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WEIGHTED GENERALIZATION OF THE TRAPEZOIDAL RULE VIA FINK IDENTITY

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ABSTRACT. The weighted Fink identity is given and used to obtain generalized weighted trapezoidal formula for n-time differentiable functions. Also, an error estimate is obtained for this formula.

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1. INTRODUCTION

The following result is known as Ostrowski inequality:

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t\right| \le \left[\frac{1}{4}+\frac{(x-\frac{a+b}{2})^2}{(b-a)^2}\right](b-a)M, \tag{1.1}$$

where $f:[a,b]\to\mathbf{R}$ is differentiable function such that $|f'(x)|\le M$, for every $x\in[a,b]$.

The Ostrowski inequality has been generalized by G. V. Milovanović and J. Pečarić [1] and A. M. Fink [2]. They considered generalizations of (1.1) in the form:

$$\left|\frac{1}{n}\Big(f(x)+\sum_{k=1}^{n-1}F_k(x)\Big)-\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t\right| \le K(n,p,x)\|f^{(n)}\|_p, \tag{1.2}$$

which is obtained from Fink identity

$$\frac{1}{n}\Big(f(x)+\sum_{k=1}^{n-1}F_k(x)\Big)-\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t = \frac{1}{n!(b-a)}\int_a^b (x-t)^{n-1}k(t,x)f^{(n)}(t)\mathrm{d}t, \tag{1.3}$$

where

$$F_k(x)=\frac{n-k}{k!}\,\frac{f^{(k-1)}(a)(x-a)^k-f^{(k-1)}(b)(x-b)^k}{b-a},$$

and $k(t,x)=t-a$, if $a\le t\le x\le b$, and $k(t,x)=t-b$, if $a\le x\le t\le b$.

Fink obtained the following result:

Theorem 1.1. *Let $f^{(n-1)}$ be absolutely continuous on $[a,b]$ with $f^{(n)}\in L_p[a,b]$. Then the inequality (1.2) holds with*

$$K(n,p,x)=\frac{\left[(x-a)^{nq+1}+(b-x)^{nq+1}\right]^{\frac{1}{q}}}{n!(b-a)}B^{\frac{1}{q}}((n-1)q+1,q+1), \tag{1.4}$$

where $1<p\le\infty$, $\frac{1}{p}+\frac{1}{q}=1$, B is Beta function, and

$$K(n,1,x)=\frac{(n-1)^{n-1}}{n^n n!(b-a)}\max\{(x-a)^n,(b-x)^n\}. \tag{1.5}$$

Remark 1.1. Milovanović and Pečarić have proved in 1976. that

$$K(n,\infty,x)=\frac{(x-a)^{n+1}+(b-x)^{n+1}}{n(n+1)!(b-a)},$$

which is the special case for $p=\infty$ in Theorem 1.1.

S. S. Dragomir and A. Sofo [4] have used Fink identity to obtain following generalization of the trapezoidal formula for $n-$time differentiable functions:

Theorem 1.2. *Let $f : [a,b] \to \mathbf{R}$ be a mapping such that its $(n-1)-th$ derivative $f^{(n-1)}$ is absolutely continuous on $[a,b]$. Then we have equality*

$$\begin{aligned}
\frac{1}{n}\Big[\frac{f(a)+f(b)}{2} &+ \sum_{k=1}^{n-1}\frac{(n-k)(b-a)^{k-1}(f^{(k-1)}(a)+(-1)^{k-1}f^{(k-1)}(b))}{2k!}\Big] \\
&- \frac{1}{b-a}\int_a^b f(y)\mathrm{d}y \qquad (1.6)\\
&= \frac{1}{2n!(b-a)}\int_a^b [(t-a)(b-t)^{n-1}-(a-t)^{n-1}(b-t)]f^{(n)}(t)\mathrm{d}t.
\end{aligned}$$

Let $\omega : [a,b] \to [0,\infty)$ be some probability density function, i.e. integrable function satisfying $\int_a^b \omega(t)\mathrm{d}t = 1$, and $W(t) = \int_a^t \omega(x)\mathrm{d}x$ for $t \in [a,b]$, $W(t) = 0$ for $t < a$ and $W(t) = 1$ for $t > b$.

In this paper we give the extension of Fink identity (1.3) for general weighted function ω instead of uniform function $\frac{1}{b-a}$. We derive the weighted trapezoidal rule using weighted Fink identity and give error estimate for this formula.

