), $G$ does as well. Hence it can be shown (see Theorem 1 of [7]) that the sequence $\{v_k\}$ converges to a classical solution $v$ of
\[
\Delta v = p(x)G(v), \quad x \in \Omega,
\]
\[
v(x) \to \infty, \quad x \to \partial \Omega.
\]
It then follows that $w_1 \leq u_{k-1} \leq u_k \leq v$. Hence, the sequence $\{u_k\}$ converges on $\Omega$ to some function $u$. A standard regularity argument for elliptic equations (See, e.g., the proof of Lemma 3 in [9]) then shows that $u$ is a classical solution to (1.1). By construction, $u$ is clearly a large solution.

Now suppose that $f$ does not satisfy (1.3); i.e. $f$ satisfies
\[
(2.7) \quad \int_1^\infty \left[ \int_0^s f(t) \, dt \right]^{-1/2} \, ds = \infty
\]
and assume, for contradiction, that $u$ is a nonnegative large solution of (1.1). Let $v_k$ be a nonnegative classical solution of
\[
(2.8) \quad \Delta v_k = p(x)H(v_k), \quad x \in \Omega,
\]
\[
v_k(x) = k, \quad x \in \partial \Omega.
\]
Then the sequence $\{v_k\}$ is nondecreasing and $v_k \leq u$ on $\Omega$. It follows that $\{v_k\}$ converges to a nonnegative function $v$ on $\Omega$. Another standard regularity argument will show that $v$ is a
classical solution of the system
\[
\Delta v = p(x)H(v), \quad x \in \Omega \\
v(x) \to \infty, \quad x \to \partial \Omega.
\]
This problem, however, has no solution because, as a consequence of (2.7), \(H\) satisfies
\[
\int_1^\infty \left[ \int_0^s H(t)dt \right]^{-1/2}ds = \infty
\]
(see Theorem 1 of [7]). Hence, equation (1.1) has no nonnegative large solution on \(\Omega\). This completes the proof. \(\blacksquare\)

3. Existence of Solutions on Unbounded Domains

We now consider the case where \(\Omega\) is unbounded and begin by letting \(\Omega = \mathbb{R}^n\). Consistent with results for nondecreasing \(f\), we require
(3.1)
\[
\int_0^\infty r\phi(r)dr < \infty,
\]
where \(\phi(r) = \max_{|x|=r} p(x)\).

**Theorem 3.1.** Suppose \(p\) is a nonnegative \(c\)-positive \(C^1_{\text{loc}}(\mathbb{R}^n)\) function which satisfies (3.1), \(f\) is Lipschitz continuous on \([0, \infty)\), \(f(0) = 0\), \(f(s) > 0\) for \(s > 0\), and \(f\) satisfies (1.2). Then (1.1) has a nonnegative entire large solution provided \(f\) satisfies (1.3).

