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ITERATED ORDER OF FAST GROWTH SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper, we investigate the growth of solutions of the differential equation $f^{(k)} + A_{k-1}(z) f^{(k-1)} + ... + A_1(z) f' + A_0(z) f = F(z)$, where $A_0(z), ..., A_{k-1}(z)$ and $F(z) \not\equiv 0$ are entire functions. Some estimates are given for the iterated order of solutions of the above equation when one of the coefficients A_s is being dominant in the sense that it has larger growth than A_j $(j \neq s)$ and F.

Key words and phrases: Linear differential equation, Growth of entire function, Iterated order.

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1. Introduction and Statement of Results

For the definition of the iterated order of an entire function, we use the same definition as in [9], ([4], p. 317), ([10], p. 129). For all $r \in \mathbf{R}$, we define $\exp_1 r := e^r$ and $\exp_{p+1} r := \exp\left(\exp_p r\right),\, p \in \mathbf{N}$. We also define for all r sufficiently large $\log_1 r := \log r$ and $\log_{p+1} r := \log\left(\log_p r\right),\, p \in \mathbf{N}$. Moreover, we denote by $\exp_0 r := r$, $\log_0 r := r$, $\log_{-1} r := \exp_1 r$ and $\exp_{-1} r := \log_1 r$. Let f be an entire function. Then the iterated $p-$order $\sigma_p(f)$ of f is defined by

$$\sigma_p(f) = \overline{\lim_{r\to+\infty}} \frac{\log_p T(r,\ f)}{\log r} = \overline{\lim_{r\to+\infty}} \frac{\log_{p+1} M(r,\ f)}{\log r} \quad (p \geq 1 \text{ is an integer}), \tag{1.1}$$

where $T(r,\ f)$ is the Nevanlinna characteristic function of f and $M(r, f) = \max_{|z|=r} |f(z)|$ (see [8]). For $p = 1$, this notation is called order and for $p = 2$ hyper-order (see [2], [4], [12]). For $k \geq 2$, we consider the non-homogeneous linear differential equation

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + ... + A_1(z) f' + A_0(z) f = F(z), \tag{1.2}$$

where $A_0(z), ..., A_{k-1}(z)$ and $F(z) \not\equiv 0$ are entire functions. It is well-known that all solutions of equation (1.2) are entire functions and if at least one coefficient $A_s(z)$ is transcendental, then at least some of the solutions are of infinite order. On the other hand, there exist equations of this form that possess one or more solutions of finite order. For example: $f(z) = e^z$ satisfies $f''' - e^z f'' - e^{-z} f' + e^z f = e^z - 1$.
Extensive work in recent years has been concerned with the growth of solutions of complex linear differential equations. Many results have been obtained for the growth of solutions of the differential equation

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + ... + A_1(z) f' + A_0(z) f = 0, \tag{1.3}$$

where $A_0(z), ..., A_{k-1}(z)$ are entire functions, see e.g. [2], [3], [9] and [11]. Examples of such results are the following two theorems:

Theorem 1.1. *([11]) Let $A_0(z), ..., A_{k-1}(z)$ be entire functions such that for some integer s, $1 \leq s \leq k-1$, we have $\sigma_p(A_j) \leq \alpha < \beta = \sigma_p(A_s) \leq +\infty$ for all $j \neq s$. Then every transcendental solution f of (1.3) satisfies $\sigma_p(f) \geq \sigma_p(A_s)$.*

Theorem 1.2. *([3]) Let $A_0(z), ..., A_{k-1}(z)$ be entire functions, where $0 < \sigma(A_0) < 1/2$, and let there exist a real constant $\beta < \sigma(A_0)$ and a set $E_\beta \subset [0, +\infty)$ with $\underline{\text{dens}}\, E_\beta = 1$ such that for all $r \in E_\beta$, we have*

$$\min_{|z|=r} |A_j(z)| \leq \exp\left(r^\beta\right) \quad (j = 1, 2, ..., k-1). \tag{1.4}$$

Then every solution $f \not\equiv 0$ of (1.3) is of infinite order with hyper-order $\sigma_2(f) \geq \sigma(A_0)$.

