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# ON PSEUDO ALMOST PERIODIC SOLUTIONS TO SOME NEUTRAL FUNCTIONAL-DIFFERENTIAL EQUATIONS

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ABSTRACT. This paper discusses the existence and uniqueness of pseudo almost periodic solutions to a class of partial neutral functional-differential equations. Under some suitable assumptions, existence and uniqueness results are obtained. An example is given to illustrate abstract results.

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## 1. INTRODUCTION

Let $(\mathbb{X}, \|\cdot\|)$ be a Banach space. The main goal of this paper is establish the existence and uniqueness of a *pseudo almost periodic* solution to the abstract neutral Cauchy problem

$$\frac{d}{dt}D(u_t) = AD(u_t) + g(t, u_t), \tag{1.1}$$

$$u_0 = \varphi \in \mathcal{B} = C([-p, 0], \mathbb{X}), \tag{1.2}$$

where $A$ is the infinitesimal generator of an uniformly exponentially stable semigroup of linear operators $(T(t))_{t\geq 0}$ on $\mathbb{X}$, $Du = u(0) - f(t, u)$, the history $u_t : [-p, 0] \mapsto \mathbb{X}$ defined by $u_t(\theta) = u(t + \theta)$ belongs to the phase space $\mathcal{B} = C([-p, 0], \mathbb{X})$, and $f, g : \mathbb{R} \times \mathcal{B} \mapsto \mathbb{X}$ are pseudo almost periodic of order $p$ in $t \in \mathbb{R}$ uniformly in the second variable.

The existence of almost automorphic, almost periodic, asymptotically almost periodic, and pseudo almost periodic solutions is one of most attractive topics in the qualitative theory of differential equations due to their significance and applications in physical sciences.

The concept of the pseudo almost periodicity (p.a.p. for short) which is the central issue in this paper was first initiated by C. Y. Zhang in the earlier nineties [25, 26]– and is a natural generalization of the well-known (Bohr) almost periodicity (a.p.). Thus this new concept is welcome to implement another existing generalization of the (Bohr) almost periodicity, the notion of asymptotically almost periodicity (a.a.p.) due to Fréchet, see, e.g., [5], and [11]. For more on these and related issues, see, e.g., [1], [2], [3], [4], [6], [7], [8], [9], [10], [17], [25] and the references therein.

Some contributions related to pseudo almost periodic solutions to abstract differential equations and partial differential equations have recently been made in [15, 16, 7, 8, 9, 10, 6, 4, 17]. Existence results concerning almost periodic and asymptotically almost periodic solutions to ordinary neutral differential equations and abstract partial neutral differential equations have recently been established in [18, 20, 14]. However, the existence of pseudo almost periodic to functional-differential equations with delay, especially, abstract partial neutral differential equations is an untreated topic and this is the main motivation of the present paper.

This paper is organized as follows. In Section 2 we introduce some notations, definitions and properties related to the theory of almost periodic and pseudo almost periodic functions. Furthermore, we establish some preliminary results on the composition of pseudo almost periodic functions of class $p$. Finally, in Section 4, we establish the existence and uniqueness of pseudo almost periodic solutions to the above-mentioned problem. We next illustrate the previous existence result by an example.

## 2. PRELIMINARIES

Let $(\mathbb{X}, \|\cdot\|)$ be a Banach space. In this section we introduce the required background and some preliminaries that we need in the sequel. In this paper, $A : D(A) \subset \mathbb{X} \to \mathbb{X}$ denotes the infinitesimal generator of an uniformly asymptotically stable semigroup of linear operators $(T(t))_{t\geq 0}$ on $\mathbb{X}$, and $M, w$ are some positive constants such that $\| T(t) \| \leq Me^{-wt}$, $t \geq 0$.

