The Australian Journal of Mathematical Analysis and Applications

http://ajmaa.org

Volume 4, Issue 1, Article 5, pp. 1-13, 2006

EXISTENCE OF SOLUTIONS FOR NEUTRAL STOCHASTIC FUNCTIONAL DIFFERENTIAL SYSTEMS WITH INFINITE DELAY IN ABSTRACT SPACE

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Received 15 November, 2005; accepted 21 September, 2006; published 29 January, 2006.

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ABSTRACT. In this paper we prove existence results for semilinear stochastic neutral functional differential systems with unbounded delay in abstract space. Our theory makes use of analytic semigroups and fractional power of closed operators and Sadovskii fixed point theorem.

Key words and phrases: Abstract neutral functional differential systems, Existence of solutions, Unbounded delay, Analytic semigroup, Sadovskii fixed point theorem, Hilbert space.

2000 *Mathematics Subject Classification.* Primary 34K40, 34K50. Secondary 34F05, 60H10.

ISSN (electronic): 1449-5910

*The work was supported by UGC, New Delhi under SAP(DRS) sanctioned No. F.510/6/DRS/2004(SAP-1) for the period 1^{st} April 2004 to 31^{st} March 2009.

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1. INTRODUCTION

Random differential and integral equations play an important role in characterizing many social, physical, biological and engineering problems. Neutral differential equations arises in many areas of applied mathematics and such equations have received attention in recent years. The existence, uniqueness, stability, invariant measures and other qualitative behaviors of solutions to stochastic differential equations have been extensively investigated by many authors (see for example [7, 8, 9, 12]). Semigroup theory gives a unified treatment of a wide class of stochastic parabolic, hyperbolic and functional differential equations, and much effort has been devoted to the study of existence results for such evolution equations.

In this paper, we are interested to study the existence of solutions of the following nonlinear neutral stochastic functional differential equation in a Hilbert space,

$$\begin{aligned} d[x(t)+F(t,x_t)] &= Ax(t)dt + G(t,x_t)dw(t), \;\; t\in J:=[0,b], \\ x(t) &= \phi(t)\in L_2(\Omega,\mathfrak{B}), \quad \text{for a .e} \quad t\in J_0:=(-\infty,0], \end{aligned} \tag{1.1}$$

where A is the infinitesimal generator of an analytic semigroup of bounded linear operators $T(t), t\geq 0$, on a separable Hilbert space H with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$. Let K be another separable Hilbert space with inner product $(\cdot,\cdot)_K$ and norm $\|\cdot\|_K$. Suppose $\{w(t)\}_{t\geq 0}$ is a given K–valued Brownian motion or Wiener process with a finite trace nuclear covariance operator $Q\geq 0$. We are also employing the same notation $\|\cdot\|$ for the norm $L(K,H)$, where $L(K,H)$ denotes the space of all bounded linear operators from K into H. The histories x_t belongs to some abstract phase space $\mathfrak{B}$ defined axiomatically (see Section 2); $F: J\times\mathfrak{B}\to H$ and $G: J\times\mathfrak{B}\to L_Q(K,H)$($L_Q(K,H)$ denotes the space of all Q-Hilbert-Schmidt operators from K into H which is going to be defined below) are the measurable mappings in $H-$norm and $L_Q(K,H)-$norm respectively.

This paper is organized as follows. In Section 2, we recall some necessary preliminaries. In Section 3 we prove the existence of a mild solution. The existence of a strong solution is proved in section 4. Finally in Section 5, an example is presented which illustrates the main theorem.

2. PRELIMINARIES

For more details on the material of this section see [2], [3] and the references therein. Throughout the paper, $(H,\|\cdot\|)$ and $(K,\|\cdot\|_K)$ denote real separable Hilbert spaces.

Let $(\Omega,\mathfrak{F},P)$ be a complete probability space furnished with complete family of right continuous increasing sub σ-algebras $\{\mathfrak{F}_t, t\in J\}$ satisfying $\mathfrak{F}_t\subset\mathfrak{F}$. An *$H$-valued random variable* is an $\mathfrak{F}$-measurable function $x(t):\Omega\to H$ and a collection of random variables $S=\{x(t,w):\Omega\to H|t\in J\}$ is called a *stochastic process*. Usually we suppress the dependence on $w\in\Omega$ and write $x(t)$ instead of $x(t,w)$ and $x(t): J\to H$ in the place of S. Let $\beta_n(t)(n=1,2,\ldots)$ be a sequence of real-valued one-dimensional standard Brownian motions mutually independent over $(\Omega,\mathfrak{F},P)$. Set

$$w(t)=\sum_{n=1}^{\infty}\sqrt{\lambda_n}\beta_n(t)\zeta_n,\; t\geq 0,$$

where $\lambda_n\geq 0$, (n=1, 2, ...) are nonnegative real numbers and $\{\zeta_n\}$ (n=1, 2, ...) is complete orthonormal basis in K. Let $Q\in L(K,K)$ be an operator defined by $Q\zeta_n=\lambda_n\zeta_n$ with finite $Tr(Q)=\sum_{n=1}^{\infty}\lambda_n<\infty$, (Tr denotes the trace of the operator). Then the above K-valued stochastic process $w(t)$ is called a Q-Wiener process. We assume that $\mathfrak{F}_t=\sigma(w(s):0\leq s\leq t)$

is the σ-algebra generated by w and $\mathfrak{F}_T = \mathfrak{F}$. Let $\varphi \in L(K, H)$ and define

$$\|\varphi\|_Q^2 = Tr(\varphi Q \varphi^*) = \sum_{n=1}^{\infty} \|\sqrt{\lambda_n}\varphi\zeta_n\|^2.$$

If $\|\varphi\|_Q < \infty$, then φ is called a Q-Hilbert-Schmidt operator. Let $L_Q(K, H)$ denote the space of all Q-Hilbert-Schmidt operators $\varphi : K \to H$. The completion $L_Q(K, H)$ of $L(K, H)$ with respect to the topology induced by the norm $\|\cdot\|_Q$ where $\|\varphi\|_Q = \langle\langle\varphi, \varphi\rangle\rangle^{1/2}$ is a Hilbert space with the above norm topology.

