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GENERALIZED HYPERGEOMETRIC FUNCTIONS DEFINED ON THE CLASS OF UNIVALENT FUNCTIONS

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ABSTRACT. Let $\mathcal{A}$ denotes the class of all analytic functions $f(z)$, normalized by the condition $f'(0) - 1 = f(0) = 0$ defined on the open unit disk Δ and S be the subclass of $\mathcal{A}$ containing univalent functions of $\mathcal{A}$. In this paper, we find the sufficient conditions for hypergeometric functions defined on S to be in certain subclasses of $\mathcal{A}$, like $k - UCV, k - ST$.

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1. INTRODUCTION

Let $\mathcal{A}$ denote the class of all analytic functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk $\Delta := \{z : |z| < 1\}$. Let S be the subclass of $\mathcal{A}$ consisting of univalent functions. A function $f \in \mathcal{A}$ is said to be starlike of order α, $0 \le \alpha < 1$ if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad (z \in \Delta).$$

This class is denoted by $S^*(\alpha)$, where $S^*(0) \equiv S^*$, the class of starlike functions. A function $f \in \mathcal{A}$ is in the class $K(\alpha)$, the class of convex univalent function of order α, if

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \quad (z \in \Delta).$$

Clearly $K(0) \equiv K$, the class of convex univalent functions. A function $f \in \mathcal{A}$ is said to be uniformly convex in Δ if $f(z)$ has the property

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \left|\frac{zf''(z)}{f'(z)}\right| \quad (z \in \Delta).$$

This class was introduced by Goodman [2]. Further Kanas and Wiśniowska [4] defined the class $k - UCV$ as

$$k - UCV := \left\{ f \in \mathcal{A} : \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > k\left|\frac{zf''(z)}{f'(z)}\right| \quad (z \in \Delta \quad \text{and} \quad 0 \le k < \infty)\right\}.$$

Rønning [5] defined a new class S_p consisting of functions $f \in \mathcal{A}$ satisfying

$$\Re\left(\frac{zf'(z)}{f(z)}\right) \ge \left|\frac{zf'(z)}{f(z)} - 1\right| \quad (z \in \Delta).$$

Kanas and Wiśniowska [6] defined the class $k - ST$ by

$$k - ST := \left\{ f \in \mathcal{A} : \Re\left(\frac{zf'(z)}{f(z)}\right) > k\left|\frac{zf'(z)}{f(z)} - 1\right| \quad (0 \le k < \infty)\right\}.$$

Let $\tau \in \mathcal{C} \setminus \{0\}$ and $-1 \le B < A \le 1$. A function $f \in \mathcal{A}$ is said to be in $R^\tau(A, B)$, if

$$\left|\frac{f'(z) - 1}{(A - B)\tau - B[f'(z) - 1]}\right| < 1 \qquad (z \in \Delta).$$

This class was introduced and studied by Dixit and Pal [7].

Ponnusamy and Rønning [8] investigated the class S^*_λ, which is defined as

$$S^*_\lambda := \left\{ f \in \mathcal{A} : \left|\frac{zf'(z)}{f(z)} - 1\right| < \lambda, \quad (z \in \Delta; \lambda > 0)\right\}.$$

Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$. The Hadamard product or convolution of $f(z)$ and $g(z)$, denoted by $(f * g)(z)$ is defined by

$$(f * g)(z) := z + \sum_{n=2}^{\infty} a_n b_n z^n.$$

Finaly we recall a sufficiently adequate special case of a convolution operator which was introduced earlier by Srivatsava [9], by using the Pochhammer symbol, defined by

$$(\lambda)_n := \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1 & (n=0) \\ \lambda(\lambda+1)(\lambda+2)....(\lambda+n-1) & (n=1,2,3...). \end{cases}$$

Let $\alpha_1, \alpha_2, \ldots, \alpha_p$ and $\beta_1, \beta_2, \ldots, \beta_q$ be complex numbers with $\beta_j \neq 0, -1, -2...$ for all $j = 1, 2, \ldots, q$. The generalized hypergeometric series is defined as

$$\begin{aligned} {}_pF_q(z) &:= {}_pF_q(\alpha_1, \alpha_2, \ldots, \alpha_p; \beta_1, \beta_2 \ldots, \beta_q; z) \\ &= \sum_{n=0}^{\infty} \frac{(\alpha_1)_n(\alpha_2)_n \ldots (\alpha_p)_n}{(\beta_1)_n(\beta_2)_n \ldots (\beta_q)_n} \frac{z^n}{n!} \qquad (p \leq q+1). \end{aligned}$$

This series converges absolutely in the entire complex plane for $p < q+1$ and in the unit disc for $p = q+1$. The condition $p \leq q+1$ is assumed throughout this paper. Note that the series ${}_pF_q(1)$ converges for $\Re(\sum_{j=1}^{q} b_j - \sum_{i=1}^{p} a_i) > 0$.