**Proof.** Using a proof similar to that of Theorem 2.2, it is a straightforward exercise to prove the existence of nonnegative solutions \(v_k\) and \(w_k\) to the following boundary value problems
(3.2)
\[
\Delta v_k = p(x)G(v_k), \quad |x| < k, \\
v_k(x) \to \infty \text{ as } |x| \to k,
\]
(3.3)
\[
\Delta w_k = p(x)H(w_k), \quad |x| < k, \\
w_k(x) \to \infty \text{ as } |x| \to k,
\]
which satisfy \(w_k \leq v_k\) on \(|x| \leq k\). It is clear, by the maximum principle, that \(v_k(x) \geq v_{k+1}(x)\) on \(|x| \leq k\), for each \(k\). By defining \(v_k(x) = \infty\) for \(|x| \geq k\), we have that the sequence \(v_k\) is monotonely decreasing on \(\mathbb{R}^n\). Furthermore, we can employ the same method to produce a nonnegative solution \(u_k\) to the boundary value problem
(3.4)
\[
\Delta u_k = p(x)f(u_k), \quad |x| < k, \\
u_k(x) \to \infty \text{ as } |x| \to k,
\]
with \(w_k \leq u_k \leq v_k\). If we can show that the sequence \(\{u_k\}\) is uniformly bounded and equicontinuous on bounded subsets, then the Ascoli-Arzela Theorem will allow us to prove that \(\{u_k\}\) has a convergent subsequence on \(\mathbb{R}^n\) which is uniformly convergent on compact sets. To do this, we let \(B(0,1) \subseteq \Omega = \mathbb{R}^n\) be the ball centered at zero with radius one. Notice that \(u_k \leq v_k\), and that the sequence \(\{v_k\}\) is decreasing. Then we have that \(u_k \leq v_2\) on \(B(0,1)\) for all \(k \geq 2\). Hence, the sequence \(u_k\) is uniformly bounded on \(\overline{B(0,1)}\). We also have that \(u_k\) is a solution to (3.4) on \(B(0,1)\), and \(u_k \in C^{2,\alpha}(B(0,1))\). Thus, by Theorem 3.9 of [4], we have, for \(k \geq 3\), the gradient bound
(3.5)
\[
\sup_{|x|<2} |\nabla u_k(x)| \leq C(\max_{|x|<2} u_k) + \sup_{|x|<2} d^2_x[p(x)f(u_k(x))],
\]
where \(C = C(n)\) and \(d_x = \text{dist}(x, \partial B(0,2))\). Furthermore, since \(d_x \geq 1\) for \(|x| \geq 1\) we have
(3.6)
\[
\sup_{|x|<1} |\nabla u_k(x)| \leq \sup_{|x|<1} d_x |\nabla u_k(x)| \leq \sup_{|x|<2} d_x |\nabla u_k(x)|,
\]
implying the sequence \( \{u_k\} \), \( k \geq 3 \), is equicontinuous on \( \overline{B}(0,1) \). Hence there exists a subsequence \( \{u_k^1\} \) of \( \{u_k\} \) which converges to a nonnegative function \( u^1 \) on the ball \( B(0,1) \subseteq \Omega \).

Now, consider the subsequence \( \{u_k^1\} \) on the ball \( B(0,2) \subseteq \Omega = \mathbb{R}^n \) centered at 0 with radius two. It is clear that the subsequence \( \{u_k^1\} \) is uniformly bounded on \( \overline{B}(0,2) \). Furthermore, \( u_k^1 \) is a solution to equation (3.4) on \( B(0,2) \), and therefore \( u_k^1 \in C^{2,\alpha}(B(0,2)) \). Thus, we have the gradient bound

\[
\sup_{|x|<3} d_x |\nabla u_k^1(x)| \leq C \left( \sup_{|x|<3} |u_k^1| + \sup_{|x|<3} d_x^2 |p(x)f(u_k^1(x))| \right),
\]

where \( C = C(n) \) and \( d_x = \text{dist}(x, \partial B(0,3)) \). Again, since \( d_x \geq 1 \) we have

\[
\sup_{|x|<2} |\nabla u_k^1(x)| \leq \sup_{|x|<3} d_x |\nabla u_k^1(x)| \leq \sup_{|x|<3} d_x |\nabla u_k^1(x)|,
\]

so that the subsequence \( \{u_k^1\} \) is also equicontinuous on \( \overline{B}(0,2) \). So, there exists a subsequence \( \{u_k^2\} \) of \( \{u_k^1\} \) which converges to a nonnegative function \( u^2 \) on the ball \( B(0,2) \subseteq \Omega \).

Continuing this line of reasoning, we have that there exist nonnegative large solutions \( u^3, u^4, u^5, \ldots \) on the balls \( B(0,3), B(0,4), B(0,5), \ldots \), respectively. Furthermore we note that

\[
u^1 = u^2 = u^3 = u^4 = u^5 = \ldots, \text{ on } B(0,1)\]

and, more generally,

\[
u^m = u^{m+1} = u^{m+2} = \ldots, \text{ on } B(0, m).
\]

Now we define the function \( u \) on \( \mathbb{R}^n \) as \( u(x) = u^i(x) \) for \( |x| < i \). Thus \( u^i(x) \to u(x) \) as \( i \to \infty \) for all \( x \in \mathbb{R}^n \) and the convergence is uniform on compact sets. Once again, a standard regularity argument will show that \( u \) is a solution to (1.1) on \( \Omega = \mathbb{R}^n \). It is easy to see that \( u \) is, in fact, a large solution since \( w \equiv \lim_{k \to \infty} w_k \) satisfies \( w \leq u \), and \( w \) is large by virtue of (3.1). (see Theorem 2 of [7].)