The purpose of this paper is to extend the above results to the non-homogeneous linear differential equation (1.2). We will prove the following theorems:

Theorem 1.3. *Let $A_0(z), ..., A_{k-1}(z)$ and $F(z) \not\equiv 0$ be entire functions such that for some integer s, $1 \leq s \leq k-1$, we have $\max\{\sigma_p(A_j)\ (j \neq s),\ \sigma_p(F)\} < \sigma_p(A_s) < +\infty$. Then every transcendental solution f of (1.2) with $\sigma_p(f) < +\infty$ satisfies $\sigma_p(f) \geq \sigma_p(A_s)$.*

The following result was proved in the case of $p = 1$ and $s = 0$ in ([6], Theorem 3).

Theorem 1.4. *Let* $A_0(z), ..., A_{k-1}(z)$ *and* $F(z) \not\equiv 0$ *be entire functions such that for some integer* s, $0 \le s \le k-1$, *we have* $\sigma_p(A_s) = +\infty$ *and* $\max\{\sigma_p(A_j)\ (j \ne s), \sigma_p(F)\} < +\infty$. *Then every solution* f *of (1.2) satisfies* $\sigma_p(f) = +\infty$.

Theorem 1.5. *Let* $A_0(z), ..., A_{k-1}(z)$, $F(z) \not\equiv 0$ *be entire functions, where* $0 < \sigma(A_0) < 1/2$ *and* $\sigma(F) = \sigma < +\infty$, *and let there exist a real constant* $\beta < \sigma(A_0)$ *and a set* $E_\beta \subset [0, +\infty)$ *with* $\underline{dens}\, E_\beta = 1$ *such that for all* $r \in E_\beta$, *we have*

$$\min_{|z|=r} |A_j(z)| \le \exp\left(r^\beta\right) \quad (j = 1, 2, ..., k-1). \tag{1.5}$$

Then every solution f *of (1.2) is of infinite order and hyper-order* $\sigma_2(f) \ge \sigma(A_0)$ *with at most one exceptional solution* f_0 *satisfying* $\sigma(f_0) < +\infty$.

2. PRELIMINARY LEMMAS

Our proofs depend mainly upon the following lemmas. Before starting these lemmas, we recall the concept of density of subsets of $[0, +\infty)$. For $E \subset [0, +\infty)$, we define the linear measure of a set E by $m(E) = \int_0^{+\infty} \chi_E(t)\, dt$, where χ_E is the characteristic function of E. The upper and the lower densities of E are defined by

$$\overline{dens} E = \overline{\lim_{r \to +\infty}} \frac{m(E \cap [0, r])}{r}, \quad \underline{dens} E = \underline{\lim_{r \to +\infty}} \frac{m(E \cap [0, r])}{r}. \tag{2.1}$$

Lemma 2.1. *(*[2]*) Let* E *be a set of complex numbers satisfying* $\overline{dens}\{|z| : z \in E\} > 0$, *and let* $A_0(z), ..., A_{k-1}(z)$ *be entire functions such that for some real constants* $0 \le \beta < \alpha$ *and* $\mu > 0$, *we have*

$$|A_0(z)| \ge \exp(\alpha |z|^\mu) \tag{2.2}$$

and

$$|A_j(z)| \le \exp(\beta |z|^\mu) \quad (j = 1, ..., k-1) \tag{2.3}$$

as $z \to \infty$ *for* $z \in E$. *Then every solution* $f \not\equiv 0$ *of (1.3) satisfies* $\sigma(f) = +\infty$ *and* $\sigma_2(f) \ge \mu$.

Lemma 2.2. *(*[7]*, p. 90) Let* f *be a nontrivial entire function, and let* $\alpha > 1$ *and* $\varepsilon > 0$ *be given constants. Then there exist a constant* $c > 0$ *and a set* $E \subset [0, +\infty)$ *having finite linear measure such that for all* z *satisfying* $|z| = r \notin E$, *we have*

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \le c\left[T(\alpha r, f)\, r^\varepsilon \log T(\alpha r, f)\right]^j \quad (j \in \mathbf{N}). \tag{2.4}$$