The operator I for functions $f \in \mathcal{A}$ is defined by

$$\begin{aligned} I^{\alpha_1,\alpha_2,\ldots,\alpha_p}_{\beta_1,\beta_2,\ldots,\beta_q}(f) &:= z \quad {}_pF_q(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z) * f(z) \\ &= z + \sum_{n=2}^{\infty} \frac{(\alpha_1)_{n-1}(\alpha_2)_{n-1} \ldots (\alpha_p)_{n-1}}{(\beta_1)_{n-1}(\beta_2)_{n-1} \ldots (\beta_q)_{n-1}(1)_{n-1}} a_n z^n. \end{aligned}$$

Merkes and Scott [10] and Ruscheweyh and Singh [11] used continued fractions to find sufficient conditions for the function $z_2F_1 = z_2F_1(a, b; c; z)$ to be in the class $S^*(\alpha), (0 \leq \alpha < 1)$ for various choices of the parameters a, b, c. Carlson and Shaffer [12] proved, convolution results in the class $S^*(\alpha)$ can be expressed in terms of a linear operator acting on a hypergeometric functions. Owa and Srivatsava [13] dealt extensively with univalent functions and starlike generalized hypergeometric functions ${}_pF_q(z)$ with $p \leq q+1$. Gangadharan *et al.* [14] investigated various mappings and inclusion properties involving such subclasses of analytic and univalent functions.

In this paper we obtain the sufficient conditions for $I^{\alpha_1,\alpha_2,\ldots,\alpha_p}_{\beta_1,\beta_2,\ldots,\beta_q}(f)$ to be in the classes $k-UCV$, $k-ST$, S^*_λ, $PM_g(\alpha)$ for $f \in S$ with appropriate restrictions on $\alpha_1, \alpha_2, \ldots, \alpha_p$ and $\beta_1, \beta_2, \ldots, \beta_q$.

2. MAIN RESULTS.

Lemma 2.1. [1] *If the function $f(z)$ of the form (1.1) is in S , then $|a_n| \leq n$.*

Lemma 2.2. [4] *A function $f(z)$ of the form (1.1) is in $k-UCV$, if*

$$\sum_{n=2}^{\infty} n(n-1)|a_n| \leq \frac{1}{k+2}.$$

Lemma 2.3. [6] *A function $f(z)$ of the form (1.1) is in $k-ST$, if*

$$\sum_{n=2}^{\infty} (n+(n-1)k)|a_n| \leq 1.$$

Lemma 2.4. [8] *A function $f(z)$ of the form (1.1) is in S^*_λ, if*

$$\sum_{n=2}^{\infty} (\lambda+n-1)|a_n| \leq \lambda.$$

Theorem 2.5. *Let $f \in S$, $\alpha_i > 0$ for all $i = 1, 2, \ldots, p$ and $\sum_{j=1}^{q} |\beta_j| > \sum_{i=1}^{p} |\alpha_i| + 3$. Then the sufficient condition for $I^{\alpha_1,\alpha_2,\ldots,\alpha_p}_{\beta_1,\beta_2\ldots,\beta_q}(f(z)) \in k-UCV$, is*

$$\left|\frac{2\alpha_1\ldots\alpha_p}{\beta_1\ldots\beta_q}\right| [{}_{p+2}F_{q+2}(|\alpha_1|+1,\ldots,|\alpha_p|+1,3,2;|\beta_1|+1,\ldots,|\beta_q|+1,2,1;1)$$
$$+{}_{p+1}F_{q+1}(|\alpha_1|+1,\ldots,|\alpha_p|+1,3;|\beta_1|+1,\ldots,|\beta_q|+1,2,;1)] \leq \frac{1}{k+2}.$$

Proof. In view of Lemma 2.2 it is enough if we prove

$$\sum_{n=2}^{\infty} n(n-1)\left|\frac{(\alpha_1)_{n-1}(\alpha_2)_{n-1}\ldots(\alpha_p)_{n-1}}{(\beta_1)_{n-1}(\beta_2)_{n-1}\ldots(\beta_q)_{n-1}(1)_{n-1}}a_n\right| \leq \frac{1}{k+2}.$$