We now extend this result to somewhat arbitrary unbounded domains.

**Theorem 3.2.** Suppose \( \Omega \) is an unbounded domain in \( \mathbb{R}^n, n \geq 3 \), with compact \( C^{2,\gamma} \) boundary and suppose there exists a sequence of bounded domains \( \{\Omega_k\} \), each with smooth boundary, such that \( \Omega_k \subseteq \Omega_{k+1} \) for all \( k = 1, 2, \ldots \) and \( \Omega = \bigcup_{k=1}^{\infty} \Omega_k \). Suppose \( p \) is a nonnegative \( c \)-positive \( C_{loc}^\alpha(\mathbb{R}^n) \) function with \( \phi(r) \equiv \max\{p(x) : |x| = r, x \in \Omega\} \) and assume that it satisfies inequality (3.7). Assume that \( f \) is Lipschitz continuous on \([0, \infty)\), \( f(0) = 0 \), \( f(s) > 0 \) for \( s > 0 \), and \( f \) satisfies (1.2). Then (1.1) has a nonnegative large solution provided \( f \) satisfies (1.3).

**Proof.** We replace the functions \( v_k \) and \( w_k \) in the proof of Theorem 3.1 with the solutions to

\[
\Delta v_k = p(x)G(v_k), \quad x \in \Omega_k,
\]

\[
v_k(x) \to \infty, \quad x \to \partial \Omega_k,
\]

\[
\Delta w_k = p(x)H(w_k), \quad x \in \Omega_k,
\]

\[
w_k(x) \to \infty, \quad x \to \partial \Omega_k,
\]

for each \( k \). The proof now follows an analogous approach to that of Theorem 3.1. We omit the details.

We now give a partial converse to Theorem 3.2.

**Theorem 3.3.** Let \( p, f, \phi, \) and \( \Omega \) be as in Theorem 3.3. In addition, suppose there exists a nonnegative function \( h \) continuous on \([0, \infty)\) and differentiable on \((0, \infty)\) such that \( 0 \leq \phi(r) \leq h^2(r) \) for all \( r \geq 0 \) and \( h \) satisfies one of the following: (a) there exists a constant \( C \) such that
$0 \leq r^{2n-2} \phi(r) \leq C, \forall r \geq 0$; or (b) \( \lim_{r \to \infty} r^{n-1} h(r) = \infty \) and \( \int_{0}^{\infty} h(r) dr < \infty \). If (1.1) has a nonnegative large solution on \( \Omega \), then \( f \) satisfies inequality (1.3).

**Proof.** Let \( u \) be a large solution of (1.1). We can now construct a proof, very similar to the proof of Theorem 5 in [7], using the equation \( \Delta v = \phi(r) H(v) \) in place of \( \Delta v = \phi(r) f(v) \) in [7] to obtain a contradiction. We omit the details.

Our final result provides necessary and sufficient conditions to ensure the existence of a large solution of (1.1) on an unbounded domain. It closely follows the corollary of [7] and therefore stated without proof.

**Corollary 3.4.** Let \( f \) and \( \Omega \) be as in Theorem 3.2, and assume \( p \) is a nonnegative c-positive \( C^\infty_{\text{loc}}(\Omega) \) function for which there exists a constant \( K \) such that

\[
(3.10) \quad p(x) \leq K|x|^{-\alpha}, \quad \alpha > 2.
\]

for \( |x| \) large and \( x \in \Omega \). Then a necessary and sufficient condition for (1.1) to have a nonnegative large solution on \( \Omega \) is that \( f \) satisfy inequality (1.3).

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