Lemma 2.3. *Let* E *be a set of complex numbers satisfying* $\overline{dens}\{|z| : z \in E\} > 0$, *and let* $A_0(z), ..., A_{k-1}(z)$ *be entire functions such that for some real constants* $0 \le \beta < \alpha$ *and* $\mu > 0$, *we have*

$$|A_0(z)| \ge \exp(\alpha |z|^\mu) \tag{2.5}$$

and

$$|A_j(z)| \le \exp(\beta |z|^\mu) \quad (j = 1, ..., k-1) \tag{2.6}$$

as $z \to \infty$ *for* $z \in E$, *and let* $F(z) \not\equiv 0$ *be an entire function with* $\sigma(F) < +\infty$. *Then every solution* f *of (1.2) satisfies* $\sigma(f) = +\infty$ *and* $\sigma_2(f) \ge \mu$ *with at most one exceptional solution* f_0 *satisfying* $\sigma(f_0) < +\infty$.

Proof. We affirm that (1.2) can only possess at most one exceptional solution f_0 such that $\sigma(f_0) < +\infty$. In fact, if f^* is a second solution with $\sigma(f^*) < +\infty$, then $\sigma(f_0 - f^*) < +\infty$. But $f_0 - f^*$ is a solution of the corresponding homogeneous equation (1.3) of (1.2). This contradicts Lemma 2.1.

Suppose that f is a solution of (1.2) with $\sigma(f) = +\infty$. Now from (1.2), it follows that

$$|A_0(z)| \leq \left|\frac{f^{(k)}}{f}\right| + |A_{k-1}(z)| \left|\frac{f^{(k-1)}}{f}\right| + ... + |A_1(z)| \left|\frac{f'}{f}\right| + \left|\frac{F}{f}\right|. \tag{2.7}$$

Then by Lemma 2.2, there exists a set $E_1 \subset [0, +\infty)$ with a finite linear measure such that for all z satisfying $|z| = r \notin E_1$, we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq r\left[T(2r, f)\right]^{k+1} \quad (j = 1, ..., k). \tag{2.8}$$

Also, by the hypothesis of Lemma 2.3, there exists a set E_2 with $\overline{dens}\{|z| : z \in E_2\} > 0$ such that for all z satisfying $z \in E_2$, we have

$$|A_0(z)| \geq \exp(\alpha |z|^{\mu}) \tag{2.9}$$

and

$$|A_j(z)| \leq \exp(\beta |z|^{\mu}) \ (j = 1, ..., k-1) \tag{2.10}$$

as $z \to \infty$. Since $\sigma(f) = +\infty$, there exists $\left\{r_n'\right\}$ $\left(r_n' \to +\infty\right)$ such that

$$\lim_{r_n' \to +\infty} \frac{\log\log M\left(r_n', f\right)}{\log r_n'} = +\infty. \tag{2.11}$$

Set the linear measure of E_1, $m(E_1) = \delta < +\infty$, then there exists a point $r_n \in \left[r_n', r_n' + \delta + 1\right] - E_1$. From

$$\frac{\log\log M(r_n, f)}{\log r_n} \geq \frac{\log\log M\left(r_n', f\right)}{\log(r_n' + \delta + 1)}$$

$$= \frac{\log\log M\left(r_n', f\right)}{\log r_n' + \log\left(1 + (\delta + 1)/r_n'\right)}, \tag{2.12}$$

it follows that

$$\varliminf_{r_n \to +\infty} \frac{\log\log M(r_n, f)}{\log r_n} = +\infty. \tag{2.13}$$

Then for a given arbitrary large $\rho > \sigma(F)$,

$$M(r_n, f) \geq \exp(r_n^{\rho}) \tag{2.14}$$

holds for sufficiently large r_n. On the other hand, for a given ε with $0 < \varepsilon < \rho - \sigma(F)$, we have

$$|F(z_n)| \leq \exp\left(r_n^{\sigma(F)+\varepsilon}\right),$$

$$\left|\frac{F(z_n)}{f(z_n)}\right| \leq \exp\left(r_n^{\sigma(F)+\varepsilon} - r_n^{\rho}\right) \to 0 \quad (r_n \to +\infty), \tag{2.15}$$

where $|f(z_n)| = M(r_n, f)$ and $|z_n| = r_n$. Hence from (2.7) -(2.10) and (2.15), it follows that for all z_n satisfying $z_n \in E_2$, $|z_n| = r_n \notin E_1$ and $|f(z_n)| = M(r_n, f)$