In view of Lemma 2.1 consider,

$$\begin{aligned}
&\sum_{n=2}^{\infty} n(n-1)\left|\frac{(\alpha_1)_{n-1}(\alpha_2)_{n-1}\ldots(\alpha_p)_{n-1}}{(\beta_1)_{n-1}(\beta_2)_{n-1}\ldots(\beta_q)_{n-1}(1)_{n-1}}a_n\right| \\
\leq\ & \sum_{n=2}^{\infty} n^2\left|\frac{(\alpha_1)_{n-1}(\alpha_2)_{n-1}\ldots(\alpha_p)_{n-1}}{(\beta_1)_{n-1}(\beta_2)_{n-1}\ldots(\beta_q)_{n-1}(1)_{n-2}}\right| \\
\leq\ & \sum_{n=2}^{\infty} n(n-1)\frac{(|\alpha_1|)_{n-1}(|\alpha_2|)_{n-1}\ldots(|\alpha_p|)_{n-1}}{(|\beta_1|)_{n-1}(|\beta_2|)_{n-1}\ldots(|\beta_q|)_{n-1}(1)_{n-2}} \\
&+\sum_{n=2}^{\infty} n\frac{(|\alpha_1|)_{n-1}(|\alpha_2|)_{n-1}\ldots(|\alpha_p|)_{n-1}}{(|\beta_1|)_{n-1}(|\beta_2|)_{n-1}\ldots(|\beta_q|)_{n-1}(1)_{n-2}} \\
=\ & \sum_{n=2}^{\infty}\frac{(|\alpha_1|)_{n-1}(|\alpha_2|)_{n-1}\ldots(|\alpha_p|)_{n-1}(2)_{n-1}(1)_{n-1}}{(|\beta_1|)_{n-1}(|\beta_2|)_{n-1}\ldots(|\beta_q|)_{n-1}(1)_{n-1}(1)_{n-2}(1)_{n-2}} \\
&+\sum_{n=2}^{\infty}\frac{(|\alpha_1|)_{n-1}(|\alpha_2|)_{n-1}\ldots(|\alpha_p|)_{n-1}(2)_{n-1}}{(|\beta_1|)_{n-1}(|\beta_2|)_{n-1}\ldots(|\beta_q|)_{n-1}(1)_{n-1}(1)_{n-2}} \\
=\ & \left|\frac{2\alpha_1\alpha_2\ldots\alpha_p}{\beta_1\beta_2\ldots\beta_q}\right|\sum_{n=2}^{\infty}\frac{(|\alpha_1|+1)_{n-2}(|\alpha_2|+1)_{n-2}\ldots(|\alpha_p|+1)_{n-2}(3)_{n-2}(2)_{n-2}}{(|\beta_1|+1)_{n-2}(|\beta_2|+1)_{n-2}\ldots(|\beta_q|+1)_{n-2}(2)_{n-2}(1)_{n-2}(1)_{n-2}} \\
&+\sum_{n=1}^{\infty}\frac{(|\alpha_1|)_{n}(|\alpha_2|)_{n}\ldots(|\alpha_p|)_{n}(2)_{n}}{(|\beta_1|)_{n}(|\beta_2|)_{n}\ldots(|\beta_q|)_{n}(1)_{n}(1)_{n-1}} \\
=\ & \left|\frac{2\alpha_1\alpha_2\ldots\alpha_p}{\beta_1\beta_2\ldots\beta_q}\right| [{}_{p+2}F_{q+2}(|\alpha_1|+1,\ldots,|\alpha_p|+1,3,2;|\beta_1|+1,\ldots,|\beta_q|+1,2,1;1) \\
&+{}_{p+1}F_{q+1}(|\alpha_1|+1,\ldots,|\alpha_p|+1,3;|\beta_1|+1,\ldots,|\beta_q|+1,2;1)].
\end{aligned}$$

As the above expression is bounded above by $\frac{1}{k+2}$, the result follows. ∎

Theorem 2.6. *Let $f \in S$. Also $\alpha_i > 0$ for all $i = 1,2,\ldots,p$ and $\sum_{j=1}^{q}|\beta_j| > \sum_{i=1}^{p}|\alpha_i| + 2$. Then the sufficient condition for $I^{\alpha_1,\alpha_2,\ldots,\alpha_p}_{\beta_1,\beta_2\ldots,\beta_q}(f(z)) \in k-ST$ is*

$$(k+1)\left|\frac{2\alpha_1\ldots\alpha_p}{\beta_1\ldots\beta_q}\right| \quad {}_{p+1}F_{q+1}(|\alpha_1|+1,\ldots,|\alpha_p|+1,3;|\beta_1|+1,\ldots,|\beta_q|+1,2;1)$$
$$+{}_{p+1}F_{q+1}(|\alpha_1|,\ldots,|\alpha_p|,2;|\beta_1|,\ldots,|\beta_q|,1;1) \leq 2.$$