2. WEIGHTED FINK IDENTITY

Theorem 2.1. *Let $f : [a,b] \to \mathbf{R}$ be such that $f^{(n-1)}$ is absolutely continuous function on $[a,b]$ for some $n \geq 1$. If $\omega : [a,b] \to [0,\infty)$ is some probability density function, then the following identity holds:*

$$\begin{aligned}
f(x) &= \int_a^b \omega(t)f(t)\mathrm{d}t - \sum_{k=1}^{n-1}F_k(x) + \sum_{k=1}^{n-1}\int_a^b \omega(t)F_k(t)\mathrm{d}t \\
&+ \frac{1}{(n-1)!(b-a)}\int_a^b (x-y)^{n-1}k(y,x)f^{(n)}(y)\mathrm{d}y \qquad (2.1)\\
&- \frac{1}{(n-1)!(b-a)}\int_a^b \Big(\int_a^b \omega(t)(t-y)^{n-1}k(y,t)\mathrm{d}t\Big)f^{(n)}(y)\mathrm{d}y.
\end{aligned}$$

Proof. First we multiply identity (1.3) by n to get

$$f(x)+\sum_{k=1}^{n-1}F_k(x)-\frac{n}{b-a}\int_a^b f(t)\mathrm{d}t = \frac{1}{(n-1)!(b-a)}\int_a^b (x-t)^{n-1}k(t,x)f^{(n)}(t)\mathrm{d}t, \qquad (2.2)$$

Now we multiply (2.2) by $\omega(x)$ and integrate it with respect to x to obtain

$$\begin{aligned}
\int_a^b \omega(x)f(x)\mathrm{d}x + \sum_{k=1}^{n-1}\int_a^b \omega(x)F_k(x)\mathrm{d}x - \Big(\int_a^b \omega(x)\mathrm{d}x\Big)\frac{n}{b-a}\int_a^b f(t)\mathrm{d}t \\
= \frac{1}{(n-1)!(b-a)}\int_a^b \Big(\int_a^b \omega(x)(x-t)^{n-1}k(t,x)\mathrm{d}y\Big)f^{(n)}(t)\mathrm{d}t. \qquad (2.3)
\end{aligned}$$

Now we subtract last identity from (2.2) to get (2.1). ∎

Remark 2.1. A. Aglić Aljinović, J. Pečarić and A. Vukelić [3] have proved the same identity (2.1) using the weighted generalization of Montgomery identity

$$f(x) = \int_a^b \omega(t)f(t)\mathrm{d}t + \int_a^b P_\omega(x,t)f'(t)\mathrm{d}t,$$

where $P_\omega(x,t)$ is the weighted Peano kernel defined by $P_\omega(x,t) = W(t)$, if $a \le t \le x$, and $P_\omega(x,t) = W(t) - 1$, for $x < t \le b$.

Remark 2.2. We can recover Fink identity from weighted Fink identity. Suppose $f : [a,b] \to \mathbf{R}$ is such that $f^{(n-1)}$ is absolutely continuous function on $[a,b]$ for some $n \ge 1$ and identity (2.1) holds for some probability density function ω on $[a,b]$.

Let's denote

$$I_n = \frac{1}{(n-1)!(b-a)} \int_a^b \Big(\int_a^b \omega(t)(t-y)^{n-1} k(y,t)\mathrm{d}t \Big) f^{(n)}(y)\mathrm{d}y.$$

By partial integration we get

$$\begin{aligned} I_n &= \frac{1}{(n-2)!(b-a)} \int_a^b \Big(\int_a^b \omega(t)(t-y)^{n-2} k(y,t)\mathrm{d}t \Big) f^{(n-1)}(y)\mathrm{d}y \\ &\quad - \frac{1}{(n-1)!(b-a)} \int_a^b \Big(\int_a^b \omega(t)(t-y)^{n-1}\mathrm{d}t \Big) f^{(n-1)}(y)\mathrm{d}y \qquad (2.4) \\ &= I_{n-1} + J_n, \qquad (2.5) \end{aligned}$$

where we denote

$$J_n = -\frac{1}{(n-1)!(b-a)} \int_a^b \Big(\int_a^b \omega(t)(t-y)^{n-1}\mathrm{d}t \Big) f^{(n-1)}(y)\mathrm{d}y.$$

By partial integration, we obtain

$$\begin{aligned} J_n &= \frac{1}{(n-1)!(b-a)}) \int_a^b \omega(t) \Big[f^{(n-2)}(a)(t-a)^{n-1} - f^{(n-2)}(b)(t-b)^{n-1} \Big] \mathrm{d}t + J_{n-1} \\ &= J'_n + J_{n-1}, \end{aligned}$$

where

$$J'_n = \frac{1}{(n-1)!(b-a)}) \int_a^b \omega(t) \Big[f^{(n-2)}(a)(t-a)^{n-1} - f^{(n-2)}(b)(t-b)^{n-1} \Big] \mathrm{d}t.$$

