1-TYPE PSEUDO-CHEBYSHEV SUBSPACES IN GENERALIZED 2-NORMED SPACES

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ABSTRACT. We construct a generalized 2-normed space from every normed space. We introduce 1-type pseudo-Chebyshev subspaces in generalized 2-normed spaces and give some results in this field.

Key words and phrases: Generalized 2-normed space, B-proximinal, 1-type pseudo-Chebyshev subspace, 2-functional.

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1. Introduction

The concept of linear 2-normed spaces has been investigated by Gähler in 1965 ([3]) and has been developed extensively in different subjects by others. Lewandowska published a series of papers on 2-normed sets and generalized 2-normed spaces in 1999-2003 ([5]-[9]). There are some works on characterization of 2-normed spaces, extension of 2-functionals and approximation in 2-normed spaces ([11], [2] and [4]). Also, there are some works in approximation theory (for example, [10]-[12]).

Let $X$ be a linear space of dimension greater than 1 over $K$, where $K$ is the real or complex numbers field. Suppose $\|\cdot,\|_2$ be a non-negative real-valued function on $X \times X$ satisfying the following conditions:

(i) $\|x, y\|_2 = 0$ if and only if $x$ and $y$ are linearly dependent vectors.

(ii) $\|x, y\|_2 = \|y, x\|_2$ for all $x, y \in X$.

(iii) $\|\lambda x, y\|_2 = |\lambda| \|x, y\|_2$ for all $\lambda \in K$ and all $x, y \in X$.

(iv) $\|x + y, z\|_2 \leq \|x, z\|_2 + \|y, z\|_2$ for all $x, y, z \in X$.

Then $\|\cdot, \cdot\|_2$ is called a 2-norm on $X$ and $(X, \|\cdot, \cdot\|_2)$ is called a linear 2-normed space.

Every 2-normed space is a locally convex topological vector space. In fact for a fixed $b \in X$, $p_b(x) = \|x, b\|_2$, $x \in X$, is a seminorm and the family $P = \{p_b : b \in X\}$ of seminorms generates a locally convex topology on $X$. But, there are no remarkable relations between normed spaces and 2-normed spaces.

We couldn’t construct any 2-norm on $X$ by a normed space $(X, \|\cdot\|)$, and this could be a motive for definition of generalized 2-normed spaces.

Definition 1.1. ([5]-[7]) Let $X$ and $Y$ be linear spaces, $D$ be a non-empty subset of $X \times Y$ such that for every $x \in X$, $y \in Y$ the sets

$$D_x = \{y \in Y : (x, y) \in D\}, \quad D^y = \{x \in X : (x, y) \in D\}$$

are linear subspaces of the spaces $Y$ and $X$, respectively. A function $\|\cdot, \cdot\| : D \longrightarrow [0, \infty)$ is called a generalized 2-norm on $D$ if it satisfies the following conditions:

$$(N_1) \quad \|x, \alpha y\| = |\alpha| \|x, y\|_2 = \|\alpha x, y\|_2,$$

for all $(x, y) \in D$ and every scalar $\alpha$.

$$(N_2) \quad \|x, y + z\|_2 \leq \|x, y\|_2 + \|x, z\|_2,$$

for all $(x, y), (x, z) \in D$.

$$(N_3) \quad \|x + y, z\|_2 \leq \|x, z\|_2 + \|y, z\|_2,$$

for all $(x, z), (y, z) \in D$.

Then, $(D, \|\cdot, \cdot\|)$ is called a 2-normed set. In particular, if $D = X \times Y$, $(X \times Y, \|\cdot, \cdot\|)$ is called a generalized 2-normed space. Moreover, if $X = Y$, then the generalized 2-normed space is denoted by $(X, \|\cdot, \cdot\|)$.