Proof. In view of Lemma 2.3 it is enough to prove that

$$\sum_{n=2}^{\infty}[n+(n-1)k]\left|\frac{(\alpha_1)_{n-1}\ldots(\alpha_p)_{n-1}}{(\beta_1)_{n-1}\ldots(\beta_q)_{n-1}(1)_{n-1}}a_n\right| \leq 1.$$

By Lemma 2.1 we have,

$$\begin{aligned}
&\sum_{n=2}^{\infty}[n+(n-1)k]\left|\frac{(\alpha_1)_{n-1}\dots(\alpha_p)_{n-1}}{(\beta_1)_{n-1}\dots(\beta_q)_{n-1}(1)_{n-1}}a_n\right| \\
\leq\ & \sum_{n=2}^{\infty}[n+(n-1)k]\frac{(|\alpha_1|)_{n-1}\dots(|\alpha_p|)_{n-1}}{(|\beta_1|)_{n-1}\dots(|\beta_q|)_{n-1}(1)_{n-1}}|a_n| \\
=\ & (k+1)\sum_{n=2}^{\infty}(n-1)\frac{(|\alpha_1|)_{n-1}\dots(|\alpha_p|)_{n-1}}{(|\beta_1|)_{n-1}\dots(|\beta_q|)_{n-1}(1)_{n-1}}|a_n| \\
&\quad+\sum_{n=2}^{\infty}\frac{(|\alpha_1|)_{n-1}\dots(|\alpha_p|)_{n-1}}{(|\beta_1|)_{n-1}\dots(|\beta_q|)_{n-1}(1)_{n-1}}|a_n| \\
\leq\ & (k+1)\sum_{n=2}^{\infty}n\frac{(|\alpha_1|)_{n-1}\dots(|\alpha_p|)_{n-1}}{(|\beta_1|)_{n-1}\dots(|\beta_q|)_{n-1}(1)_{n-2}} \\
&\quad+\sum_{n=2}^{\infty}n\frac{(|\alpha_1|)_{n-1}\dots(|\alpha_p|)_{n-1}}{(|\beta_1|)_{n-1}\dots(|\beta_q|)_{n-1}(1)_{n-1}} \\
=\ & (k+1)\sum_{n=2}^{\infty}\frac{(|\alpha_1|)_{n-1}\dots(|\alpha_p|)_{n-1}(2)_{n-1}}{(|\beta_1|)_{n-1}\dots(|\beta_q|)_{n-1}(1)_{n-1}(1)_{n-2}} \\
&\quad+\sum_{n=2}^{\infty}\frac{(|\alpha_1|)_{n-1}\dots(|\alpha_p|)_{n-1}(2)_{n-1}}{(|\beta_1|)_{n-1}\dots(|\beta_q|)_{n-1}(1)_{n-1}(1)_{n-1}} \\
=\ & (k+1)\left|\frac{2\alpha_1\dots\alpha_p}{\beta_1\dots\beta_q}\right|\sum_{n=2}^{\infty}\frac{(|\alpha_1|+1)_{n-2}\dots(|\alpha_p|+1)_{n-2}(3)_{n-2}}{(|\beta_1|+1)_{n-2}\dots(|\beta_q|+1)_{n-2}(2)_{n-2}(1)_{n-2}} \\
&\quad+\sum_{n=1}^{\infty}\frac{(|\alpha_1|)_{n}\dots(|\alpha_p|)_{n}(2)_{n}}{(|\beta_1|)_{n}\dots(|\beta_q|)_{n}(1)_{n}(1)_{n}} \\
=\ & (k+1)\left|\frac{2\alpha_1\dots\alpha_p}{\beta_1\dots\beta_q}\right|_{p+1}F_{q+1}(|\alpha_1|+1,\dots,|\alpha_p|+1,3;|\beta_1|+1,\dots,|\beta_q|+1,2;1) \\
&\quad+{}_{p+1}F_{q+1}(|\alpha_1|,\dots,|\alpha_p|,2;|\beta_1|,\dots,|\beta_q|,1;1)-1.
\end{aligned}$$

Since the above quantity is bounded above by 1, result is proved. ∎

Theorem 2.7. *Let $f\in S$. Also $\alpha_i>0$ for all $i=1,2,\dots,p$ and $\sum_{j=1}^{q}|\beta_j|>\sum_{i=1}^{p}|\alpha_i|+2$. Then the sufficient condition for $I^{\alpha_1,\alpha_2,\dots,\alpha_p}_{\beta_1,\beta_2\dots,\beta_q}(f(z))\in S^*_\lambda$, is*

$$\begin{aligned}
&{}_{p+1}F_{q+1}(|\alpha_1|,\dots,|\alpha_p|,2;|\beta_1|,\dots,|\beta_q|,1;1) \\
+\ & \frac{1}{\lambda}\left|\frac{2\alpha_1\dots\alpha_p}{\beta_1\dots\beta_q}\right|_{p+1}F_{q+1}(|\alpha_1|+1,\dots,|\alpha_p|+1,3;|\beta_1|+1,\dots,|\beta_q|+1,2;1)\leq 2.
\end{aligned}$$