By iteration we get

$$J_n = \sum_{k=2}^{n} J'_k + J_1.$$

Further,

$$J_1 = -\frac{1}{b-a} \int_a^b f(t)\mathrm{d}t,$$

and

$$I_1 = \int_a^b \omega(t) f(t)\mathrm{d}t - \frac{1}{b-a} \int_a^b f(t)\mathrm{d}t.$$

Finally,

$$\begin{aligned} I_n &= I_1 + \sum_{k=1}^{n-1} (n-k) J'_{k+1} + (n-1) J_1 \\ &= \sum_{k=1}^{n-1} \int_a^b \omega(t) F_k(t)\mathrm{d}t + \int_a^b \omega(t) f(t)\mathrm{d}t - \frac{n}{b-a} \int_a^b f(t)\mathrm{d}t. \end{aligned}$$

That implies that first, third and last member in (2.1) can be replaced by $\frac{n}{b-a}\int_a^b f(t)\mathrm{d}t$ so we get

(2.6)

$$f(x) = -\sum_{k=1}^{n-1} F_k(x) + \frac{n}{b-a}\int_a^b f(t)\mathrm{d}t + \frac{1}{(n-1)!(b-a)}\int_a^b (x-y)^{n-1}k(y,x)f^{(n)}(y)\mathrm{d}y.$$

Dividing this by n we obtain (1.3).

Remark 2.3. In [3] is shown that if we take $\omega(t) = \frac{1}{b-a}$, for $t \in [a,b]$, formula (2.1) becomes Fink identity (1.3), which allows us to call (2.1) weighted Fink identity. Previous remark is the generalization of this result.

3. TRAPEZODIAL FORMULA AND ITS EXTENSION

Let $f : [a,b] \to \mathbf{R}$ be a mapping such that its $(n-1)-th$ derivative $f^{(n-1)}$ is absolutely continuous on $[a,b]$. Then, by Theorem 1.2. we have generalized trapezoidal formula (1.6). We can write

$$\begin{aligned} & \sum_{k=1}^{n-1} \frac{(n-k)(b-a)^{k-1}(f^{(k-1)}(a)+(-1)^{k-1}f^{(k-1)}(b))}{2k!} \\ = & \frac{(n-1)(f(a)+f(b))}{2} \\ + & \sum_{k=2}^{n-1} \frac{(n-k)(b-a)^{k-1}(f^{(k-1)}(a)+(-1)^{k-1}f^{(k-1)}(b))}{2k!}, \end{aligned} \tag{3.1}$$

so we get from (1.6)

$$\begin{aligned} \frac{f(a)+f(b)}{2} & + \frac{1}{n}\sum_{k=2}^{n-1} \frac{(n-k)(b-a)^{k-1}[f^{(k-1)}(a)+(-1)^{k-1}f^{(k-1)}(b)]}{2k!} \\ & - \frac{1}{b-a}\int_a^b f(y)\mathrm{d}y \\ & = \frac{1}{2n!(b-a)}\int_a^b [(t-a)(b-t)^{n-1}+(a-t)^{n-1}(t-b)]f^{(n)}(t)\mathrm{d}t. \end{aligned} \tag{3.2}$$

In [3] is obtained the following weighted trapezoidal formula:

$$\begin{aligned} \frac{1}{2}[f(a)+f(b)] = & \int_a^b \omega(t)f(t)\mathrm{d}t + \sum_{k=1}^{n-1}\int_a^b \omega(t)F_k(t)\mathrm{d}t \\ - & \frac{1}{2}\sum_{k=1}^{n-1}\frac{n-k}{k!}\left[f^{(k-1)}(a)(b-a)^{k-1}+f^{(k-1)}(b)(a-b)^{k-1}\right] \\ + & \frac{1}{2(n-1)!(b-a)}\int_a^b [(a-y)^{n-1}(y-b)+(b-y)^{n-1}(y-a)]f^{(n)}(y)\mathrm{d}y \\ - & \frac{1}{(n-1)!(b-a)}\int_a^b \Big(\int_a^b \omega(t)(t-y)^{n-1}k(y,t)\mathrm{d}t\Big) f^{(n)}(y)\mathrm{d}y. \end{aligned} \tag{3.3}$$

Remark 3.1. We can recover generalized trapezoidal formula (3.2) from identity (3.3) using Remark 2.2.

If we take $\omega(t) = \frac{1}{b-a}$ in (3.3) and apply Remark 2.2, we obtain generalized trapezoidal rule (3.2). So, it is logical to call identity (3.3) generalized weighted trapezoidal formula.