We suppose that $0 \in \rho(A)$ and that the semigroup $T(\cdot)$ is uniformly bounded, that is to say, $\|T(t)\| \leq \bar{M}_1$, for some constant $\bar{M}_1 \geq 1$ and every $t \geq 0$. For $0 < \alpha \leq 1$, it is possible to define the fractional power operator $(-A)^\alpha$, as a closed linear operator on its domain $D\big((-A)^\alpha\big)$. Furthermore, the subspace $D\big((-A)^\alpha\big)$ is dense in H and the expression

$$\|x\|_\alpha = \|(-A)^\alpha x\|, \quad x \in D\big((-A)^\alpha\big),$$

defines a norm on $D\big((-A)^\alpha\big)$. Hereafter we represent by H_α the space $D\big((-A)^\alpha\big)$ endowed with the norm $\|\cdot\|_\alpha$. Then the following properties are well known ([10]).

Lemma 2.1. *Suppose that the preceding conditions are satisfied.*

(a) *Let* $0 < \alpha \leq 1$. *Then* H_α *is a Banach space.*
(b) *If* $0 < \beta \leq \alpha$ *then* $H_\alpha \hookrightarrow H_\beta$, *the imbedding is continuous.*
(c) *For every* $0 < \alpha \leq 1$, *there exists a positive constant* M_α *such that*

$$\|(-A)^\alpha T(t)\| \leq \frac{M_\alpha}{t^\alpha}, \quad 0 < t \leq b. \tag{2.1}$$

To study the system (1.1), we assume that the histories $x_t : J_0 \to H$, $x_t(\theta) = \{x(t+\theta)(w) : \theta \in (-\infty, 0]\}$ belong to some abstract phase space $\mathfrak{B}$, which is defined axiomatically. In this work, we will employ an axiomatic definition of the phase space $\mathfrak{B}$ introduced by Hale and Kato [4]. Thus, the space $\mathfrak{B}$ will be a linear space of $\mathfrak{F}_0$-measurable functions mapping from J_0 into H, endowed with a seminorm $\|\cdot\|_{\mathfrak{B}}$. We will assume that $\mathfrak{B}$ satisfies the following axioms:

(ai) If $x : (-\infty, b) \to H,\ b > 0$, is continuous on $[0, b)$ and x_0 in $\mathfrak{B}$, then for every $t \in [0, b)$ the following conditions hold:
1. x_t is in $\mathfrak{B}$,
2. $\|x(t)\| \leq L\|x_t\|_{\mathfrak{B}}$,
3. $\|x_t\|_{\mathfrak{B}} \leq K(t)\sup\{\|x(s)\| : 0 \leq s \leq t\} + N(t)\|x_0\|_{\mathfrak{B}}$, where $L > 0$ is a constant; $K, N : [0, \infty) \to [0, \infty)$, K is continuous, N is locally bounded and L, K, N are independent of $x(\cdot)$.

(aii) For the function $x(\cdot)$ in (ai), x_t is a $\mathfrak{B}$-valued continuous function on $[0, b)$.
(aiii) The space $\mathfrak{B}$ is complete.

The collection of all strongly-measurable, square-integrable H-valued random variables, denoted by $L_2(\Omega, \mathfrak{F}, P; H) \equiv L_2(\Omega; H)$, is a Banach space equipped with norm $\|x(\cdot)\|_{L_2} = \big(E\|x(\cdot; w)\|_H^2\big)^{\frac{1}{2}}$, where the expectation, E is defined by $E(h) = \int_\Omega h(w)dP$. Let $J_1 = (-\infty, b]$ and $C(J_1, L_2(\Omega; H))$ be the Banach space of all continuous maps from J_1 into $L_2(\Omega; H)$ satisfying the condition $\sup_{t\in J_1} E\|x(t)\|^2 < \infty$.

Let Z be the closed subspace of all continuous process x that belong to the space $C\big(J_1, L_2(\Omega; H)\big)$ consisting of $\mathfrak{F}_t$-adapted measurable processes such that the $\mathfrak{F}_0$-adapted processes $\phi \in L_2(\Omega; \mathfrak{B})$. Let $\|\cdot\|_Z$ be a seminorm in Z defined by

$$\|x\|_Z = \big(\sup_{t\in J}\|x_t\|_{\mathfrak{B}}^2\big)^{\frac{1}{2}}$$

where

$$\|x_t\|_{\mathfrak{B}} \leq \bar{N}E\|\phi\|_{\mathfrak{B}} + \bar{K}\sup\{E\|x(s)\| : 0 \leq s \leq b\},$$

$\bar{N} = \sup_{t\in J}\{N(t)\}, \bar{K} = \sup_{t\in J}\{K(t)\}$. It is easy to verify that Z furnished with the norm topology as defined above, is a Banach space.

The consideration of this paper is based in the following fixed point theorem ([11]).

Theorem 2.2. *(Sadovskii's fixed point theorem). Let Φ be a condensing operator on a Banach space H, that is, Φ is continuous and takes bounded sets into bounded sets, and $\alpha(\Phi(B)) \leq \alpha(B)$ for every bounded set B of H with $\alpha(B) > 0$. If $\Phi(\Omega) \subset \Omega$ for a convex, closed and bounded set Ω of H, then Φ has a fixed point in H (where $\alpha(\cdot)$ denotes Kuratowski's measure of non-compactness).*

3. MAIN RESULT

Before stating and proving our main result, we give first the definition of the mild solution.