To deal with pseudo almost periodic solutions we will need to introduce some classical and new concepts. In what follows, $(\mathbb{Z}, \|\cdot\|_{\mathbb{Z}})$ and $(\mathbb{W}, \|\cdot\|_{\mathbb{W}})$ stand for Banach spaces. In addition to that $C(\mathbb{R}, \mathbb{Z})$ and $BC(\mathbb{R}, \mathbb{Z})$ denote the collection of continuous functions, and the Banach space of bounded continuous functions from $\mathbb{R}$ into $\mathbb{Z}$ equipped with the sup norm defined by $\|u\|_\infty := \sup_{t\in\mathbb{R}} \|u(t)\|_{\mathbb{Z}}$, respectively. Similar definitions apply for both $C(\mathbb{R} \times \mathbb{Z}, \mathbb{W})$ and $BC(\mathbb{R} \times \mathbb{Z}, \mathbb{W})$. The notation $B_r(x, \mathbb{Z})$ stands for the open ball centered at $x \in \mathbb{Z}$ with radius $r > 0$.

**Definition 1.** *A function $f \in C(\mathbb{R}, \mathbb{Z})$ is almost periodic (a.p.) if for each $\varepsilon > 0$ there exists a relatively dense subset of $\mathbb{R}$ denoted by $\mathcal{H}(\varepsilon, f, \mathbb{Z})$ (i.e., there exists $\delta > 0$ such that $[a, a+\delta] \cap \mathcal{H}(\varepsilon, f, \mathbb{Z}) \neq \{\emptyset\}$ for each $a \in \mathbb{R}$) such that $\|f(t+\tau) - f(t)\|_{\mathbb{Z}} < \varepsilon$ for each $t \in \mathbb{R}$ and each $\tau \in \mathcal{H}(\varepsilon, f, \mathbb{Z})$. The collection of those functions will be denoted by $AP(\mathbb{Z})$.*

We need the next Lemma in the sequel.

**Lemma 1.** [24, p. 25] *A function $f \in C(\mathbb{R}, \mathbb{Z})$ is almost periodic if and only if the set of functions $\{\sigma_\tau f : \tau \in \mathbb{R}\}$, where $(\sigma_\tau f)(t) = f(t+\tau)$, is relatively compact in $C(\mathbb{R}, \mathbb{Z})$.*

Similarly, a function $F \in C(\mathbb{R} \times \mathbb{Z}, \mathbb{W})$ is called almost periodic in $t \in \mathbb{R}$ uniformly in $z \in \mathbb{Z}$ if for each $\varepsilon > 0$ and for all compact $K \subset \mathbb{Z}$, if there exists a relatively dense subset of $\mathbb{R}$ denoted by $\mathcal{H}(\varepsilon, F, K)$ such that $\|F(t+\tau, z) - F(t, z)\|_{\mathbb{W}} < \varepsilon$ for all $t \in \mathbb{R}$, $z \in K$, and $\tau \in \mathcal{H}(\varepsilon, F, K)$. The collection of those functions will be denoted $AP(\mathbb{Z}, \mathbb{W})$.

The notation $PAP_0(\mathbb{Z})$ stands for the space of functions

$$PAP_0(\mathbb{Z}) := \left\{ u \in BC(\mathbb{R}, \mathbb{Z}) : \lim_{r \to \infty} \frac{1}{2r} \int_{-r}^{r} \| u(t) \|_{\mathbb{Z}} \, dt = 0 \right\}.$$

To study the issues related to delay we need to introduce the new space of functions defined for each $p \in \mathbb{R}$ by

$$PAP_0(\mathbb{Z}, p) := \left\{ u \in BC(\mathbb{R}, \mathbb{Z}) : \lim_{r \to \infty} \frac{1}{2r} \int_{-r}^{r} \left( \sup_{\theta \in [t-p, t]} \| u(\theta) \|_{\mathbb{Z}} \right) dt = 0 \right\}.$$

In addition to the above-mentioned spaces, the present setting requires the introduction of the following new function spaces

$$PAP_0(\mathbb{Z}, \mathbb{W}) = \left\{ u \in BC(\mathbb{R} \times \mathbb{Z}, \mathbb{W}) : \lim_{r \to \infty} \frac{1}{2r} \int_{-r}^{r} \|u(t, z)\|_{\mathbb{W}} dt = 0 \right\}, \quad \text{and}$$

$$PAP_0(\mathbb{Z}, \mathbb{W}, p) := \left\{ u \in BC(\mathbb{R} \times \mathbb{Z}, \mathbb{W}) : \lim_{r \to \infty} \frac{1}{2r} \int_{-r}^{r} \left( \sup_{\theta \in [t-p, t]} \| u(\theta, z) \|_{\mathbb{W}} \right) dt = 0 \right\},$$

where in both cases the limit (as $r \mapsto \infty$) is uniform in $z \in \mathbb{Z}$.