Proof. In view of Lemma 2.4, it is enough to prove that

$$\sum_{n=2}^{\infty}(\lambda+n-1)\left|\frac{(\alpha_1)_{n-1}\dots(\alpha_p)_{n-1}}{(\beta_1)_{n-1}\dots(\beta_q)_{n-1}(1)_{n-1}}a_n\right|\leq\lambda$$

By using Lemma 2.1 we have

$$\begin{aligned}
&\sum_{n=2}^{\infty}(\lambda+n-1)\left|\frac{(\alpha_1)_{n-1}\dots(\alpha_p)_{n-1}}{(\beta_1)_{n-1}\dots(\beta_q)_{n-1}(1)_{n-1}}a_n\right| \\
\leq\ & \lambda\sum_{n=2}^{\infty}\frac{(|\alpha_1|)_{n-1}\dots(|\alpha_p|)_{n-1}(2)_{n-1}}{(|\beta_1|)_{n-1}\dots(|\beta_q|)_{n-1}(1)_{n-1}(1)_{n-1}} \\
&+\sum_{n=2}^{\infty}\frac{(|\alpha_1|)_{n-1}\dots(|\alpha_p|)_{n-1}(2)_{n-1}}{(|\beta_1|)_{n-1}\dots(|\beta_q|)_{n-1}(1)_{n-1}(1)_{n-2}} \\
=\ & \lambda\sum_{n=1}^{\infty}\frac{(|\alpha_1|)_{n}\dots(|\alpha_p|)_{n}(2)_{n}}{(|\beta_1|)_{n}\dots(|\beta_q|)_{n}(1)_{n}(1)_{n}} \\
&+\left|\frac{2\alpha_1\dots\alpha_p}{\beta_1\dots\beta_q}\right|\sum_{n=2}^{\infty}\frac{(|\alpha_1|+1)_{n-2}\dots(|\alpha_p|+1)_{n-2}(3)_{n-2}}{(|\beta_1|+1)_{n-2}\dots(|\beta_q|+1)_{n-2}(2)_{n-2}(1)_{n-2}} \\
=\ & \lambda[{}_{p+1}F_{q+1}(|\alpha_1|,\dots,|\alpha_p|,2;|\beta_1|,\dots,|\beta_q|,1;1)-1] \\
&+\left|\frac{2\alpha_1\dots\alpha_p}{\beta_1\dots\beta_q}\right|\quad {}_{p+1}F_{q+1}(|\alpha_1|+1,\dots,|\alpha_p|+1,3;|\beta_1|+1,\dots,|\beta_q|+1,2;1).
\end{aligned}$$

Hence the result follows, as the above expression is bounded above by λ. ∎

3. THE CLASS $PM_g(\alpha)$

Definition 3.1. [15] Let P be the class of all analytic functions $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ with $a_n\geq 0$. Let $g(z)=z+\sum_{n=2}^{\infty}b_nz^n$ be a fixed analytic function in Δ with $b_n>0$, for all $n\geq 2$. Define the class $PM_g(\alpha)$ by,

$$PM_g(\alpha):=\left\{f\in P:\Re\left(\frac{z(f*g)'(z)}{(f*g)(z)}\right)<\alpha\quad(1<\alpha<3/2;z\in\Delta)\right\}$$

This class was introduced by Ravichandran *et al.*.

Lemma 3.1. [15] *Let the function* $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ *be in* P. *Then* $f(z)\in PM_g(\alpha)$ *if and only if*

$$\sum_{n=2}^{\infty}(n-\alpha)a_nb_n\leq\alpha-1\quad(1<\alpha<3/2).$$

Lemma 3.2. [4, 6] *Let the function* $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ *is in the class* $k-UCV$ *then the following coefficient inequality holds true.*

$$|a_n|\leq\frac{(P_1)_{n-1}}{n!}\quad(n\in\mathrm{N}-\{1\})$$

where $P_1=P_1(k)$ *is the coefficient of* z *in the function*

$$p_k(z)=1+\sum_{n=1}^{\infty}P_n(k)z^n,$$

which is the extremal function for the class $\mathcal{P}(p_k)$ *related to the class* $k-UCV$ *by the range of expression*

$$1+\frac{zf''(z)}{f'(z)}\quad(z\in\Delta).$$

Similarly for a function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *belonging to the class* $k - ST$*, the following coefficient inquality holds true*

$$|a_n| \le \frac{(P_1)_{n-1}}{(n-1)!} \quad (n \in \mathrm{N} - \{1\})$$

where $P_1 = P_1(k)$.