Definition 3.1. An $\mathfrak{F}_t$-adapted stochastic process $x(t) : J_1 \to H$ is a mild solution of the abstract Cauchy problem (1.1) if $x_0 = \phi \in L^2(\Omega, \mathfrak{B})$ on J_0 satisfying $\|\phi\|^2_{\mathfrak{B}} < \infty$; the restriction of $x(\cdot)$ to the interval $[0, b)$ is continuous stochastic processes, for each $s \in [0, t)$ the function $AT(t-s)f(s, x_s)$ is integrable and the following integral equation is verified :

$$\begin{aligned} x(t) &= T(t)[\phi(0) + F(0,\phi)] - F(t, x_t) - \int_0^t AT(t-s)f(s, x_s)ds \\ &+ \int_0^t T(t-s)G(s, x_s)dw(s), \text{ for a. e. } t \in J. \end{aligned} \tag{3.1}$$

Theorem 3.1. *Assume that:*

(H1) *the semigroup $T(t)$ is compact for $t > 0$, and there exists $M_T \geq 1$ such that*

$$\|T(t)\| \leq M_T, \quad \text{for all } t \geq 0;$$

(H2) *$F : J \times \mathfrak{B} \to H$ is a continuous function, and there exists a constant $\beta \in (0, 1)$ and $M_F, \overline{M}_F > 0$ such that the function F is H_β-valued, and satisfies the Lipschitz condition:*

$$\|(-A)^\beta F(s_1, \phi_1) - (-A)^\beta F(s_2, \phi_2)\| \leq M_F(|s_1 - s_2| + \|\phi_1 - \phi_2\|_{\mathfrak{B}}),$$

for $0 \leq s_1, s_2 \leq b, \phi_1, \phi_2 \in \mathfrak{B}$, and the inequality

$$\|(-A)^\beta F(t, \phi)\| \leq \overline{M}_F(||\phi||_{\mathfrak{B}} + 1) \tag{3.2}$$

holds for $t \in J, \phi \in \mathfrak{B}$;

(H3) *The function $G : J \times \mathfrak{B} \to L(K, H)$ satisfies the following conditions :*

(i) for each $t \in J$, the function $G(t, \cdot) : \mathfrak{B} \to L(K, H)$ is continuous and for each $\phi \in \mathfrak{B}$ the function $G(\cdot, \phi) : J \to L(K, H)$ is $\mathfrak{F}_t$-measurable;

(ii) for each positive number q, there is a positive function $h_q \in L^1(J)$ such that

$$\sup_{\|\phi\|^2 \leq q} E\|G(t, \phi)\|^2_Q \leq h_q(t) \quad \textit{and} \quad \liminf_{q\to\infty} \frac{1}{q}\int_0^b h_q(s)ds = \gamma < \infty.$$

Then the Cauchy problem (1.1) has a mild solution on J provided that

$$L_0 := (2M_F)^2 \left(M_0^2 + \frac{(M_{1-\beta}b^\beta)^2}{2\beta - 1}\right) < 1, \tag{3.3}$$

$$16\left[(2M_0\overline{M}_F)^2 + \frac{(2M_{1-\beta}\overline{M}_F b^\beta)^2}{2\beta - 1} + Tr(Q)M_T^2\gamma\right] < 1, \tag{3.4}$$

where $M_0 = \|(-A)^{-\beta}\|$ *and* $M_{1-\beta}$ *is defined in (2.1).*

Proof. Let $\mathfrak{B}_b$ be the space of all functions $x : (-\infty, b] \to H$ such that $x_0 \in \mathfrak{B}$ and the restriction $x : J \to H$ is continuous. Let $\|\cdot\|_b$ be the seminorm in $\mathfrak{B}_b$ defined by

$$\|x\|_b = \|x_0\|_{\mathfrak{B}} + \sup\{\|x(s)\| : 0 \le s \le b\}, \ x \in \mathfrak{B}_b.$$

Let $Z_b = C\big(J_1, L_2(\Omega; \mathfrak{B}_b)\big)$. Consider the map $\Phi : Z_b \to Z_b$ defined by Φx, the set of $h \in Z_b$ such that

$$h(t) = \begin{cases} \phi(t), & \text{if} \quad t \in J_0, \\ T(t)[\phi(0) + F(0,\phi)] - F(t, x_t) - \int_0^t AT(t-s)F(s, x_s)ds & \\ + \int_0^t T(t-s)G(s, x_s)dw(s), & \text{for a. e. } t \in J. \end{cases}$$

We shall show that the operator Φ has a fixed point, which then is a solution of the system (1.1).

For $\phi \in Z$, let $y(\cdot) : (-\infty, b) \to Z_b$ be the function defined by

$$y(t) = \begin{cases} \phi(t), & \text{if } t \in (-\infty, 0], \\ T(t)\phi(0), & \text{if } t \in J. \end{cases}$$

Set $x(t) = z(t) + y(t), -\infty < t \le b$. It is clear that x satisfies (3.1) if and only if z satisfies $z_0 = 0$ and

$$\begin{aligned} z(t) &= T(t)F(0,\phi) - F(t, z_t + y_t) - \int_0^t AT(t-s)F(s, z_s + y_s)ds \\ &\quad + \int_0^t T(t-s)G(s, z_s + y_s)dw(s), \ \ t \in J. \end{aligned}$$

Let $\mathcal{B}_b^0 = \{z \in \mathfrak{B}_b : z_0 = 0 \in \mathfrak{B}\}$. For any $z \in \mathfrak{B}_b^0$ we have

$$\|z\|_b = \|z_0\|_{\mathfrak{B}} + \sup\{\|z(s)\| : 0 \le s \le b\} = \sup\{\|z(s)\| : 0 \le s \le b\}.$$