Definition 1.2. ([5]-[7]) Let $X$ be a linear space, $\chi$ be a non-empty subset of $X \times X$ such that $\chi = \chi^{-1}$ and the set $\chi^y = \{x \in X : (x, y) \in \chi\}$ is a linear subspace of $X$, for all $y \in X$. A function $\|\cdot, \cdot\| : \chi \longrightarrow [0, \infty)$ is called a generalized symmetric 2-norm on $\chi$ if it satisfies the following conditions:

$$(S_1) \quad \|x, y\| = \|y, x\|_2,$$

for all $(x, y) \in \chi$.

$$(S_2) \quad \|x, \alpha y\| = |\alpha| \|x, y\|_2,$$

for all $(x, y) \in \chi$ and every scalar $\alpha$.

$$(S_3) \quad \|x + y, z\|_2 \leq \|x, z\|_2 + \|y, z\|_2,$$

for all $(x, y), (x, z) \in \chi$.
Then, \((\chi, \|\cdot\|, \|\cdot\|)\) is called a generalized symmetric 2-normed set. In particular, if \(\chi = X \times X\), the function \(\|\cdot\|\) is called a generalized symmetric 2-norm on \(\chi\) and \((X, \|\cdot\|, \|\cdot\|)\) is called a generalized symmetric 2-normed space.

**Definition 1.3.** (5) Let \((X \times Y, \|\cdot\|, \|\cdot\|)\) be a generalized 2-normed space.

(a) The family \(\beta\) of all sets defined by \(\bigcap_{i=1}^{n} \{x \in X : \|x, y\| < \varepsilon\}\), where \(n \in \mathbb{N}\), \(y_1, ..., y_n \in Y\) and \(\varepsilon > 0\), forms a complete system of neighborhoods of zero for a locally convex topology in \(Y\).

(b) The family \(\beta\) of all sets defined by \(\bigcap_{i=1}^{n} \{y \in Y : \|x, y\| < \varepsilon\}\), where \(n \in \mathbb{N}\), \(x_1, ..., x_n \in X\) and \(\varepsilon > 0\), forms a complete system of neighborhoods of zero for a locally convex topology in \(X\).

We will denote the above topologies by the symbols \(\tau(X, Y)\) and \(\tau(Y, X)\), respectively. In the case when \(X = Y\), we will denote these topologies by \(\tau_1(X) = \tau(X, Y)\) and \(\tau_2(X) = \tau(Y, X)\).

Let us consider the linear spaces \(X\) and \(Y\) and let \(D \subseteq X \times Y\) be a 2-normed set and \(Z\) be a normed space. A map \(f : D \longrightarrow Z\) is called 2-linear if it satisfies the following conditions:

(i) \(f(x_1 + x_2, y_1 + y_2) = f(x_1, y_1) + f(x_1, y_2) + f(x_2, y_1) + f(x_2, y_2)\), for all \(x_1, x_2, y_1, y_2 \in X\) such that \(x_1, x_2 \in D^n \cap D^{m}\).

(ii) \(f(\delta x, \lambda y) = \delta \lambda f(x, y)\), for all scalars \(\delta, \lambda\) and all \((x, y) \in D\).

A 2-linear map \(f\) is said to be bounded if there exists a non-negative real number \(M\) such that \(\|f(x, y)\| \leq M\|x, y\|\) for all \((x, y) \in D\). Also, the norm of a 2-linear map \(f\) is defined by \(\|f\| = \inf\{M \geq 0 : \|f(x, y)\| \leq M\|x, y\|\\) for all \((x, y) \in D]\).

We denote by \(< b >\) the subspace of \(X\) generated by the element \(b \in X\). For a generalized 2-normed space \((X \times Y, \|\cdot\|, \|\cdot\|)\), a subspace \(W\) of \(X\) and \(b \in Y\), we denote by \(W^b\) the Banach space of all \(K\)-valued bounded 2-linear maps on \(W \times < b >\).

Let \((X \times Y, \|\cdot\|, \|\cdot\|)\) be a generalized 2-normed space, \(W\) be a subspace of \(X\) and \(b \in Y\).