Lemma 3.3. [7] *If* $f \in R^{\tau}[A, B]$*, then the following coefficient inequality holds true.*

$$|a_n| \le \frac{A-B}{n}|\tau|.$$

Note that if $g(z)$ is also univalent then we have the following interesting sufficient conditions.

Theorem 3.4. *Let* $\alpha_i > 0$ *for all* $i = 1, 2, \ldots, p$ *and* $\sum_{j=1}^{q} \beta_j > \sum_{i=1}^{p} \alpha_i + 2$*. Then the sufficient condition for* $z_p F_q(z) \in PM_g(\alpha)$ *is*

$$\begin{aligned} &\frac{2\alpha_1 \ldots \alpha_p}{(\alpha-1)\beta_1 \ldots \beta_q}[{}_{p+1}F_{q+1}(\alpha_1+1, \ldots, \alpha_p+1, 3; \beta_1+1, \ldots, \beta_q+1, 2; 1)] \\ + \quad &[{}_{p+1}F_{q+1}(\alpha_1, \ldots, \alpha_p, 2; \beta_1, \ldots, \beta_q, 1; 1) - 1] \le 1. \end{aligned}$$

Proof. In view of Lemma 3.1 it is enough if we prove

$$\sum_{n=2}^{\infty}(n-\alpha)\frac{(\alpha_1)_{n-1}\ldots(\alpha_p)_{n-1}}{(\beta_1)_{n-1}\ldots(\beta_q)_{n-1}(1)_{n-1}}b_n \le \alpha - 1.$$

By using Lemma 2.1 we have

$$\begin{aligned} &\sum_{n=2}^{\infty}(n-\alpha)\frac{(\alpha_1)_{n-1}\ldots(\alpha_p)_{n-1}}{(\beta_1)_{n-1}\ldots(\beta_q)_{n-1}(1)_{n-1}}b_n \\ \le \quad &\sum_{n=2}^{\infty}\frac{n^2(\alpha_1)_{n-1}\ldots(\alpha_p)_{n-1}}{(\beta_1)_{n-1}\ldots(\beta_q)_{n-1}(1)_{n-1}} \\ &\qquad -\alpha\sum_{n=2}^{\infty}\frac{n(\alpha_1)_{n-1}\ldots(\alpha_p)_{n-1}}{(\beta_1)_{n-1}\ldots(\beta_q)_{n-1}(1)_{n-1}} \\ = \quad &\sum_{n=2}^{\infty}\frac{n(n-1)(\alpha_1)_{n-1}\ldots(\alpha_p)_{n-1}}{(\beta_1)_{n-1}\ldots(\beta_q)_{n-1}(1)_{n-1}} \\ &\qquad +(\alpha-1)\sum_{n=2}^{\infty}\frac{n(\alpha_1)_{n-1}\ldots(\alpha_p)_{n-1}}{(\beta_1)_{n-1}\ldots(\beta_q)_{n-1}(1)_{n-1}} \\ = \quad &\sum_{n=2}^{\infty}\frac{n(\alpha_1)_{n-1}\ldots(\alpha_p)_{n-1}}{(\beta_1)_{n-1}\ldots(\beta_q)_{n-1}(1)_{n-2}} \\ &\qquad +(\alpha-1)\sum_{n=1}^{\infty}\frac{(\alpha_1)_n\ldots(\alpha_p)_n(2)_n}{(\beta_1)_n\ldots(\beta_q)_n(1)_n(1)_n} \\ = \quad &\sum_{n=2}^{\infty}\frac{(\alpha_1)_{n-1}\ldots(\alpha_p)_{n-1}(2)_{n-1}}{(\beta_1)_{n-1}\ldots(\beta_q)_{n-1}(1)_{n-1}(1)_{n-2}} \\ &\qquad +(\alpha-1)\sum_{n=1}^{\infty}\frac{(\alpha_1)_n\ldots(\alpha_p)_n(2)_n}{(\beta_1)_n\ldots(\beta_q)_n(1)_n(1)_n} \end{aligned}$$

$$= \frac{2\alpha_1 \dots \alpha_p}{\beta_1 \dots \beta_q}[{}_{p+1}F_{q+1}(\alpha_1+1,\dots,\alpha_p+1,3;\beta_1+1,\dots,\beta_q+1,2;1)]$$
$$+(\alpha-1)[{}_{p+1}F_{q+1}(\alpha_1,\dots,\alpha_p,2;\beta_1,\dots,\beta_q,1;1)-1].$$