Thus if $Z_b^0 = C\big(J_1, L_2(\Omega; \mathfrak{B}_b^0)\big)$, then $(Z_b^0, \|\cdot\|_b)$ is a Banach space. Set

$$B_q = \{z \in Z_b^0 : ||z||_b^2 \le q\} \ \text{ for some } q \ge 0;$$

then, $B_q \subseteq Z_b^0$ is uniformly bounded and, for $z \in B_q$, we remark that

$$\begin{aligned} \|z_t + y_t\|_{\mathfrak{B}}^2 &\le 4\big(\|z_t\|_{\mathfrak{B}}^2 + \|y_t\|_{\mathfrak{B}}^2\big) \\ &\le 16\Big(K(t)\sup_{0 \le s \le t} E\|z(s)\|^2 + N(t)E\|z_0\|_{\mathfrak{B}}^2 \\ &\quad + K(t)\sup_{0 \le s \le t} E\|y(s)\|^2 + N(t)E\|y_0\|_{\mathfrak{B}}^2\Big) \\ &\le 16\Big(\bar{K}^2(q + M_T^2\|\phi(0)\|_{\mathfrak{B}}^2) + \bar{N}^2\|\phi\|_{\mathfrak{B}}^2\Big) := k. \end{aligned} \tag{3.5}$$

Let the operator $\mathcal{Q} : Z_b^0 \to Z_b^0$ be defined by $\mathcal{Q}z$, the set of $\bar{h} \in Z_b^0$ such that

$$\bar{h}(t) = \begin{cases} 0, & t \in J_0, \\ T(t)F(0,\phi) - F(t, z_t + y_t) - \int_0^t AT(t-s)F(s, z_s + y_s)ds & \\ + \int_0^t T(t-s)G(s, z_s + y_s)dw(s), & t \in J. \end{cases}$$

Obviously the operator Φ has a fixed point is equivalent to $\mathcal{Q}$ has one, so it turns out to prove that $\mathcal{Q}$ has a fixed point. For each positive number k, let

$$B_k = \{z \in Z_b^0 : z(0) = 0, ||z||_b^2 \le k, \ 0 \le t \le b\} \ \text{ for some } k \ge 0;$$

then for each k, $B_k \subseteq Z_b^0$ is clearly a bounded closed convex set. In addition to the familiar Young, Hölder and Minkowskii inequalities, the inequality of the form $\left(\sum_{i=1}^n a_i\right)^m \leq n^m \sum_{i=1}^n a_i^m$, where a_i are nonnegative constants $(i = 1, 2, \dots, n)$ and $m, n \in \mathbb{N}$ is helpful to establishing various estimates. From (2.1) and (3.2) together with the Hölder inequality, yields the following relation:

$$\begin{aligned} \|(-A)T(t-s)F(s, z_s + y_s)ds\|^2 &= \| \textstyle\int_0^t (-A)^{1-\beta} T(t-s)(-A)^\beta F(s, v(s))ds\|^2 \\ &\leq 4 \textstyle\int_0^t \frac{M_{1-\beta}^2}{(t-s)^{2(1-\beta)}} \overline{M}_F{}^2 (k+1) ds. \end{aligned} \tag{3.6}$$

It follows that $(-A)T(t-s)F(s, v(s))ds$ is integrable on J, so $\mathcal{Q}$ is well defined on B_k.

Similarly from $(H3)(ii)$ together with the Ito's formula, a computation can be performed to obtain the following:

$$\begin{aligned} E\| \textstyle\int_0^t T(t-s)G(s, z_s + y_s))dw(s)\|^2 &\leq Tr(Q) M_T^2 \textstyle\int_0^t E\|G(s, u(s))\|_Q^2 ds \\ &\leq Tr(Q) M_T^2 \textstyle\int_0^t h_k(s) ds. \end{aligned} \tag{3.7}$$

We claim that there exists a positive number k such that $\mathcal{Q}B_k \subseteq B_k$. If it is not true, then for each positive number k, there is a function $z^{(k)}(\cdot) \in B_k$, but $\mathcal{Q}z^{(k)} \notin B_k$, but $\|\mathcal{Q}z^{(k)}(t)\|^2 > k$ for some $t \in J$. However from the equations (3.2), (3.6) and (3.7), we have

$$\begin{aligned} k &\leq E\|\mathcal{Q}z^{(k)}(t)\|^2 \\ &= E\Big\| T(t)F(0,\phi) - F(t, z_t^{(k)} + y_t) - \int_0^t AT(t-s)F(s, z_s^{(k)} + y_s)ds \\ &\quad + \int_0^t T(t-s)G(s, z_s^{(k)} + y_s)dw(s)\Big\|^2 \\ &\leq 16\Big\{ M_T^2 E\|F(0,\phi)\|^2 + E\|F(t, z_t^{(k)} + y_t)\|^2 \\ &\quad + b\int_0^t \|(-A)^{1-\beta}T(t-s)\|^2 E\|(-A)^\beta F(s, z_s^{(k)} + y_s)\|^2 ds \\ &\quad + Tr(Q)M_T^2 \int_0^t E\|G(s, z_s^{(k)} + y_s)\|_Q^2 ds \Big\}. \end{aligned}$$

Then

$$\begin{aligned} k \leq \|\mathcal{Q}z^{(k)}(t)\|_Z^2 &\leq 16\Big\{ (2M_T\overline{M}_F)^2(1 + \|\phi\|_Z^2) + \big(2M_0\overline{M}_F\big)^2(k+1) \\ &\quad + \frac{1}{2\beta - 1}(2M_{1-\beta}\overline{M}_F b^\beta)^2(k+1) + Tr(Q)M_T^2 k \frac{1}{k}\int_0^b h_k(s)ds \end{aligned}$$

$$\begin{aligned} &\leq M^* + 16\Big\{ \big(2M_0\overline{M}_F\big)^2 k + \tfrac{1}{2\beta-1}(2M_{1-\beta}\overline{M}_F b^\beta)^2 k \\ &\quad + Tr(Q)M_T^2 k \tfrac{1}{k} \textstyle\int_0^b h_k(s)ds \Big\}, \end{aligned} \tag{3.8}$$

where

$$M^* = 16\Big\{ (2M_T\overline{M}_F)^2(1 + \|\phi\|_Z^2) + \big(2M_0\overline{M}_F\big)^2 + \frac{1}{2\beta - 1}(2M_{1-\beta}\overline{M}_F b^\beta)^2 \Big\}.$$

Dividing on both sides of the equation (3.8) by k and taking the lower limit as $k \to \infty$, we get

$$16\left[(2M_0\overline{M}_F)^2 + \frac{(2M_{1-\beta}\overline{M}_F b^{\beta})^2}{2\beta-1} + Tr(Q)M_T^2\gamma\right] \geq 1.$$

This contradicts (3.4) and hence for some positive number k, $\mathcal{Q}B_k \subseteq B_k$.