Theorem 3.1. *Let $f : [a,b] \to \mathbf{R}$ be a mapping such that its $(n-1)-th$ derivative $f^{(n-1)}$ is absolutely continuous on $[a,b]$, and let $\omega : [a,b] \to [0,\infty)$ be probability density function. Then we can write the generalized weighted trapezoidal formula in the next way*

$$
\begin{aligned}
& f(a)\Big[\frac{1}{2} - \Big(1-\frac{1}{n}\Big)\int_a^b \omega(t)\frac{t-a}{b-a}\mathrm{d}t\Big] + f(b)\Big[\frac{1}{2} + \Big(1-\frac{1}{n}\Big)\int_a^b \omega(t)\frac{t-b}{b-a}\mathrm{d}t\Big] \\
= & \ \frac{1}{n}\Bigg[\int_a^b \omega(t)f(t)\mathrm{d}t + \sum_{k=2}^{n-1}\int_a^b \omega(t)F_k(t)\mathrm{d}t \\
- & \ \frac{1}{2}\sum_{k=2}^{n-1}\frac{n-k}{k!}\Big[f^{(k-1)}(a)(b-a)^{k-1} + f^{(k-1)}(b)(a-b)^{k-1}\Big]\Bigg] \\
+ & \ \frac{1}{2n!(b-a)}\int_a^b [(a-y)^{n-1}(y-b) + (b-y)^{n-1}(y-a)]f^{(n)}(y)\mathrm{d}y \\
- & \ \frac{1}{n!(b-a)}\int_a^b \Big(\int_a^b \omega(t)(t-y)^{n-1}k(y,t)\mathrm{d}t\Big)f^{(n)}(y)\mathrm{d}y.
\end{aligned} \tag{3.4}
$$

Proof. We write

$$
\begin{aligned}
& \frac{1}{2}\sum_{k=1}^{n-1}\frac{n-k}{k!}\Big[f^{(k-1)}(a)(b-a)^{k-1} + f^{(k-1)}(b)(a-b)^{k-1}\Big] \\
= & \ (n-1)\frac{f(a)+f(b)}{2} + \frac{1}{2}\sum_{k=2}^{n-1}\frac{n-k}{k!}\Big[f^{(k-1)}(a)(b-a)^{k-1} + f^{(k-1)}(b)(a-b)^{k-1}\Big],
\end{aligned} \tag{3.5}
$$

and

$$
\begin{aligned}
& \sum_{k=1}^{n-1}\int_a^b \omega(t)F_k(t)\mathrm{d}t \\
= & \ \frac{n-1}{b-a}\left[f(a)\int_a^b \omega(t)(t-a)dt - f(b)\int_a^b \omega(t)(t-b)dt\right] + \sum_{k=2}^{n-1}\int_a^b \omega(t)F_k(t)\mathrm{d}t
\end{aligned} \tag{3.6}
$$

and put it in (3.3) to get

$$
\begin{aligned}
& \frac{1}{2}[f(a)+f(b)] = \int_a^b \omega(t)f(t)\mathrm{d}t \\
+ & \ \frac{n-1}{b-a}\left[f(a)\int_a^b \omega(t)(t-a)dt - f(b)\int_a^b \omega(t)(t-b)dt\right] + \sum_{k=2}^{n-1}\int_a^b \omega(t)F_k(t)\mathrm{d}t \\
- & \ (n-1)\frac{f(a)+f(b)}{2} - \frac{1}{2}\sum_{k=2}^{n-1}\frac{n-k}{k!}\Big[f^{(k-1)}(a)(b-a)^{k-1} + f^{(k-1)}(b)(a-b)^{k-1}\Big] \\
+ & \ \frac{1}{2(n-1)!(b-a)}\int_a^b [(a-y)^{n-1}(y-b) + (b-y)^{n-1}(y-a)]f^{(n)}(y)\mathrm{d}y \\
- & \ \frac{1}{(n-1)!(b-a)}\int_a^b \Big(\int_a^b \omega(t)(t-y)^{n-1}k(y,t)\mathrm{d}t\Big)f^{(n)}(y)\mathrm{d}y.
\end{aligned} \tag{3.7}
$$

Putting all terms with $f(a)$ and $f(b)$ to the left side, and dividing by n, we obtain (3.4). ∎

Now we establish error estimate for generalized weighted trapezoidal formula.