As the above expression is bounded above by $\alpha - 1$, the result follows. ∎

Theorem 3.5. *Let $f \in R^\tau[A,B]$ and $\alpha_i > 0$ for all $i = 1,2,\dots,p$ with $\sum_{j=1}^{q}\beta_j > \sum_{i=1}^{p}\alpha_i + 1$. Then the sufficient condition for $I^{\alpha_1\alpha_2\dots\alpha_p}_{\beta_1\beta_2\dots\beta_q}(f(z)) \in PM_g(\alpha)$ is,*

$$\frac{(A-B)|\tau|}{\alpha-1}[{}_{p+1}F_{q+1}(\alpha_1,\dots,\alpha_p,2;\beta_1,\dots,\beta_q,1;1)$$
$$- \quad \alpha_p F_q(\alpha_1,\dots,\alpha_p;\beta_1,\dots,\beta_q;1)+1] \le 1.$$

Proof. In view of Lemma 3.1 it is enough to prove that

$$\sum_{n=2}^{\infty}(n-\alpha)\frac{(\alpha_1)_{n-1}\dots(\alpha_p)_{n-1}}{(\beta_1)_{n-1}\dots(\beta_q)_{n-1}(1)_{n-1}}a_n b_n \le \alpha - 1.$$

By using the coefficient estimates of the classes S and $R^\tau[A,B]$, we have

$$\sum_{n=2}^{\infty}(n-\alpha)\frac{(\alpha_1)_{n-1}\dots(\alpha_p)_{n-1}}{(\beta_1)_{n-1}\dots(\beta_q)_{n-1}(1)_{n-1}}a_n b_n$$
$$\le \sum_{n=2}^{\infty}(n-\alpha)\frac{(\alpha_1)_{n-1}\dots(\alpha_p)_{n-1}}{(\beta_1)_{n-1}\dots(\beta_q)_{n-1}(1)_{n-1}}(A-B)|\tau|$$
$$= (A-B)|\tau|\left[\sum_{n=2}^{\infty} n\frac{(\alpha_1)_{n-1}\dots(\alpha_p)_{n-1}}{(\beta_1)_{n-1}\dots(\beta_q)_{n-1}(1)_{n-1}} - \alpha\sum_{n=2}^{\infty}\frac{(\alpha_1)_{n-1}\dots(\alpha_p)_{n-1}}{(\beta_1)_{n-1}\dots(\beta_q)_{n-1}(1)_{n-1}}\right]$$
$$= (A-B)|\tau|\left[\sum_{n=1}^{\infty}\frac{(\alpha_1)_n\dots(\alpha_p)_n(2)_n}{(\beta_1)_n\dots(\beta_q)_n(1)_n(1)_n} - \alpha\sum_{n=1}^{\infty}\frac{(\alpha_1)_n\dots(\alpha_p)_n}{(\beta_1)_n\dots(\beta_q)_n(1)_n}\right]$$
$$= (A-B)|\tau|[{}_{p+1}F_{q+1}(\alpha_1,\dots,\alpha_p,2;\beta_1,\dots,\beta_q,1;1)-1$$
$$-\alpha_p F_q(\alpha_1,\dots,\alpha_p;\beta_1,\dots,\beta_q;1)+\alpha].$$

The above expression is bounded above by $\alpha - 1$. Hence the result follows. ∎

Theorem 3.6. *Let $f \in k-UCV$ and $\alpha_i > 0$ for all $i = 1,2,\dots,p$ with $\sum_{j=1}^{q}\beta_j > \sum_{i=1}^{p}\alpha_i + P_1$. Then the sufficient condition for $I^{\alpha_1\alpha_2\dots\alpha_p}_{\beta_1\beta_2\dots\beta_q}(f(z)) \in PM_g(\alpha)$ is,*

$$\left(\frac{\alpha_1\dots\alpha_p P_1}{\beta_1\dots\beta_q}\right) \quad {}_{p+1}F_{q+1}(\alpha_1+1,\dots,\alpha_p+1,P_1+1;\beta_1+1,\dots,\beta_q+1,2;1)$$
$$+(1-\alpha)[{}_{p+1}F_{q+1}(\alpha_1,\dots,\alpha_p,P_1;\beta_1,\dots,\beta_q,1;1)-1] \le \alpha - 1.$$

Proof. In view of Lemma 3.1 it is enough to prove that

$$\sum_{n=2}^{\infty}(n-\alpha)\frac{(\alpha_1)_{n-1}\dots(\alpha_p)_{n-1}}{(\beta_1)_{n-1}\dots(\beta_q)_{n-1}(1)_{n-1}}a_n b_n \le \alpha - 1.$$