Next we will show that the operator $\mathcal{Q}$ has a fixed point on B_k, which implies problem (1.1) has a mild solution. To this end, we decompose $\mathcal{Q}$ as $\mathcal{Q} = \mathcal{Q}_1 + \mathcal{Q}_2$, where the operators $\mathcal{Q}_1, \mathcal{Q}_2$ are defined on B_k, respectively, by

$$(\mathcal{Q}_1 z)(t) = T(t)F(0,\phi) - F(t, z_t + y_t) - \int_0^t AT(t-s)F(s, z_s + y_s)ds, \quad t \in J,$$

and

$$(\mathcal{Q}_2 z)(t) = \int_0^t T(t-s)G(s, z_s + x_s)ds \quad t \in J.$$

We will show that $\mathcal{Q}_1$ is a contraction, while $\mathcal{Q}_2$ is a compact operator.

To prove that $\mathcal{Q}_1$ is a contraction we take $z^{(1)}, z^{(2)} \in B_k$. Then for each $t \in J$, by condition $(H2)$ and equation (3.3), we have

$$\begin{aligned}
E\|(\mathcal{Q}_1 z^{(1)})(t) \;-\;& (\mathcal{Q}_1 z^{(2)}(t)\|^2 \leq 4E\|F(t, z_t^{(1)} + y_t) - F(t, z_t^{(2)} + y_t)\|^2 \\
&+4b\int_0^t E\|AT(t-s)\big[F(s, z_s^{(1)} + y_s) - F(s, z_s^{(2)} + y_s)\big]\|^2 ds \\
\leq\;& (2M_0M_F)^2 E\|z_t^{(1)} - z_t^{(2)}\|_{\mathfrak{B}}^2 + 4b\int_0^t M_F^2 \frac{M_{1-\beta}^2}{(t-s)^{2(1-\beta)}} E\|z_s^{(1)} - z_s^{(2)}\|_{\mathfrak{B}}^2 ds \\
\leq\;& (2M_F)^2\left(M_0^2 + \frac{(M_{1-\beta}b^{\beta})^2}{2\beta-1}\right)\sup_{0\leq s\leq b} E\|z^{(1)}(s) - z^{(2)}(s)\|^2 \\
=\;& L_0 \sup_{0\leq s\leq b} E\|z^{(1)}(s) - z^{(2)}(s)\|_{\mathfrak{B}}^2.
\end{aligned}$$

Thus,

$$\|\mathcal{Q}_1 z^{(1)} - \mathcal{Q}_1 z^{(2)}\|_Z^2 \leq L_0\|z^{(1)} - z^{(2)}\|_Z^2,$$

and so $\mathcal{Q}_1$ is a contraction, since $L_0 < 1$.

To prove that $\mathcal{Q}_2$ is compact, first we prove that $\mathcal{Q}_2$ is continuous on B_k. Let $\{z^{(n)}\} \subseteq B_k$ with $z^{(n)} \to z$ in B_k. Then for each $s \in J$, $z_s^{(n)} \to z_s$, by $(H3)(i)$, we have

$$G(s, z_s^{(n)} + y_s) \to G(s, z_s + y_s), n \to \infty.$$

Since

$$E\|G(s, z_s^{(n)} + y_s) - G(s, z_s + y_s)\|^2 \leq 2h_k(s),$$

by the dominated convergence theorem, we have

$$E\|\mathcal{Q}_2 z^{(n)} - \mathcal{Q}_2 z\|^2 = \sup_{0\leq t\leq b} E\left\|\int_0^t T(t-s)[G(s, z_s^{(n)} + y_s) - G(s, z_s + y_s)]dw(s)\right\|^2 \to 0,$$

as $n \to \infty$, that is, $\mathcal{Q}_2$ is continuous.

Next we prove that the family $\{\mathcal{Q}_2 z : z \in B_k\}$ is an equicontinuous family of functions. To do this, let $\epsilon > 0$ small, and $0 < t_1 < t_2$. Then we have

$$\begin{aligned} E\|(\mathcal{Q}_2 z)(t_1) \quad - \quad & (\mathcal{Q}_2 z)(t_2)\|^2 \\ \leq \quad & Tr(Q) \int_0^{t_1-\varepsilon} \|T(t_2 - s) - T(t_1 - s)\|^2 E\|G(s, z_s + x_s)\|^2 ds \\ & + Tr(Q) \int_{t_1-\varepsilon}^{t_1} \|T(t_2 - s) - T(t_1 - s)\|^2 E\|G(s, z_s + x_s)\|^2 ds \\ & + Tr(Q) \int_{t_1}^{t_2} \|T(t_2 - s)\|^2 E\|G(s, z_s + x_s)\|^2 ds. \end{aligned}$$

Therefore

$$\begin{aligned} \|(\mathcal{Q}_2 z)(t_1) - (\mathcal{Q}_2 z)(t_2)\|_Z^2 \quad \leq \quad & Tr(Q) \int_0^{t_1-\varepsilon} \|T(t_2 - s) - T(t_1 - s)\|^2 h_k(s) ds \\ & + Tr(Q) \int_{t_1-\varepsilon}^{t_1} \|T(t_2 - s) - T(t_1 - s)\|^2 h_k(s) ds \\ & + Tr(Q) \int_{t_1}^{t_2} \|T(t_2 - s)\|^2 h_k(s) ds. \end{aligned}$$

The right hand side tends to zero independently of $z \in B_k$ as $t_2 \to t_1$, with ε sufficiently small, since the compactness of $T(t)$ for $t > 0$ implies the continuity in the uniform operator topology. Hence, $\mathcal{Q}_2$ maps B_k into a equicontinuous family of functions.