Theorem 3.2. *Let $f : [a,b] \to \mathbf{R}$ be a mapping such that $f^{(n-1)}$ is absolutely continuous on $[a,b]$ for some $n \geq 1$, $f^{(n)} \in L_p[a,b]$ for $1 \leq p \leq \infty$, $\frac{1}{p}+\frac{1}{q}=1$, and let $\omega : [a,b] \to [0,\infty)$ be probability density function. Define*

$$
\begin{aligned}
(3.8)\qquad T(a,b,\omega,n) &:= f(a)\Big[\frac{1}{2}-\Big(1-\frac{1}{n}\Big)\int_a^b \omega(t)\frac{t-a}{b-a}\mathrm{d}t\Big] \\
&+ f(b)\Big[\frac{1}{2}+\Big(1-\frac{1}{n}\Big)\int_a^b \omega(t)\frac{t-b}{b-a}\mathrm{d}t\Big] \\
&- \frac{1}{n}\Bigg[\int_a^b \omega(t)f(t)\mathrm{d}t+\sum_{k=2}^{n-1}\int_a^b \omega(t)F_k(t)\mathrm{d}t \\
&- \frac{1}{2}\sum_{k=2}^{n-1}\frac{n-k}{k!}\Big[f^{(k-1)}(a)(b-a)^{k-1}+f^{(k-1)}(b)(a-b)^{k-1}\Big]\Bigg],
\end{aligned}
$$

then

$$
(3.9)\qquad T(a,b,\omega,n)=\frac{n-1}{2n!(b-a)}\int_a^b\Big[\int_a^b (2W(t)-1)k(y,t)(t-y)^{n-2}\mathrm{d}t\Big]f^{(n)}(y)\mathrm{d}y.
$$

Further,

$$
(3.10)\quad |T(a,b,\omega,n)| \leq \frac{n-1}{2n!(b-a)}\Big[\int_a^b\Big|\int_a^b (2W(t)-1)k(y,t)(t-y)^{n-2}\mathrm{d}t\Big|^q\mathrm{d}y\Big]^{\frac{1}{q}}\|f^{(n)}\|_p,
$$

for $p>1$, and

$$
(3.11)\qquad |T(a,b,\omega,n)| \leq \frac{n-1}{2n!(b-a)}\Big\|\int_a^b (2W(t)-1)k(y,t)(t-y)^{n-2}\mathrm{d}t\Big\|_\infty \cdot \|f^{(n)}\|_1.
$$

The constant $\frac{n-1}{2n!(b-a)}\Big[\int_a^b\Big|\int_a^b(2W(t)-1)k(y,t)(t-y)^{n-2}\mathrm{d}t\Big|^q\mathrm{d}y\Big]^{\frac{1}{q}}$ is sharp for $1<p\leq\infty$, and the constant $\frac{n-1}{2n!(b-a)}\Big\|\int_a^b(2W(t)-1)k(y,t)(t-y)^{n-2}\mathrm{d}t\Big\|_\infty$ is the best possible for $p=1$.

The following lemma will be useful in the proof of Theorem 3.2.

Lemma 3.3. *Let f and ω be as in Theorem 3.2. The following identities hold:*

$$
(3.12)\quad (n-1)\int_a^b W(t)(t-y)^{n-2}k(y,t)\mathrm{d}t=(y-a)(b-y)^{n-1}-\int_a^b \omega(t)k(y,t)(t-y)^{n-1}\mathrm{d}t,
$$

and

(3.13)

$$
(n-1)\int_a^b (W(t)-1)(t-y)^{n-2}k(y,t)\mathrm{d}t=(y-b)(a-y)^{n-1}-\int_a^b \omega(t)k(y,t)(t-y)^{n-1}\mathrm{d}t.
$$

Proof. We use the partial integration formula for the integrals on the left side of the identities. ∎

Proof of Theorem 3.2. By the definition of $T(a,b,\omega,n)$ and identity (3.4) it follows that

$$\begin{aligned} T(a,b,\omega,n) &\\ = & \frac{1}{2n!(b-a)}\int_a^b [(a-y)^{n-1}(y-b)+(b-y)^{n-1}(y-a)]f^{(n)}(y)\mathrm{d}y \\ - & \frac{1}{n!(b-a)}\int_a^b \Big(\int_a^b \omega(t)(t-y)^{n-1}k(y,t)\mathrm{d}t\Big) f^{(n)}(y)\mathrm{d}y \\ = & \frac{1}{2n!(b-a)}\int_a^b \Big[(a-y)^{n-1}(y-b)+(b-y)^{n-1}(y-a) \\ - & 2\int_a^b \omega(t)(t-y)^{n-1}k(y,t)\mathrm{d}t\Big] f^{(n)}(y)\mathrm{d}y. \end{aligned} \tag{3.14}$$

Applying Lemma 3.3 twice we obtain

$$T(a,b,\omega,n) = \frac{n-1}{2n!(b-a)}\int_a^b \Big[\int_a^b (2W(t)-1)k(y,t)(t-y)^{n-2}\mathrm{d}t\Big] f^{(n)}(y)\mathrm{d}y.$$