(i) \(w_0 \in W\) is called b-best approximation of \(x \in X\) in \(W\), if

\[\|x - w_0, b\| = \inf\{\|x - w, b\| : w \in W\}.\]

The set of all b-best approximations of \(x \in W\) is denoted by \(P^b_W(x)\).

(ii) \(W\) is called b-proximinal if for every \(x \in X \setminus (\overline{W}W)\), there exists \(w_0 \in W\) such that \(\|x - w_0, b\| = \inf\{\|x - w, b\| : w \in W\}\), where \(\overline{W}\) denotes the closure of \(W\) in the seminormed space \((X, p_0)\).

Note that, \(W\) is b-proximinal if and only if \(P^b_W(x) \neq \emptyset\) for all \(x \in X \setminus \overline{W}\).

The following basic lemma is important in the proof of main results.

**Proposition 1.1.** (3): Theorem 3.6). Let \((X, \|\cdot\|, \|\cdot\|)\) be a 2-normed space, \(W\) be a subspace of \(X\) and \(b \in X\). If \(x_0 \in X\) is such that

\[\delta = \inf\{\|x_0 - w, b\| : w \in W\} > 0,\]

then there exists a bounded 2-linear map \(F : X \times < b > \longrightarrow K\) such that \(F|_{W \times < b >} = 0\), \(F(x_0, b) = 1\) and \(\|F\| = \frac{1}{\delta}\).

By review of [3], we find that the following similar Lemma holds for generalized 2-normed spaces.
Lemma 1.2. Let $(X \times Y, \|., .\|)$ be a generalized 2-normed space, $W$ be a subspace of $X$ and $b \in Y$. If $x_0 \in X$ is such that
\[
\delta = \inf \{ \|x_0 - w, b\| : w \in W \} > 0,
\]
then there exists a bounded 2-linear map $F : X \times <b> \rightarrow K$ such that $F|_{W \times <b>} = 0$, $F(x_0, b) = 1$ and $\|F\| = \frac{1}{\delta}$.

Lemma 1.3. Let $(X \times Y, \|., .\|)$ be a generalized 2-normed space, $W$ be a subspace of $X$, $b \in Y$ and $x \in X \setminus W$, where $W$ denotes the closure of $W$ in the seminormed space $(X, p_b)$. Then, $M \subseteq P_b^d(x)$ if and only if there exists $f \in X^d_b$ such that $f|_{W \times <b>} = 0$, $\|f\| = 1$ and $f(x_0 - m, b) = \|x_0 - m, b\|$ for all $m \in M$.

Proof. First suppose that there exists $f \in X^d_b$ such that $f|_{W \times <b>} = 0$, $\|f\| = 1$ and $f(x_0 - m, b) = \|x_0 - m, b\|$ for all $m \in M$. Then,
\[
\|x_0 - m, b\| = f(x_0 - m, b) = f(x_0, b) = f(x_0 - w, b)
\]
\[
\leq \|f\| \|x_0 - w, b\| = \|x_0 - w, b\|
\]
for all $m \in M$ and all $w \in W$. Hence, $m \in P_b^d(x)$ for all $m \in M$. Conversely, fix $m_0 \in M$. Then,
\[
\delta = \|x_0 - m_0, b\| = \inf \{ \|x_0 - w, b\| : w \in W \} > 0.
\]
By Lemma 1.2, there exists $g \in X^d_b$ such that $g|_{W \times <b>} = 0$, $g(x_0, b) = 1$ and $\|g\| = \frac{1}{\delta}$. Now for $f = \delta g$ we have, $f|_{W \times <b>} = 0$, $f(x_0 - m, b) = \|x_0 - m, b\|$ and $\|f\| = 1$. Note that, $f(x_0 - m, b) = \|x_0 - m, b\|$ for all $m \in M$. 