It remains to prove that $V(t) = \{\mathcal{Q}_2 z)(t) : z \in B_k\}$ is relatively compact in H. Let $0 < t \leq b$ be fixed and $0 < \varepsilon < t$. For $z \in B_k$ we define

$$\begin{aligned} (\mathcal{Q}_2^\varepsilon z)(t) \quad = \quad & \int_0^{t-\varepsilon} T(t - s) G(s, z_s + x_s) dw(s) \\ = \quad & T(\varepsilon) \int_0^{t-\varepsilon} T(t - \varepsilon - s) G(s, z_s + x_s) dw(s). \end{aligned}$$

Then by the compactness of $T(t)$ $(t > 0)$, we obtain $V_\varepsilon(t) = \{(\mathcal{Q}_2^\varepsilon z)(t) : z \in B_k\}$ is relatively compact in H for every ε, $0 < \varepsilon < t$. Moreover, for every $z \in B_k$, we have

$$\|(\mathcal{Q}_2 z)(t) - (\mathcal{Q}_2^\varepsilon z)(t)\|_Z^2 \quad \leq \quad 4Tr(Q) M_T^2 \int_{t-\varepsilon}^{t} h_k(s) ds.$$

Therefore there are relatively compact sets arbitrary close to the set $V(t)$, hence the set $V(t)$ is also relatively compact in H.

Thus, by Arzelá-Ascoli theorem $\mathcal{Q}_2$ is a compact operator. These arguments enable us to conclude that $\mathcal{Q} = \mathcal{Q}_1 + \mathcal{Q}_2$ is a condensing map on B_k, and by the fixed point theorem of Sadovskii there exists a fixed point $z(\cdot)$ for $\mathcal{Q}$ on B_k. The proof is completed and we deduce that Φ has a fixed point and therefore the Cauchy problem (1.1) has a mild solution on J. ∎

4. EXISTENCE OF STRONG SOLUTION

Definition 4.1. An $\mathfrak{F}_t$-adapted stochastic process $x(\cdot) : J_1 \to H$ is said to be a strong solution of the Cauchy problem (1.1), if $x(t)$ and $F(t, x_t)$ are differentiable a. e. on $(0, b]$; $x'(t)$ and $F'(t, x_t)$ belong to $L^1([0, b); H)$; $x_0 = \phi$ and equation in (1.1) is satisfied for a. e. t in $[0, b)$.

Theorem 4.1. *Let $\phi \in D(A) \subset \mathfrak{B}$. Suppose the conditions $(H1)$ and $(H2)$ are satisfied with $F(J \times H) \subset D(A)$, and the function $(-A)F(0, \cdot) : H \to D(A)$ maps bounded sets into bounded sets. In addition assume that the following conditions:*

(H4) $G(\cdot, \cdot) : J \times D(A) \to L(K, H)$ *is Lipschitz continuous, that is, there exists a constant* M_G *such that*

$$\|G(s_1, \phi_1) - G(s_2, \phi_2)\|_Q \leq M_G(|s_1 - s_2| + \|\phi_1 - \phi_2\|_{\mathfrak{B}}),$$

for $0 \leq s_1, s_2 \leq b, \phi_1, \phi_2 \in D(A)$. *Moreover, thre is an* $\overline{M}_G > 0$ *such that*

$$\|G(t, \phi)\|_Q \leq \overline{M}_G(||\phi||_{\mathfrak{B}} + 1) \tag{4.1}$$

holds for any $(t, \phi) \in J \times D(A)$*;*

(H5) *There hold (3.3) and the inequality*

$$16\left[(2M_0\overline{M}_F)^2 + \frac{(2M_{1-\beta}\overline{M}_F b^\beta)^2}{2\beta - 1} + (2M_T\overline{M}_G)^2 Tr(Q)b\right] < 1.$$

Then the mild solution of Cauchy problem (1.1) is also a strong solution.

Proof. We prove this theorem by using the fixed point theorem of Sadovskii again. Denote Φx the set $h \in Z_b$ defined as in the proof of the Theorem 3.1. Define the set

$$S(\rho) = \{x \in Z_b : z(0) = \phi(0),\ \|x(t)\|_Z^2 \leq \rho, \|x(t) - x(s)\|_Z^2 \leq L^*|t - s|^2, t, s \in J\}$$

for some positive constants ρ and L^* large enough. Then $S(\rho)$ is a non-empty bounded, closed and convex subset of Z_b and the operator Φ defined in the proof of Theorem 3.1 is well defined on $S(\rho)$. We will show that Φ maps $S(\rho)$ into $S(\rho)$. Let $x \in S(\rho)$. Then we have

$$\begin{aligned}
& E\|(\Phi x)(t_2) - (\Phi x)(t_1)\|^2 \\
\leq\ & 16\Big\{\|T(t_2) - T(t_1)\|^2 E\|\phi(0) + F(0, \phi)\|^2 + E\|F(t_2, x_{t_2}) - F(t_1, x_{t_1})\|^2 \\
& + E\Big\|\int_0^{t_2} (-A)T(t_2 - s)F(s, x_s)ds - \int_0^{t_1} (-A)T(t_1 - s)F(s, x_s)ds\Big\|^2 \\
& + E\Big\|\int_0^{t_2} T(t_2 - s)G(s, x_s)dw(s) - \int_0^{t_1} T(t_1 - s)G(s, x_s)dw(s)\Big\|^2\Big\} \\
\leq\ & 16\Big\{\|T(t_2) - T(t_1)\|^2 E\|\phi(0) + F(0, \phi)\|^2 + E\|(-A)^{-\beta}[(-A)^\beta F(t_2, x_{t_2}) \\
& - (-A)^\beta F(t_1, x_{t_1})\|^2 + E\Big\|\int_0^{t_1} (-A)^{1-\beta}T(t_1 - s)[(-A)^\beta F(s + t_2 - t_1, x_{s+t_2-t_1}) \\
& - (-A)^\beta F(s, x_s)]ds + \int_0^{t_2-t_1} (-A)^{1-\beta}T(t_2 - s)[(-A)^\beta F(s, x_s)ds\Big\|^2 \\
& + E\Big\|\int_0^{t_1} T(t_1 - s)[G(s + t_2 - t_1, x_{s+t_2-t_1}) - G(s, x_s)]dw(s) \\
& + \int_0^{t_2-t_1} T(t_1 - s)G(s, x_s)dw(s)\Big\|^2\Big\}.
\end{aligned}$$