$$\exp(\alpha |z_n|^{\mu}) \leq |z_n| \left[T(2|z_n|, f)\right]^{k+1} \left[1 + (k-1)\exp(\beta |z_n|^{\mu})\right]$$

$$+ o(1) \tag{2.16}$$

as $z_n \to \infty$. Now set $E = \{|z_n| : z_n \in E_2\} \backslash E_1 \subset [0, +\infty)$, then $\overline{dens}\, E > 0$ and

$$\exp\left(\alpha r_n^{\mu}\right) \leq d r_n \exp\left(\beta r_n^{\mu}\right) \left[T\left(2r_n, f\right)\right]^{k+1} \tag{2.17}$$

as $|z_n| = r_n \to +\infty$ in E, where $d\,(>0)$ is some constant. Therefore,

$$\sigma_2(f) = \overline{\lim_{r_n \to +\infty}} \frac{\log\log T(r_n, f)}{\log r_n} \geq \mu. \tag{2.18}$$

Lemma 2.4. *([5]) Let f be an entire function of order σ, where $0 < \sigma < 1/2$, and let $\varepsilon > 0$ be a given constant. Then there exists a set $E \subset [0, +\infty)$ with $\overline{dens}\, E \geq 1 - 2\sigma$ such that for all z satisfying $|z| = r \in E$, we have*

$$|f(z)| \geq \exp\left(r^{\sigma - \varepsilon}\right). \tag{2.19}$$

Lemma 2.5. *([9]) Let f be a meromorphic function for which $i(f) = p \geq 1$ and $\sigma_p(f) = \sigma$, and let $k \geq 1$ be an integer. Then for any $\varepsilon > 0$,*

$$m\left(r, \frac{f^{(k)}}{f}\right) = O\left(\exp_{p-2} r^{\sigma+\varepsilon}\right), \tag{2.20}$$

outside of a possible exceptional set E of finite linear measure.

To avoid some problems caused by the exceptional set we recall the following Lemma.

Lemma 2.6. *([1], [9]) Let $g : [0, +\infty) \to \mathbf{R}$ and $h : [0, +\infty) \to \mathbf{R}$ be monotone non-decreasing functions such that $g(r) \leq h(r)$ outside of an exceptional set E of finite linear measure. Then for any $\alpha > 1$, there exists $r_0 > 0$ such that $g(r) \leq h(\alpha r)$ for all $r > r_0$.*

3. PROOF OF THEOREM 1.3

Let $\max\{\sigma_p(A_j)\ (j \neq s), \sigma_p(F)\} = \beta < \sigma_p(A_s) = \alpha$. Suppose that f is a transcendental solution of (1.2) with $\sigma = \sigma_p(f) < +\infty$. It follows from (1.2) that

$$A_s(z) = \frac{F(z)}{f^{(s)}} - \frac{f^{(k)}}{f^{(s)}} - A_{k-1}(z)\frac{f^{(k-1)}}{f^{(s)}} - \dots - A_{s+1}(z)\frac{f^{(s+1)}}{f^{(s)}}$$

$$- A_{s-1}(z)\frac{f^{(s-1)}}{f^{(s)}} - \dots - A_1(z)\frac{f'}{f^{(s)}} - A_0(z)\frac{f}{f^{(s)}}. \tag{3.1}$$

Applying Lemma 2.5, we have

$$m\left(r, \frac{f^{(j+1)}}{f}\right) = O\left(\exp_{p-2} r^{\sigma+\varepsilon}\right) \quad (j = 0, \dots, k-1), \tag{3.2}$$

holds for all r outside a set $E \subset (0, +\infty)$ with a linear measure $m(E) = \delta < +\infty$. Since $N\left(r, f^{(j+1)}\right) \leq (j+2)\, N(r, f)$, it holds for $j = 0, \dots, k-1$ that

$$T\left(r, f^{(j+1)}\right) = m\left(r, f^{(j+1)}\right) + N\left(r, f^{(j+1)}\right)$$