In view of the above, it is clear that $PAP_0(\mathbb{Z}, p)$ and $PAP_0(\mathbb{Z}, \mathbb{W}, p)$ are continuously embedded in the spaces $PAP_0(\mathbb{Z})$ and $PAP_0(\mathbb{Z}, \mathbb{W})$, respectively. Furthermore, it is not hard to see that $PAP_0(\mathbb{Z}, p)$ and $PAP_0(\mathbb{W}, \mathbb{Z}, p)$ are closed in $PAP_0(\mathbb{Z})$ and $PAP_0(\mathbb{W}, \mathbb{Z})$, respectively. Consequently, using [17, Lemma 1.2], one obtains the following:

**Lemma 2.** *The spaces $PAP_0(\mathbb{Z}, p)$ and $PAP_0(\mathbb{W}, \mathbb{Z}, p)$ endowed with the uniform convergence topology are Banach spaces.*

**Definition 2.** *A function $f \in BC(\mathbb{R}, \mathbb{Z})$ is called pseudo almost periodic (p.a.p.) if $f = g + \varphi$, where $g \in AP(\mathbb{Z})$ and $\varphi \in PAP_0(\mathbb{Z})$. The class of those functions will be denoted by $PAP(\mathbb{Z})$.*

**Definition 3.** *A function $F \in BC(\mathbb{R} \times \mathbb{Z}, \mathbb{W})$ is called uniformly pseudo almost periodic (u.p.a.p.) if $F = G + \Phi$, where $G \in AP(\mathbb{Z}, \mathbb{W})$ and $\Phi \in PAP_0(\mathbb{Z}, \mathbb{W})$. The class of those functions will be denoted by $UPAP(\mathbb{Z}, \mathbb{W})$.*

We need to introduce two new notions of pseudo almost periodicity that we will use in the sequel.

**Definition 4.** *A function $F \in BC(\mathbb{R}, \mathbb{Z})$ is called pseudo almost periodic of class p (p.a.p.p.) if $F = G + \varphi$, where $G \in AP(\mathbb{Z})$ and $\varphi \in PAP_0(\mathbb{Z}, p)$. The class of those functions will be denoted by $PAP(\mathbb{Z}, p)$.*

**Definition 5.** *A function $F \in BC(\mathbb{R} \times \mathbb{Z}, \mathbb{W})$ is called uniformly pseudo almost periodic of class p (u.p.a.p.p.) if $F = G + \varphi$, where $G \in AP(\mathbb{R} \times \mathbb{Z}, \mathbb{W})$ and $\varphi \in PAP_0(\mathbb{Z}, \mathbb{W}, p)$. The class of those functions will be denoted by $UPAP(\mathbb{Z}, \mathbb{W}, p)$.*

## 3. PRELIMINARY RESULTS

To establish our main result on the existence of pseudo almost periodic solutions we need to prove some preliminaries results related to the composition of pseudo almost periodic functions of class $p$. Basically, those results are inspired from ideas and estimates given in [17] and their complete proofs can be found in Diagana and Hernández [8].

**Theorem 3.1.** *Let $F \in UPAP(\mathbb{Z}, \mathbb{W}, p)$ and let $h \in PAP(\mathbb{W}, p)$. Assume that there exists a function $L_F : \mathbb{R} \mapsto [0, \infty)$ satisfying*

$$\|F(t, z_1) - F(t, z_2)\|_{\mathbb{W}} \leq L_F(t) \|z_1 - z_2\|_{\mathbb{Z}}, \quad \forall t \in \mathbb{R}, \ \forall z_1, z_2 \in \mathbb{Z}. \tag{3.1}$$

*If*

$$\limsup_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} \left( \sup_{\theta \in [t-p,t]} L_F(\theta) \right) dt < \infty, \quad \textit{and} \tag{3.2}$$

$$\lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} \left( \sup_{\theta \in [t-p,t]} L_F(\theta) \right) \xi(t) dt = 0 \tag{3.3}$$

*for each $\xi \in PAP_0(\mathbb{R})$, then the function $t \mapsto F(t, h(t))$ belongs to $PAP(\mathbb{W}, p)$.*

**Remark 3.1.** Note that assumptions (3.2) and (3.3) are verified by many functions. Examples include constants functions, functions in $PAP(\mathbb{R}, p)$, and functions of $L^1(\mathbb{R})$ which are decreasing on $[0, \infty)$ and nondecreasing on $[-\infty, 0)$ among others.