By conditions $(H1),(H2),(H4)$ and the boundedness of $(-A)F(0,\cdot)$ it yields that

$$\begin{aligned}
&\|(\Phi x)(t_2)-(\Phi x)(t_1)\|_Z^2\\
\leq\ & 16\Big\{M_T^2A\|\phi(0)+F(0,\phi)\|_Z^2|t_2-t_1|^2+\Big[(2M_0M_F)^2+\frac{(2M_{1-\beta}M_Fb^\beta)^2}{2\beta-1}\Big]\\
&\times[|t_2-t_1|^2+L^*|t_2-t_1|^2]+\frac{(M_{1-\beta}\overline{M}_F)^2}{2\beta-1}(\rho+1)|t_2^{2\beta}-t_1^{2\beta}|\\
&+Tr(Q)(2M_TM_G)^2[|t_2-t_1|^2+L^*|t_2-t_1|^2]+Tr(Q)\overline{M}_G^2(\rho+1)|t_2-t_1|^2\Big\}\\
\leq\ & \left\{C^*+64\left[\left(M_0^2+\frac{(M_{1-\beta}b^\beta)^2}{2\beta-1}\right)M_F^2+Tr(Q)(M_TM_G)^2\right]L^*\right\}|t_2-t_1|^2\\
\leq\ & L^*|t_2-t_1|^2,
\end{aligned}$$

where $L^*=C^*+N^*L^*$,

$$\begin{aligned}
C^* =\ & 16\Big\{M_T^2A\|\phi(0)+F(0,\phi)\|_Z^2+4\Big[\Big(M_0^2+\frac{(M_{1-\beta}b^\beta)^2}{2\beta-1}\Big)M_F^2\\
&+Tr(Q)(M_TM_G)^2\Big]+Tr(Q)\overline{M}_G^2(\rho+1)+\frac{(M_{1-\beta}\overline{M}_F)^2}{2\beta-1}(\rho+1)|t_2^{2\beta-1}-t_1^{2\beta-1}|\Big\},
\end{aligned}$$

and

$$N^*=64\left[\left(M_0^2+\frac{(M_{1-\beta}b^\beta)^2}{2\beta-1}\right)M_F^2+Tr(Q)(M_TM_G)^2\right],$$

is a constant independent of L^* and $x\in S(\rho)$. So it follows that

$$\|(\Phi x)(t_2)-(\Phi x)(t_1)\|^2\leq L^*|t_2-t_1|^2, t_2,t_1\in J$$

as long as L^* is large enough ($\geq\frac{C^*}{(1-N^*)}$). Thus, Φ has a fixed point x which is a mild solution of equation (1.1). For this $x(\cdot)$, let

$$\begin{aligned}
f(t) &= F(t,x_t),\\
m(t) &= T(t)[\phi(0)+F(0,\phi)],\\
n(t) &= \int_0^t(-A)T(t-s)F(s,x_s)ds,\\
o(t) &= \int_0^t T(t-s)G(s,x_s)dw(s).
\end{aligned}$$

Then they are all Lipschitz continuous. Since x is Lipschitz continuous on J and taking values in the Hilbert space H, we see that $x(\cdot)$ is a. e. differentiable on $(0,b]$ and that $x'(\cdot)\in L^1(J;H)$. The same argument shows that f, n and o also have this property. On the other hand, by the standard arguments (also see [5, Lemma 3.1]) we can obtain that $n(t)\in D(A), o(t)\in D(A)$, and

$$\begin{aligned}
n'(t) &= -AF(t,x_t)ds+A\int_0^t(-A)T(t-s)F(s,x_s)ds,\\
o'(t) &= G(t,x_t)+A\int_0^t T(t-s)G(s,x_s)dw(s).
\end{aligned}$$

So we have that x' satisfies a. e. that

$$\begin{aligned}\frac{d}{dt}[x(t)+F(t,x_t)] &= \frac{d}{dt}T(t)[\phi(0)+F(0,\phi)]+n'(t)+o'(t)\\ &= AT(t)[\phi(0)+F(0,\phi)]-AF(t,x_t)+An(t)\\ &\quad + G(t,x_t)dw(t)+Ao(t)\\ &= Ax(t)+G(t,x_t)dw(t).\end{aligned}$$

This shows that $x(\cdot)$ is also a strong solution of the Cauchy problem (1.1). Thus the proof is completed. ∎

5. EXAMPLE

In this section an example is presented for the existence of mild and strong solutions of the following partial neutral stochastic differential equation:

$$(5.1)\qquad \begin{aligned} d\Big[v(t,x) &+ \int_{-\infty}^{t}\int_0^{\pi}\mu_1(s-t,y,x)v(s,y)dyds\Big] = \tfrac{\partial^2}{\partial x^2}v(t,x)dt\\ &+ \int_{-\infty}^{t}\mu_2(s-t)v(s,x)d\beta(s), \quad 0\le x\le\pi,\ t\in J,\\ v(t,0) &= v(t,\pi)=0, \qquad t\ge 0,\\ v(t,x) &= \phi(t,x), \qquad t\in J_0, \quad 0\le x\le \pi.\end{aligned}$$

To write the above system (5.1) into the abstract form of (1.1), let $H = L^2([0,\pi])$ and A be defined by $A\xi = -\big(\frac{\partial^2}{\partial x^2}\big)\xi$, with domain $D(A) = \Big\{\xi\in H : \xi, \frac{d\xi}{dx}$ are absolutely continuous, and $\Big(\frac{d^2}{dx^2}\Big)\xi\in H, \xi(0)=\xi(\pi)=0\Big\}$. Then $-A$ generates a strongly continuous semigroup $T(\cdot)$ which is compact, analytic, and self-adjoint.