2. 1-type Pseudo-Chebyshev Subspaces

Definition 2.1. Let $(X \times Y, \|., .\|)$ be a generalized 2-normed space, $W$ be a subspace of $X$ and $b \in Y$.

(i) $W$ is called $b$-pseudo Chebyshev if for every $x \in X \setminus W$, where $W$ denotes the closure of $W$ in the seminormed space $(X, p_b)$, $P_b^d(x)$ is non-empty and finite dimensional.

(ii) $W$ is called 1-type pseudo-Chebyshev if $W$ is $b$-pseudo Chebyshev for every $0 \neq b \in Y$.

Example 2.1. Let $X = \mathbb{R}^3$, $W = \{(x, y, 0) : x, y \in \mathbb{R}\}$ and
\[
\| (x_1, x_2, x_3), (y_1, y_2, y_3) \| = \max \{ |x_1y_2 - x_2y_1| + |x_1y_3 - x_3y_1|, |x_1y_2 - x_2y_1| + |x_2y_3 - x_3y_2| \}
\]
for all $(x_1, x_2, x_3), (y_1, y_2, y_3) \in X$. Then, $\|., .\|$ is a 2-norm on $X$ and $W$ is 1-type pseudo-Chebyshev subspace.

Example 2.2. Let $W$ be a pseudo-Chebyshev subspace of a normed space $(X, \|., \|_1)$ and let $(Y, \|., \|_2)$ be an arbitrary normed space. Then, $\|x, y\| = \|x\|_1 \|y\|_2$ is a generalized 2-norm on $X \times Y$ and $W$ is 1-type pseudo-Chebyshev subspace.

Proposition 2.1. Let $(X \times Y, \|., .\|)$ be a generalized 2-normed space, $W$ be a subspace of $X$ and $b \in Y$. Then, $W$ is $b$-pseudo Chebyshev subspace of $X$ if and only if there do not exist $f \in X^d_b$, $x_0 \in X \setminus W$, where $W$ denotes the closure of $W$ in the seminormed space $(X, p_b)$, and infinitely many linearly independent elements $w_1, w_2, \ldots$ in $W$ such that $f|_{W \times <b>} = 0$, $\|f\| = 1$ and $f(x_0 - w_n, b) = \|x_0 - w_n, b\|$, for all $n \geq 1$. 

AJMAA, Vol. 4, No. 1, Art. 9, pp. 1-7, 2007
Proof. Suppose that $W$ is not b-pseudo Chebyshev subspace. Then, there exists $x \in X \setminus \overline{W}$, such that $P^b_w(x)$ is not finite dimensional. Fix $w_0 \in P^b_w(x)$. Then, there exist infinitely many elements $w_1, w_2, \ldots \in P^b_w(x)$ such that $w_0 - w_1, w_0 - w_2, \ldots$ are infinitely many linearly independent elements of $W$. Put $x_0 = x - w_0$ and $g_n = w_n - w_0$ for all $n \geq 1$ and note that, $g_1, g_2, \ldots$ are infinitely many linearly independent elements of $P^b_w(x)$. By Lemma 1.3, there exists $f \in X^b_w$ such that $f|_{W^\times <b>} = 0$, $\|f\| = 1$ and $f(x_0 - g_n, b) = \|x_0 - g_n, b\|$ for all $n \geq 1$. This is a contradiction. 

Corollary 2.2. Let $(X \times Y, \|\cdot, \cdot\|)$ be a generalized 2-normed space and let $W$ be a subspace of $X$. Then, $W$ is 1-type pseudo-Chebyshev subspace if and only if there do not exist $0 \neq b_0 \in Y$, $x_0 \in X \setminus \overline{W}$, $f_{b_0} \in X^b_w$, where $W$ denotes the closure of $W$ in the seminormed space $(X, p_{b_0})$, and infinitely many linearly independent elements $w_1, w_2, \ldots \in W$ such that $\|f_{b_0}\| = 1$, $f_{b_0}|_{W^\times <b>} = 0$ and $f_{b_0}(x_0 - w_n, b_0) = \|x_0 - w_n, b_0\|$ for all $n \geq 1$.