$$\leq m\left(r, \frac{f^{(j+1)}}{f}\right) + m(r, f) + (j+2)\, N(r, f)$$

$$\leq (j+2)\, T(r, f) + m\left(r, \frac{f^{(j+1)}}{f}\right). \tag{3.3}$$

By (3.3), we can obtain from (3.1) and (3.2) that

$$T(r, A_s) \leq T(r, F) + cT(r, f) + \sum_{j \neq s} T(r, A_j)$$

$$+ O\left(\exp_{p-2} r^{\sigma+\varepsilon}\right) \ (r \notin E), \tag{3.4}$$

where c is a constant. Since $\sigma_p(A_s) = \alpha$, there exists $\left\{r_n'\right\}$ $\left(r_n' \to +\infty\right)$ such that

$$\lim_{r_n' \to +\infty} \frac{\log_p T\left(r_n', A_s\right)}{\log r_n'} = \alpha. \tag{3.5}$$

Since $m(E) = \delta < +\infty$, there exists a point $r_n \in \left[r_n', r_n' + \delta + 1\right] - E$. From

$$\frac{\log_p T(r_n, A_s)}{\log r_n} \geq \frac{\log_p T\left(r_n', A_s\right)}{\log\left(r_n' + \delta + 1\right)} = \frac{\log_p T\left(r_n', A_s\right)}{\log r_n' + \log\left(1 + (\delta+1)/r_n'\right)} \tag{3.6}$$

we get

$$\underline{\lim}_{r_n \to +\infty} \frac{\log_p T(r_n, A_s)}{\log r_n} \geq \alpha. \tag{3.7}$$

So for any given $\varepsilon \left(0 < 2\varepsilon < \alpha - \beta\right)$, and for $j \neq s$

$$T(r_n, A_j) \leq \exp_{p-1} r_n^{\beta+\varepsilon}, \ T(r_n, F) \leq \exp_{p-1} r_n^{\beta+\varepsilon} \tag{3.8}$$

and

$$T(r_n, A_s) > \exp_{p-1} r_n^{\alpha-\varepsilon} \tag{3.9}$$

hold for sufficiently large r_n. By (3.4), (3.8) and (3.9) we obtain for sufficiently large r_n

$$\exp_{p-1} r_n^{\alpha-\varepsilon} \leq k \exp_{p-1} r_n^{\beta+\varepsilon} + cT(r_n, f) + O\left(\exp_{p-2} r_n^{\sigma+\varepsilon}\right). \tag{3.10}$$

Therefore,

$$\overline{\lim}_{r_n \to +\infty} \frac{\log_p T(r_n, f)}{\log r_n} \geq \alpha - \varepsilon \tag{3.11}$$

and since $\varepsilon > 0$ is arbitrary, we get $\sigma_p(f) \geq \sigma_p(A_s) = \alpha$. This proves Theorem 1.3.

4. PROOF OF THEOREM 1.4

Setting $\max\left\{\sigma_p(A_j) \ \ (j \neq s), \sigma_p(F)\right\} = \beta$, then for a given $\varepsilon > 0$, we have

$$T(r, A_j) \leq \exp_{p-1} r^{\beta+\varepsilon} \ \ (j \neq s), \ T(r, F) \leq \exp_{p-1} r^{\beta+\varepsilon} \tag{4.1}$$

for sufficiently large r. Now we can write from (1.2)

$$A_s(z) = \frac{F(z)}{f^{(s)}} - \frac{f^{(k)}}{f^{(s)}} - A_{k-1}(z)\frac{f^{(k-1)}}{f^{(s)}} - ... - A_{s+1}(z)\frac{f^{(s+1)}}{f^{(s)}}$$

$$- A_{s-1}(z)\frac{f^{(s-1)}}{f^{(s)}} - ... - A_1(z)\frac{f'}{f^{(s)}} - A_0(z)\frac{f}{f^{(s)}}. \tag{4.2}$$