**Theorem 3.2.** *If $u \in PAP(\mathbb{Z}, p)$, then $t \to u_t$ belongs to $PAP(C([-p, 0], \mathbb{Z}), p)$.*

**Theorem 3.3.** *Let $u \in PAP_0(\mathbb{Z}, p)$. If $v$ is the function defined by*

$$v(t) := \int_{-\infty}^{t} T(t-s)u(s)ds, \quad \forall t \in \mathbb{R},$$

*then $v \in PAP_0(\mathbb{Z}, p)$.*

## 4. EXISTENCE RESULTS

In this section we study the existence of pseudo almost periodic solution for the neutral system. The next definitions of a mild solution is inspired by semigroup theory.

**Definition 1.** *A continuous function $u : [\sigma, \sigma + a) \to \mathbb{X}$, $a > 0$, is a mild solution of the neutral system (1.1) on $[\sigma, \sigma + a)$, if $u_\sigma = \varphi$ and*

$$u(t) = T(t-\sigma)(\varphi(0) + f(\sigma, \varphi)) + f(t, u_t) + \int_{\sigma}^{t} T(t-s)g(s, u_s)ds, \quad t \in [\sigma, \sigma + a).$$

Since the semigroup is uniformly exponentially stable, the next concept of a pseudo almost periodic mild solution is clear.

**Definition 2.** *A function $u \in BC(\mathbb{R}, \mathbb{X})$ is a mild pseudo almost periodic solution of neutral system (1.1), if for each $t \in \mathbb{R}$ and*

$$u(t) = f(t, u_t) + \int_{-\infty}^{t} T(t-s)g(s, u_s)ds, \quad t \in \mathbb{R}.$$

To discuss the existence of pseudo almost periodic solutions we suppose that $f, g : \mathbb{R} \times \mathcal{B} \mapsto \mathbb{X}$ belong to $UPAP(\mathcal{B}, \mathbb{X}, p)$, where $\mathcal{B} = C([-p, 0], \mathbb{X})$ is the (Banach) space of all continuous functions from $[-p, 0]$ into $\mathbb{X}$ equipped with the norm $\|\phi\|_{\mathcal{B}} = \sup_{t\in[-p,0]} \|\phi(t)\|$.

**Theorem 4.1.** *The functions $f, g : \mathbb{R}\times\mathcal{B} \to \mathbb{X}$ are continuous and there exist a constant $L_f > 0$ and a continuous function $L_g : \mathbb{R} \to [0, \infty)$ such that*

$$\begin{aligned} \| f(t, \psi_1) - f(t, \psi_2) \| &\leq L_f \| \psi_1 - \psi_2 \|_{\mathcal{B}}, \\ \| g(t, \psi_1) - g(t, \psi_2) \| &\leq L_g(t) \| \psi_1 - \psi_2 \|_{\mathcal{B}}, \end{aligned}$$

*for all $t \in \mathbb{R}$ and every $\psi_i \in \mathcal{B}(i = 1, 2)$. If*

$$\Theta = L_f + M \sup_{t\in\mathbb{R}} \int_{-\infty}^{t} e^{-w(t-s)} L_g(s)ds < 1, \tag{4.1}$$

*then there exists a unique pseudo almost periodic mild solution to (1.1).*

**Proof:** Let $\Gamma : PAP(\mathbb{X}, p) \to C(\mathbb{R}, \mathbb{X})$ be the operator defined by

$$\Gamma u(t) := -f(t, u_t) + \int_{-\infty}^{t} T(t - s)g(s, u_s)ds, \quad t \in \mathbb{R}. \tag{4.2}$$