Let's denote

$$C(y) = \frac{n-1}{2n!(b-a)}\int_a^b (2W(t)-1)k(y,t)(t-y)^{n-2}\mathrm{d}t.$$

By Hölder inequality we obtain

$$|T(a,b,\omega,n) \leq \Big(\int_a^b |C(y)|^q \mathrm{d}y\Big)^{\frac{1}{q}} \|f^{(n)}\|_p,$$

for $p > 1$, and

$$|T(a,b,\omega,n)| \leq \|C\|_\infty \cdot \|f^{(n)}\|_1.$$

For the proof of the sharpness of the constant $\Big(\int_a^b |C(y)|^q \mathrm{d}y\Big)^{\frac{1}{q}}$ we take the function f such that

$$f^{(n)}(y) = \mathrm{sgn}C(y)\cdot |C(y)|^{\frac{1}{1-p}},$$

for $1 < p < \infty$, and

$$f^{(n)}(y) = \mathrm{sgn}C(y),$$

for $p = \infty$. For $p = 1$ we shall prove that

$$\Big|\int_a^b C(y)f^{(n)}(y)\mathrm{d}y\Big| \leq \max_{y\in[a,b]} |C(y)| \Big(\int_a^b |f^{(n)}(y)|\mathrm{d}y\Big) \tag{3.15}$$

is the best possible inequality. Suppose that $|C(y_0)|$ is maximum of C on $[a,b]$. First, we assume that $C(y_0) > 0$. For ϵ small enough define $f_\epsilon(y)$ by

$$f_\epsilon(y) = \begin{cases} 0, & a \leq y \leq y_0 \\ \frac{1}{\epsilon n!}(y-y_0)^n, & y_0 \leq y \leq y_0+\epsilon \\ \frac{1}{n!}(y-y_0)^{n-1}, & y_0+\epsilon \leq y \leq b. \end{cases}$$

then for ϵ small enough

$$\Big|\int_a^b C(y)f^{(n)}(y)\mathrm{d}y\Big| = \Big|\int_{y_0}^{y_0+\epsilon} C(y)\frac{1}{\epsilon}\mathrm{d}y\Big| = \frac{1}{\epsilon}\int_{y_0}^{y_0+\epsilon} C(y)\mathrm{d}y.$$

Now, from inequality (3.15) we have

$$\frac{1}{\epsilon}\int_{y_0}^{y_0+\epsilon} C(y)\mathrm{d}y \le C(y_0)\int_{y_0}^{y_0+\epsilon}\frac{1}{\epsilon}\mathrm{d}y = C(y_0).$$

Since,

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_{y_0}^{y_0+\epsilon} C(y)\mathrm{d}y = C(y_0)$$

the statement follows. For the case $C(y_0) < 0$, we take

$$f_\epsilon(y) = \begin{cases} \frac{1}{n!}(y-y_0-\epsilon)^{n-1}, & a \le y \le y_0 \\ -\frac{1}{\epsilon n!}(y-y_0-\epsilon)^n, & y_0 \le y \le y_0+\epsilon \\ 0, & y_0+\epsilon \le y \le b \end{cases}$$

and the rest of the proof is the same as above. ∎

Corollary 3.4. *Let* $f : [a,b] \to \mathbf{R}$ *be a mapping such that* $f^{(n-1)}$ *is absolutely continuous on* $[a,b]$ *for some* $n \ge 1$*,* $f^{(n)} \in L_p[a,b]$ *for* $1 \le p \le \infty$*,* $\frac{1}{p}+\frac{1}{q}=1$*, and let* $\omega : [a,b] \to [0,\infty)$ *be probability density function. Then we have*

$$|T(a,b,\omega,n)| \le K(a,b,n,p)\|f^{(n)}\|_p, \tag{3.16}$$

where

$$K(a,b,n,p) = \begin{cases} \frac{(b-a)^{n-1+\frac{1}{q}}}{n!}B^{\frac{1}{q}}(q+1,(n-1)q+1) & \text{for} \quad p>1, q>1 \\ \frac{(b-a)^{n-\frac{1}{2}}}{\sqrt{2(2n+1)!n!}}\sqrt{2(2n-2)!+(n!)^2} & \text{for} \quad p=q=2 \\ \frac{(b-a)^n}{n(n+1)!} & \text{for} \quad p=\infty \\ \frac{\max_Y\{(b-Y)(Y-a)^{n-1}+(b-Y)^{n-1}(Y-a)\}}{2n!(b-a)} & \text{for} \quad p=1, \end{cases} \tag{3.17}$$

where $Y \in [a, \frac{a+b}{2}]$ *is solution of the polynomial equation*

$$(b-y)^{n-1}-(y-a)^{n-1}+(n-1)[(b-y)(y-a)^{n-2}-(y-a)(b-y)^{n-2}] = 0.$$

Proof. Let's suppose $1 < p < \infty$ (that implies $1 < q < \infty$). By Theorem 3.2.