3. $(b, \varepsilon)$-pseudo Chebyshev subspaces

Definition 3.1. Let $(X \times Y, \|\cdot, \cdot\|)$ be a generalized 2-normed space, $W$ be a subspace of $X$, $0 \neq b \in Y$ and $\varepsilon > 0$ be given.

(i) $w_0 \in W$ is called $(b, \varepsilon)$-best approximation of $x \in X$ in $W$, if $\|x - w_0, b\| = \inf\{\|x - w, b\| : w \in W\} + \varepsilon$.

The set of all $b$-best approximations of $x$ in $W$ is denoted by $P^b_{W, \varepsilon}(x)$.

(ii) $W$ is called $(b, \varepsilon)$-pseudo Chebyshev if $P^b_{W, \varepsilon}(x)$ is finite dimensional for every $x \in X$.

Theorem 3.1. Let $(X \times Y, \|\cdot, \cdot\|)$ be a generalized 2-normed space, $W$ be a subspace of $X$, $w_0 \in W$, $0 \neq b \in Y$ and $\varepsilon > 0$ be given. Then, $w_0 \in P^b_{W, \varepsilon}(x)$ if and only if there exist $f \in X^b_w$ such that $f|_{W^\times <b>} = 0$, $\|f\| = 1$ and $f(x - w_0, b) \geq \|x - w_0, b\| - \varepsilon$.

Proof. First suppose that there exist $f \in X^b_w$ such that $f|_{W^\times <b>} = 0$, $\|f\| = 1$ and $f(x - w_0, b) = \|x - w_0, b\| - \varepsilon$. Then, $\|x - w_0, b\| \leq f(x - w_0, b) + \varepsilon = f(x - w, b) + \varepsilon \leq \|x - w, b\| + \varepsilon$ for all $w \in W$. Hence, $w_0 \in P^b_{W, \varepsilon}(x)$. Conversely, Let $w_0 \in P^b_{W, \varepsilon}(x)$. If $x \in \overline{W}$, where $W$ denotes the closure of $W$ in the seminormed space $(X, p_{b_0})$, choose $w_0 \in W$ such that $\|x - w_0, b\| < \varepsilon$. Then, every $f \in X^b_w$ with $f|_{W^\times <b>} = 0$ and $\|f\| = 1$, satisfies $f(x - w_0, b) \geq \|x - w_0, b\| - \varepsilon$. If $x \in X \setminus \overline{W}$, $\delta = \inf\{\|x - w, b\| : w \in W\} > 0$. Then by Lemma 1.2, there exists $g \in X^b_w$ such that $g|_{W^\times <b>} = 0$, $g(x, b) = 1$ and $\|g\| = \frac{1}{\delta}$. Put $f = \delta g$. Then, $f \in X^b_w$, $f|_{W^\times <b>} = 0$, $\|f\| = 1$ and $f(x - w_0, b) + \varepsilon = \delta + \varepsilon \geq \|x - w_0, b\|$.

Lemma 3.2. Let $(X \times Y, \|\cdot, \cdot\|)$ be a generalized 2-normed space, $W$ be a subspace of $X$, $\varepsilon > 0$ be given and $0 \neq b \in Y$. Then, $M \subseteq P^b_{W, \varepsilon}(x)$ if and only if there exists $f \in X^b_w$ such that $f|_{W^\times <b>} = 0$, $\|f\| = 1$ and $f(x_0 - m, b) \geq \|x_0 - m, b\| - \varepsilon$ for all $m \in M$.