Clearly, $\Gamma u$ is well-defined and continuous. Moreover, from Theorems 3.1, 3.2 and 3.3 it easily follows that $\Gamma u \in PAP(\mathbb{X}, p)$ whenever $u \in AP(\mathbb{X}, p)$. On the other hand, for $u, v \in PAP(\mathbb{X}, p)$ we get

$$\begin{aligned} \| \Gamma u(t) - \Gamma v(t) \| &\leq L_f \| u_t - v_t \|_{\mathcal{B}} + M \int_{-\infty}^{t} e^{-w(t-s)} L_g(s) \| u_s - v_s \|_{\mathcal{B}} \, ds \\ &\leq \left( L_f + M \sup_{t\in\mathbb{R}} \int_{-\infty}^{t} e^{-w(t-s)} L_g(s)ds \right) \| u - v \|_{\infty} \\ &\leq \Theta \| u - v \|_{\infty} \qquad (4.3) \end{aligned}$$

which prove that $\Gamma$ is a strict contraction. Consequently, by the Banach fixed-point principle there exists a unique mild solution to (1.1), which obviously is pseudo almost periodic. ■

## 5. EXAMPLE

This section is devoted to a concrete example related to the existence of pseudo almost periodic solutions to the neutral system (1.1)-(1.2).

In what follows, we set $\mathbb{X} = (L^2[0, \pi], \| \cdot \|_2)$ and define $A$ the linear operator

$$D(A) := \{u \in \mathbb{X} : u'' \in L^2[0, \pi],\ u(0) = u(\pi) = 0\}, \quad Au := u'', \quad \forall u \in D(A).$$

It is well known that $A$ is the infinitesimal generator of an analytic semigroup $(T(t))_{t\geq 0}$ on $\mathbb{X}$. Moreover, $A$ has a discrete spectrum, with eigenvalues of the form $-n^2, n \in \mathbb{N}$, whose corresponding (normalized) eigenfunctions are given by $z_n(\xi) := \sqrt{\frac{2}{\pi}} \sin(n\xi)$.

The corresponding semigroup to $A$ is defined by $T(t)u = \sum_{n=1}^{\infty} e^{-n^2 t} \langle u, z_n\rangle z_n$ for each $u \in \mathbb{X}$, with the estimate $\|T(t)\| \leq e^{-t}$ for every $t \geq 0$.

Consider the first-order neutral system

$$
\begin{aligned}
\frac{\partial}{\partial t}\left[u(t,\xi)+\int_{-p}^{0} b(s)u(t+s,\xi)ds\right] &= \frac{\partial^2}{\partial \xi^2}\left[u(t,\xi)+\int_{-p}^{0} b(s)u(t+s,\xi)ds\right]+a_0(\xi)u(t,\xi)\\
&\quad +\int_{-p}^{0} a(s)u(t+s,\xi)ds, \quad (t,\xi)\in\mathbb{R}\times[0,\pi], \qquad (5.1)\\
u(t,0) &= u(t,\pi)=0, \qquad (5.2)
\end{aligned}
$$

where $a, b, a_0$ are continuous functions.

By defining the substituting operators

$$f(t,\psi)(\xi) := \int_{-p}^{0} b(s)\psi(s,\xi)dds \qquad (5.3)$$

$$g(t,\psi)(\xi) := a_0(\xi)\psi(0,\xi)+\int_{-p}^{0} a(s)\psi(s,\xi)ds, \qquad (5.4)$$

the system (5.1)-(5.2) can be rewritten as a system of the form (1.1)-(1.2). Moreover, it is clear that $f, g$ are bounded linear operators and that

$$\|f(t,\psi)\| \leq \left[\sqrt{\int_{-p}^{0} b^2(s)ds}\right] . \|\psi\|_{\mathcal{B}}, \qquad (5.5)$$

$$\|g(t,\psi)\| \leq \left(\|a_0\|_\infty+\sqrt{\int_{-p}^{0} a^2(s)ds}\right) . \|\psi\|_{\mathcal{B}}, \qquad (5.6)$$

for all $t\in\mathbb{R}$ and $\psi_i\in\mathcal{B}(i=1,2)$.

The next result is a straightforward consequence of Theorem 4.1.

**Proposition 1.** *Under the previous assumptions, the neutral system (5.1)-(5.2) has a unique pseudo almost periodic mild solution whenever*

$$\sqrt{\int_{-p}^{0} b^2(s)ds} + \left(\|a_0\|_\infty+\sqrt{\int_{-p}^{0} a^2(s)ds}\right) < 1. \qquad (5.7)$$

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