In view of Lemma 2.1 and Lemma 3.2 we have

$$\begin{aligned}
&\sum_{n=2}^{\infty}(n-\alpha)\frac{(\alpha_1)_{n-1}\dots(\alpha_p)_{n-1}}{(\beta_1)_{n-1}\dots(\beta_q)_{n-1}(1)_{n-1}}a_nb_n\\
\leq\ &\sum_{n=2}^{\infty}(n-\alpha)\frac{(\alpha_1)_{n-1}\dots(\alpha_p)_{n-1}}{(\beta_1)_{n-1}\dots(\beta_q)_{n-1}(1)_{n-1}}\frac{n(P_1)_{n-1}}{n!}\\
\leq\ &\sum_{n=2}^{\infty}(n-\alpha)\frac{(\alpha_1)_{n-1}\dots(\alpha_p)_{n-1}(P_1)_{n-1}}{(\beta_1)_{n-1}\dots(\beta_q)_{n-1}(1)_{n-1}(1)_{n-1}}\\
=\ &\sum_{n=2}^{\infty}(n-\alpha)\frac{(\alpha_1)_{n-1}\dots(\alpha_p)_{n-1}(P_1)_{n-1}}{(\beta_1)_{n-1}\dots(\beta_q)_{n-1}(1)_{n-1}(1)_{n-1}}\\
=\ &\sum_{n=1}^{\infty}\frac{(n+1-\alpha)(\alpha_1)_{n}\dots(\alpha_p)_{n}(P_1)_{n}}{(\beta_1)_{n}\dots(\beta_q)_{n}(1)_{n}(1)_{n}}\\
=\ &\sum_{n=1}^{\infty}\frac{(\alpha_1)_{n}\dots(\alpha_p)_{n}(P_1)_{n}}{(\beta_1)_{n}\dots(\beta_q)_{n}(1)_{n}(1)_{n-1}}\\
&\qquad+(1-\alpha)\sum_{n=1}^{\infty}\frac{(\alpha_1)_{n}\dots(\alpha_p)_{n}(P_1)_{n}}{(\beta_1)_{n}\dots(\beta_q)_{n}(1)_{n}(1)_{n}}\\
=\ &\left(\frac{\alpha_1\dots\alpha_pP_1}{\beta_1\dots\beta_q}\right)\sum_{n=1}^{\infty}\frac{(\alpha_1+1)_{n-1}\dots(\alpha_p+1)_{n-1}(P_1+1)_{n-1}}{(\beta_1+1)_{n-1}\dots(\beta_q+1)_{n-1}(2)_{n-1}(1)_{n-1}}\\
&\qquad+(1-\alpha)\sum_{n=1}^{\infty}\frac{(\alpha_1)_{n}\dots(\alpha_p)_{n}(P_1)_{n}}{(\beta_1)_{n}\dots(\beta_q)_{n}(1)_{n}(1)_{n}}\\
=\ &\left(\frac{\alpha_1\dots\alpha_pP_1}{\beta_1\dots\beta_q}\right)\ {}_{p+1}F_{q+1}(\alpha_1+1,\dots,\alpha_p+1,P_1+1;\beta_1+1,\dots,\beta_q+1,2;1)\\
&\qquad+(1-\alpha)[{}_{p+1}F_{q+1}(\alpha_1,\dots,\alpha_p,P_1;\beta_1,\dots,\beta_q,1;1)-1].
\end{aligned}$$

Since the above expression is bounded above by $\alpha-1$, the result follows. ∎

On similar lines we have the following theorem, the proof of which is omitted.

Theorem 3.7. *Let $f\in k-ST$ and $\alpha_i>0$ for all $i=1,2,\dots,p$ with $\sum_{j=1}^{q}\beta_j>\sum_{i=1}^{p}\alpha_i+P_1+1$ then $I^{\alpha_1\alpha_2\dots\alpha_p}_{\beta_1\beta_2\dots\beta_q}(f(z))\in PM_g(\alpha)$, if*

$$\begin{aligned}
&\frac{2\alpha_1\dots\alpha_pP_1}{\beta_1\dots\beta_q}\ {}_{p+2}F_{q+2}(\alpha_1+1,\dots,\alpha_p+1,P_1+1,3;\beta_1+1,\dots,\beta_q+1,2,2;1)\\
&\qquad+(1-\alpha)[{}_{p+2}F_{q+2}(\alpha_1,\dots,\alpha_p,P_1,2;\beta_1,\dots,\beta_q,1,1;1)-1]\leq\alpha-1.
\end{aligned}$$

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