Proof. Let $M \subseteq P^b_{W, \varepsilon}(x)$ and choose $w_0 \in P^b_{W, \varepsilon}(x)$ with $\|x - w_0, b\| = \lambda + \varepsilon$, where $\lambda = \inf\{\|x - w, b\| : w \in W\}$. By Theorem 3.1, there exist $f \in X^b_w$ such that $f|_{W^\times <b>} = 0$, $\|f\| = 1$ and $f(x - w_0, b) \geq \|x - w_0, b\| - \varepsilon$. Then, $f(x - m, b) = f(x - w_0, b) \geq \|x - w_0, b\| - \varepsilon = \lambda \geq \|x - m, b\| - \varepsilon$, for all $m \in M$.

Theorem 3.3. $(X \times Y, \|\cdot, \cdot\|)$ be a generalized 2-normed space, $W$ be a subspace of $X$, $0 \neq b \in Y$ and $\varepsilon > 0$ be given. Then, $W$ is $(b, \varepsilon)$-pseudo Chebyshev subspace if and only if there do not exist $f \in X^b_w$, $x \in X$ and infinitely many linearly independent elements $w_1, w_2, \ldots$ in $W$ such that $\|x, b\| \leq 1$, $f|_{W^\times <b>} = 0$, $\|f\| = 1$ and $f(x - w_n, b) \geq \|x - w_n, b\| - \varepsilon$, for all $n \geq 1$. 

AJMAA, Vol. 4, No. 1, Art. 9, pp. 1-7, 2007
Proof. First assume that there exist \( f \in X^b_0, x \in X \) and infinitely many linearly independent elements \( w_1, w_2, \ldots \) in \( W \) such that \( \| x, b \| \leq 1, f|_{W \times \langle b \rangle} = 0, \| f \| = 1 \) and \( f(x - w_n, b) \geq \| x - w_n, b \| - \varepsilon \), for all \( n \geq 1 \). Then, \( w_n \in P^b_{W,\varepsilon}(x) \) for all \( n \geq 1 \). It follows that \( \dim P^b_{W,\varepsilon}(x) = \infty \) and hence \( W \) is not \((b,\varepsilon)\)-pseudo Chebyshev subspace. Now, suppose that \( W \) is not \((b,\varepsilon)\)-pseudo Chebyshev subspace. Since \( P^0_{W,\varepsilon}(\lambda x) = \lambda P^0_{W,\varepsilon}(x) \) and \( P^0_{W,\varepsilon}(x) \subseteq P^0_{W,\varepsilon}(\varepsilon) \) for all \( 0 < \varepsilon \leq \varepsilon_2 \), \( x \in X \) and \( \lambda > 0 \), there exist \( x_0 \in X \) with \( \| x_0, b \| \leq 1 \) such that \( \dim P^0_{W,\varepsilon}(x) = \infty \). Hence, \( P_{W,\varepsilon}(x_0) \) contains infinitely many linearly independent elements \( g_1, g_2, \ldots \). By Lemma 3.2, there exists \( f \in X^b_0 \) such that \( \| f \| = 1, f|_{W \times \langle b \rangle} = 0 \) and \( f(x_0 - g_n, b) \geq \| x_0 - g_n, b \| - \varepsilon \) for all \( n \geq 1 \).

**Definition 3.2.** Let \((X 	imes Y, \| . \|, \| . \|)\) be a generalized 2-normed space, \( 0 \neq b \in Y, \varepsilon > 0 \) be given and \( f \in X^b_0 \). Define

\[
M^b_{f,\varepsilon} = \{ x \in X : f(x, b) \geq \| x, b \| - \varepsilon, \| x, b \| \leq 1 + \varepsilon \}.
\]

**Theorem 3.4.** Let \((X 	imes Y, \| . \|, \| . \|)\) be a generalized 2-normed space, \( W \) be a subspace of \( X \), \( 0 \neq b \in Y \) and \( \varepsilon > 0 \) be given. If \( M^b_{f,\varepsilon} \) is finite dimensional for all \( f \in X^b_0 \) with \( \| f \| = 1 \) and \( f|_{W \times \langle b \rangle} = 0 \), then \( W \) is \((b,\varepsilon)\)-pseudo Chebyshev subspace.