Hence by (3.3) and (4.2) we obtain that

$$T(r, A_s) \leq T(r, F) + cT(r, f) + \sum_{j=0}^{k-1} m\left(r, \frac{f^{(j+1)}}{f}\right) + \sum_{j \neq s} T(r, A_j), \tag{4.3}$$

where c is a constant. If $\sigma = \sigma_p(f) < +\infty$, then

$$m\left(r, \frac{f^{(j+1)}}{f}\right) = O\left(\exp_{p-2} r^{\sigma+\varepsilon}\right) \ \ (j = 0, ..., k-1) \tag{4.4}$$

holds for all r outside a set $E \subset (0,+\infty)$ with a linear measure $m(E) = \delta < +\infty$. For sufficiently large r, we have

$$T(r,f) \leq \exp_{p-1} r^{\sigma+\varepsilon}. \tag{4.5}$$

Thus

$$T(r,A_s) \leq k \exp_{p-1} r^{\beta+\varepsilon} + c \exp_{p-1} r^{\sigma+\varepsilon} + O\left(\exp_{p-2} r^{\sigma+\varepsilon}\right) \tag{4.6}$$

for $r \notin E$ and sufficiently large r. By Lemma 2.6, we have for any $\alpha > 1$

$$T(r,A_s) \leq k \exp_{p-1} (\alpha r)^{\beta+\varepsilon} + c \exp_{p-1} (\alpha r)^{\sigma+\varepsilon} + O\left(\exp_{p-2} (\alpha r)^{\sigma+\varepsilon}\right) \tag{4.7}$$

for sufficiently large r. Therefore,

$$\sigma_p(A_s) \leq \max\{\beta+\varepsilon,\ \sigma+\varepsilon\} < +\infty. \tag{4.8}$$

This contradicts the fact that $\sigma_p(A_s) = +\infty$.

5. PROOF OF THEOREM 1.5

Let $\beta < \sigma(A_0)$ and let f be a solution of (1.2). Suppose that $\beta < \alpha < \sigma(A_0)$ and that there is a set $E_\beta \subset [0,+\infty)$ of lower density 1 satisfying (1.5). Set

$$E_1 = \left\{ z : |z| = r \in E_\beta \text{ and } |A_j(z)| = \min_{|z|=r} |A_j(z)| \ \ (j = 1,2,...,k-1) \right\}. \tag{5.1}$$

Then $\underline{\text{dens}}\{|z| : z \in E_1\} = 1$ and

$$|A_j(z)| \leq \exp\left(r^\beta\right) \quad (j = 1,2,...,k-1) \tag{5.2}$$

for all $z \in E_1$. Also, from Lemma 2.4, there is a set $E_2 \subset [0,+\infty)$ of positive upper density such that for all z satisfying $|z| \in E_2$, we have

$$|A_0(z)| \geq \exp(r^\alpha). \tag{5.3}$$

Now let $E = \{z \in E_1 : |z| \in E_2\}$. Then with a set E and the number α, $A_0(z), ..., A_{k-1}(z)$ and $F(z)$ satisfy the hypothesis of Lemma 2.3 respectively. Hence we conclude by Lemma 2.3 that every solution f of equation (1.2) satisfies $\sigma(f) = +\infty$ and

$$\sigma_2(f) = \overline{\lim_{r\to+\infty}} \frac{\log\log T(r,f)}{\log r} \geq \alpha \tag{5.4}$$

with at most one exceptional solution f_0 satisfying $\sigma(f_0) < +\infty$. Thus the result of the theorem follows since α is arbitrary.

Next, we give an example that illustrates Theorem 1.5.

Example 5.1. *Let $P_1(z)$,..., $P_{k-1}(z)$ be nonconstant polynomials, and let $h_1(z)$,..., $h_{k-1}(z)$ be entire functions satisfying $\sigma(h_j) < \deg P_j$ $(j = 1,...,k-1)$. Let $A_0(z)$, $F(z) \not\equiv 0$ be entire functions, where $0 < \sigma(A_0) < 1/2$ and $\sigma(F) = \sigma < +\infty$. Then, by Theorem 1.5, every solution f of the equation*

$$f^{(k)} + h_{k-1}(z) e^{P_{k-1}(z)} f^{(k-1)} + ... + h_1(z) e^{P_1(z)} f' + A_0(z) f = F(z) \tag{5.5}$$

is of infinite order and $\sigma_2(f) \geq \sigma(A_0)$ with at most one exceptional solution f_0 satisfying $\sigma(f_0) < +\infty$ since

$$\min_{|z|=r} \left| h_j(z) e^{P_j(z)} \right| \to 0 \quad (j = 1,...,k-1)$$

as $r \to +\infty$.

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