$$|T(a,b,\omega,n)| \le \frac{n-1}{2n!(b-a)}\left[\int_a^b \left|\int_a^b (2W(t)-1)k(y,t)(t-y)^{n-2}\mathrm{d}t\right|^q \mathrm{d}y\right]^{\frac{1}{q}} \cdot \|f^{(n)}\|_p \tag{3.18}$$

Now, let us establish some upper bounds for the integral

$$\int_a^b \left|\int_a^b (2W(t)-1)k(y,t)(t-y)^{n-2}\mathrm{d}t\right|^q \mathrm{d}y.$$

Since $0 \le W(t) \le 1$, for every $t \in [a,b]$, so $|2W(t)-1| \le 1$ and we have

$$\begin{aligned}
& \int_a^b \Big| \int_a^b (2W(t)-1)k(y,t)(t-y)^{n-2}\mathrm{d}t \Big|^q \mathrm{d}y \\
\le\ & \int_a^b \Big(\int_a^b |k(y,t)| \cdot |t-y|^{n-2}\mathrm{d}t \Big)^q \mathrm{d}y \\
=\ & \int_a^b \Big(\int_a^y (b-y)\cdot(y-t)^{n-2}\mathrm{d}t + \int_y^b (y-a)\cdot(t-y)^{n-2}\mathrm{d}t \Big)^q \mathrm{d}y \\
=\ & \int_a^b \Big((b-y)\frac{(y-a)^{n-1}}{n-1} + (y-a)\frac{(b-y)^{n-1}}{n-1} \Big)^q \mathrm{d}y \\
=\ & \frac{1}{(n-1)^q} \int_a^b \Big((b-y)(y-a)^{n-1} + (y-a)(b-y)^{n-1} \Big)^q \mathrm{d}y \\
& \{\text{using inequality} \quad |u+v|^q \le 2^{q-1}(|u|^q + |v|^q)\} \\
\le\ & \frac{2^{q-1}}{(n-1)^q} \Big(\int_a^b (b-y)^q (y-a)^{(n-1)q} \mathrm{d}y + \int_a^b (y-a)^q (b-y)^{(n-1)q} \mathrm{d}y \Big) \\
=\ & \frac{2^q}{(n-1)^q} \cdot (b-a)^{nq+1} \cdot B(q+1, (n-1)q+1),
\end{aligned}$$

where the last identity is obtained by the substitution $y = a + u(b-a)$ and the symmetry of the Beta function,

$$B(u,v) = B(v,u) = \int_0^1 t^{u-1}(1-t)^{v-1}\mathrm{d}t, \quad u,v > 0.$$

Further,

$$\left[\int_a^b \Big| \int_a^b (2W(t)-1)k(y,t)(t-y)^{n-2}\mathrm{d}t \Big|^q \mathrm{d}y \right]^{\frac{1}{q}} \le \frac{2}{n-1}(b-a)^{n+\frac{1}{q}} B^{\frac{1}{q}}(q+1, (n-1)q+1),$$

and from (3.18) we deduce

$$|T(a,b,\omega,n)| \le \frac{(b-a)^{n-1+\frac{1}{q}}}{n!} B^{\frac{1}{q}}(q+1,(n-1)q+1) \cdot \|f^{(n)}\|_p, \tag{3.19}$$

and the first part of the theorem is proved. For $p = q = 2$ we have by Theorem 3.2

$$|T(a,b,\omega,n)| \le \frac{n-1}{2n!(b-a)} \left[\int_a^b \Big| \int_a^b (2W(t)-1)k(y,t)(t-y)^{n-2}\mathrm{d}t \Big|^2 \mathrm{d}y \right]^{\frac{1}{2}} \cdot \|f^{(n)}\|_2, \tag{3.20}$$

and same as in previous case, $|2W(t)-1| \le 1$, so we calculate

$$\begin{aligned}
& \int_a^b \Big[\int_a^b |k(y,t)| \cdot |t-y|^{n-2} \mathrm{d}t\Big]^2 \mathrm{d}y \\
= & \int_a^b \Big[\int_a^y (b-y)(y-t)^{n-2}\mathrm{d}t + \int_y^b (y-a)(t-y)^{n-2}\mathrm{d}t\Big]^2 \mathrm{d}y \\
= & \int_a^b \Big[(b-y)\frac{(y-a)^{n-1}}{n-1} + (y-a)\frac{(b-y)^{n-1}}{n-1}\Big]^2 \mathrm{d}y \\
= & \frac{1}{(n-1)^2}\int_a^b \Big[(b-y)^2(y-a)^{2n-2} + 2(b-y)^n(y-a)^n + (y-a)^2(b-y)^{2n-2}\Big]\mathrm{d}y \\
= & \frac{1}{(n-1)^2}\Big[(b-a)^{2n+1}\Big(2B(3,2n-1)+2B(n+1,n+1)\Big)\Big] \\
= & \frac{2(b-a)^{2n+1}}{(n-1)^2(2n+1)!} \cdot [2(2n-2)! + (n!)^2].
\end{aligned}$$