**Proof.** Assume that \( W \) is not \((b,\varepsilon)\)-pseudo Chebyshev subspace. Then by Theorem 3.3 there exist \( f \in X^b_0, x_0 \in X \) with \( \| x_0, b \| \leq 1 \) and infinitely many linearly independent elements \( w_1, w_2, \ldots \) in \( W \) such that \( \| f \| = 1, f|_{W \times \langle b \rangle} = 0 \), and \( f(x_0 - w_n, b) \geq \| x_0 - w_n, b \| - \varepsilon \) for all \( n \geq 1 \). Since \( \| x_0 - w_n, b \| \leq \| x_0 - w_n, b \| + \varepsilon = \| f(x_0, b) \| + \varepsilon \leq 1 + \varepsilon \), \( x_0 - w_n \in M^b_{f,\varepsilon} \) for all \( n \geq 1 \). This is a contradiction.

**Definition 3.3.** Let \((X 	imes Y, \| . \|, \| . \|)\) be a generalized 2-normed space, \( 0 \neq b \in Y, \varepsilon > 0 \) be given and let \( M \) be a subspace of \( X^b_0 \). For each \( x \in X \), put

\[
D^{M,b}_{x,\varepsilon} = \{ y \in X : f(y, b) = f(x, b) \text{ for all } f \in M \text{ & } \| y, b \| \leq \| x, b \| + \varepsilon \},
\]

where \( \| x, b \| = \sup \{ \| f(x, b) \| : \| f \| \leq 1, f \in M \} \).

It is clear that \( D^{M,b}_{x,\varepsilon} \) is a non-empty, closed and convex subset of \((X, p_b)\), for all \( x \in X \).

We say that \( M \) has the property \((b,\varepsilon) - F^* \) if \( D^{M,b}_{x,\varepsilon} \) is finite dimensional for all \( x \in X \).

**Theorem 3.5.** Let \((X 	imes Y, \| . \|, \| . \|)\) be a generalized 2-normed space, \( W \) be a subspace of \( X \), \( \varepsilon > 0 \) be given, \( 0 \neq b \in Y \) and let \( M_0 = \{ f \in X^b_0 : f|_{W \times \langle b \rangle} = 0 \} \). Then, \( W \) is \((b,\varepsilon)\)-pseudo Chebyshev subspace if and only if \( M_0 \) has the property \((b,\varepsilon) - F^* \).

**Proof.** If \( \dim D^{M_0}_{x,\varepsilon} = \infty \) for some \( x \in X \), then there exist infinitely many linearly independent elements \( y_1, y_2, \ldots \) in \( D^{M_0}_{x,\varepsilon} \). Hence, \( y_1 - y_n \in W \) for all \( n \geq 1 \) and

\[
\| y_1 - (y_1 - y_n), b \| = \| y_n, b \| \leq \| x, b \| = \| y_1 - (y_1 - y_n), b \| + \varepsilon
\]

for all \( n \geq 1 \). Therefore, \( y_1 - y_n \in P^b_{W,\varepsilon}(y_1) \) for all \( n \geq 1 \). It follows that \( W \) is not \((b,\varepsilon)\)-pseudo Chebyshev subspace. Now, suppose that \( \dim P^b_{W,\varepsilon}(x_0) = \infty \) for some \( x_0 \in X \). Then, there exist infinitely many linearly independent elements \( g_1, g_2, \ldots \) in \( P^b_{W,\varepsilon}(x_0) \). It is easy to see that, \( \| x_0 - g_n, b \| \leq \| x_0 - g_n, b \| + \varepsilon = \| x_0, b \| + \varepsilon \) for all \( n \geq 1 \). It follows that \( x_0 - g_n \in D^{M_0,b}_{x_0,\varepsilon} \) for all \( n \geq 1 \), which is a contradiction.
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