First of all, note that there exists a complete orthonormal set $\{\zeta_n\}, (n=1,2,3,\dots)$ of eigenvectors of A with $\zeta_n(x) = \sqrt{2/\pi}\ \sin\ nx$. Then, the following properties hold:

(a) If $\zeta\in D(A)$, then $A\zeta = \sum_{n=1}^{\infty} n^2\langle\zeta,\zeta_n\rangle\zeta_n$.
(b) For each $\zeta\in H,\ A^{-1/2}\zeta = \sum_{n=1}^{\infty}\frac{1}{n}\langle\zeta,\zeta_n\rangle\zeta_n$. In particular, $\|A^{-1/2}\|^2 = 1$.
(c) The operator $A^{-1/2}$ is given by $A^{1/2}\zeta = \sum_{n=1}^{\infty} n\langle\zeta,\zeta_n\rangle\zeta_n$ on the space $D[A^{1/2}] = \{\zeta(\cdot)\in H, \sum_{n=1}^{\infty} n\langle\zeta,\zeta_n\rangle\zeta_n\in H\}$.

Here we take the phase space $\mathfrak{B} = C(J_0,H)\times L^2(g;H)$, which contains all classes of functions $\phi : J_0\to H$ such that ϕ is $\mathfrak{F}_0$-measurable and $g(\cdot)\|\phi(\cdot)\|^2$ is integrable on J_0 where $g:(-\infty,0)\to\mathbb{R}$ is a positive integrable function and there exists a nonnegative and locally bounded function η on J_0 such that $g(\tau+\theta)\le\eta(\tau)g(\theta)$, for $\tau\le 0$ and $\theta\in(-\infty,0)\setminus R_\tau$, where $R_\tau\subseteq(-\infty,0)$ is a set with Lebesgue measure 0. The seminorm in $\mathfrak{B}$ is defined by

$$\|\phi\|_{\mathfrak{B}} = \|\phi(0)\| + \Big(\int_{-\infty}^{0} g(\theta)\|\phi(\theta)\|^2 d\theta\Big)^{1/2}.$$

The general form of phase space $\mathfrak{B} = C((-\infty,-r],H)\times L^p(g;H),\ r\ge 0,\ 1\le p<\infty$ has been discussed in [6] (here in particular, we are taking r = 0, p = 2). From [6], under some conditions, $(\mathfrak{B},\|\phi\|_{\mathfrak{B}})$ is a Banach space which satisfies (ai)-(aiii) with

$$K(t) = 1 + \Big(\int_{-t}^{0} g(\theta)d\theta\Big)^{1/2} \text{ and } N(t) = \eta(-t)^{1/2} \text{ for all } t\ge 0,$$

(for details see [6]). We assume the following conditions hold:

(i) The function μ_1 is $\mathfrak{F}_t$-measurable and

$$\int_0^\pi \int_{-\infty}^0 \int_0^\pi \big(\mu_1^2(\theta, y, x)/g(\theta)\big) dy d\theta dx < \infty.$$

(ii) The function $(\partial/\partial x)\mu_1(\theta, y, x)$ is measurable, $\mu_1(\theta, y, 0) = \mu_1(\theta, y, \pi) = 0$, and let

$$N_1 = \int_0^\pi \int_{-\infty}^0 \int_0^\pi \frac{1}{g(\theta)} \Big(\frac{\partial}{\partial x}\mu_1(\theta, y, x)\Big)^2 dy d\theta dx < \infty.$$

(iii) The function μ_2 is $\mathfrak{F}_t$-measurable with $\int_{-\infty}^0 \big(\mu_2^2(\theta)/g(\theta)\big) d\theta < \infty$.

(iv) The function ϕ defined by $\phi(\theta)(x) = \phi(\theta, x)$ belong to $\mathfrak{B}$.

(v) $\beta(t)$ denotes a one-dimensional standard Brownian motion.

We define $F, G : J \times \mathfrak{B} \to H$ by $F(t, \phi) = \Psi_1(\phi)$ and $G(t, \phi) = \Psi_2(\phi)$, where

$$\begin{aligned} \Psi_1(\phi) &= \int_{-\infty}^0 \int_0^\pi \mu_1(\theta, y, x)\phi(\theta, y) dy d\theta, \\ \Psi_2(\phi) &= \int_{-\infty}^0 \mu_2(\theta)\phi(\theta, x) d\theta. \end{aligned}$$

Then, system (5.1) is the abstract formulation of the system (1.1). Moreover, from (i) and (iii) it is clear that Ψ_1 and Ψ_2 are bounded linear operators on $\mathfrak{B}$. Furthermore, $\Psi_1(\phi) \in D[A^{1/2}]$, and $\|A^{1/2}\Psi_1\|^2 \le N_1$. In fact, from the definition of Ψ_1 and (ii) it follows that

$$\langle \Psi_1(\phi), \zeta_n \rangle = \frac{1}{n}\left(\frac{2}{\pi}\right)^{1/2} \langle \Psi(\phi), cos(nx) \rangle,$$

where Ψ is defined by

$$\Psi(\phi) = \int_{-\infty}^0 \int_0^\pi \frac{\partial}{\partial x}\mu_1(\theta, y, x)\phi(\theta, y) dy d\theta.$$

From (ii) we know that $\Psi : \mathfrak{B} \to H$ is a bounded linear operator with $\|\Psi\|^2 \le N_1$. Hence $\|A^{1/2}\Psi_1(\phi)\|^2 = \|\Psi(\phi)\|^2$, which iimplies the assertion. Therefore, under the above and if in addition assumptions (3.3) holds, then from Theorem 3.1, the Cauchy problem (5.1) has a mild solution on J and it has also a strong solution if (3.3) and (H5) hold by Theorem 4.1 provided that $N_1 < 1$.

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