From (3.20) we have

$$|T(a,b,\omega,n)| \le \frac{(b-a)^{n-\frac{1}{2}}}{\sqrt{2(2n+1)!}n!}\sqrt{2(2n-2)!+(n!)^2} \cdot \|f^{(n)}\|_2.$$

For $p=\infty$ and $q=1$ we have by Theorem 3.2

$$(3.21)\quad |T(a,b,\omega,n)| \le \frac{n-1}{2n!(b-a)}\left[\int_a^b \Big|\int_a^b (2W(t)-1)k(y,t)(t-y)^{n-2}\mathrm{d}t\Big|\mathrm{d}y\right] \cdot \|f^{(n)}\|_\infty,$$

and we calculate

$$\begin{aligned}
& \int_a^b \Big|\int_a^b (2W(t)-1)k(y,t)(t-y)^{n-2}\mathrm{d}t\Big|\mathrm{d}y \\
\le & \int_a^b \Big(\int_a^b |k(y,t)| \cdot |t-y|^{n-2}\mathrm{d}t\Big)\mathrm{d}y \\
= & \int_a^b \Big(\int_a^y (b-y)(y-t)^{n-2}\mathrm{d}t + \int_a^b (y-a)(t-y)^{n-2}\mathrm{d}t\Big)\mathrm{d}y \\
= & \frac{1}{n-1}\int_a^b [(b-y)(y-a)^{n-1} + (y-a)(b-y)^{n-1}]\mathrm{d}y \\
= & \frac{2(b-a)^{n+1}}{(n-1)n(n+1)}.
\end{aligned}$$

So, from (3.21) we obtain

$$|T(a,b,\omega,n)| \le \frac{(b-a)^n}{n(n+1)!} \cdot \|f^{(n)}\|_\infty.$$

For the last case, $p=1$ and $q=\infty$, we also apply Theorem 3.2

$$(3.22)\qquad |T(a,b,\omega,n)| \le \frac{n-1}{2n!(b-a)} \sup_{y\in[a,b]} \Big|\int_a^b (2W(t)-1)k(y,t)(t-y)^{n-2}\mathrm{d}t\Big| \|f^{(n)}\|_1.$$

Now we establish some upper bounds for

$$\sup_{y\in[a,b]} \Big|\int_a^b (2W(t)-1)k(y,t)(t-y)^{n-2}\mathrm{d}t\Big|.$$

We have

$$\begin{aligned}
&\sup_{y\in[a,b]}\Big|\int_a^b (2W(t)-1)k(y,t)(t-y)^{n-2}\mathrm{d}t\Big| \\
\leq\ &\sup_{y\in[a,b]}\int_a^b |k(y,t)||t-y|^{n-2}\mathrm{d}t \\
=\ &\sup_{y\in[a,b]}\Big[\int_a^y (b-y)(y-t)^{n-2}\mathrm{d}t+\int_y^b (y-a)(b-y)^{n-2}\mathrm{d}t\Big] \\
=\ &\frac{1}{n-1}\sup_{y\in[a,b]}[(b-y)(y-a)^{n-1}+(y-a)(b-y)^{n-1}] \\
=\ &\frac{1}{n-1}\max_{Y}[(b-y)(y-a)^{n-1}+(y-a)(b-y)^{n-1}]
\end{aligned}$$

where last assertion follows from the fact that function

$$y\mapsto[(b-y)(y-a)^{n-1}+(y-a)(b-y)^{n-1}]$$

is symmetric with respect to $\frac{a+b}{2}$ so the local extrema for that function are $Y\in[a,\frac{a+b}{2}]$ which are solution of the polynomial equation

$$(b-y)^{n-1}-(y-a)^{n-1}+(n-1)[(b-y)(y-a)^{n-2}-(y-a)(b-y)^{n-2}]=0.$$

Finally,

$$|T(a,b,\omega,n)|\leq\frac{\max_Y\{(b-Y)(Y-a)^{n-1}+(b-Y)^{n-1}(Y-a)\}}{2n!(b-a)}\|f^{(n)}\|_1 \tag{3.23}$$

which ends the proof of the theorem. ∎

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