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POWER AND EULER-LAGRANGE NORMS

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ABSTRACT. We introduce the notions of power and Euler-Lagrange norms by replacing the triangle inequality, in the definition of norm, by appropriate inequalities. We prove that every usual norm is a power norm and vice versa. We also show that every norm is an Euler-Lagrange norm and that the converse is true under certain condition.

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1. INTRODUCTION AND PRELIMINARIES

We introduce the notions of power norm and Euler-Lagrange by replacing the triangle inequality, in the definition of norm, by interesting inequalities. The reader is referred to [2] for undefined terms and notations.

We shall need the following lemma [1]. For the sake of completeness we state its proof.

Lemma 1.1. *Let $\mathcal{X}$ be a real or complex linear space. Let $\|\cdot\| : \mathcal{X} \to [0,\infty)$ be a mapping satisfying (1) and (2) in the definition of a q-norm. Then $\|\cdot\|$ is a norm if and only if the set $B = \{x \mid \|x\| \leq 1\}$ is convex.*

Proof. If $\|\cdot\|$ is a norm, then B is clearly a convex set. Conversely, let B be convex and $x, y \in \mathcal{X}$. We can assume that $x \neq 0, y \neq 0$. Putting $x' = \frac{x}{\|x\|}$ and $y' = \frac{y}{\|y\|}$ we have $x', y' \in B$.

Now $\lambda x' + (1-\lambda) y' \in B$ for all $0 \leq \lambda \leq 1$. In particular, for $\lambda = \frac{\|x\|}{\|x\|+\|y\|}$ we obtain

$$\|\frac{x}{\|x\|+\|y\|} + \frac{y}{\|x\|+\|y\|}\| = \|\lambda x' + (1-\lambda) y'\| \leq 1.$$

So that $\|x+y\| \leq \|x\| + \|y\|$. ∎

2. POWER NORM

We start this section with the definition of power norm by using a more general inequality than the triangle inequality.

Definition 2.1. Let $\mathcal{X}$ be a real or complex linear space, q, p, r be non-negative fixed numbers such that $q \geq 2$ and $\frac{p}{r} = \alpha + \sqrt{\alpha^2 - 1}$ with $\alpha = 2^{q-1} - 1$. A mapping $\|\cdot\| : \mathcal{X} \to [0,\infty)$ is called a power norm on $\mathcal{X}$ if it satisfies the following conditions:

(1) $\|x\| = 0 \Leftrightarrow x = 0$,
(2) $\|\lambda x\| = \|\lambda\|\|x\|$ for all $x \in \mathcal{X}$ and all scalar λ,
(3) $\frac{\|x_1+x_2\|^q}{p+r} \leq \frac{\|x_1\|^q}{p} + \frac{\|x_2\|^q}{r}$, for all $x, y \in \mathcal{X}$.

Remark 2.1. Let $q \geq 2$ be given. The condition $\frac{p}{r} = \alpha + \sqrt{\alpha^2 - 1}$, where $\alpha = 2^{q-1} - 1$ implies that $\frac{p}{r}$ satisfies the equation $x^2 + (2 - 2^q)x + 1 = 0$, which is converted, in turn, to $(p+r)^2 = 2^q pr$.

Our first result reads as follows.

Proposition 2.1. *Every usual norm is a power norm.*

Proof. The function $f(t) = \frac{1}{p} + \frac{t^q}{r} - \frac{(1+t)^q}{p+r}$ has the nonnegative derivative $f'(t) = \frac{q}{r}t^{q-1} - \frac{q}{p+r}(1+t)^{q-1}$ on the interval $[1,\infty)$ and thus it is monotonically increasing. In fact, the condition $\frac{p}{r} \geq \alpha$ implies that for $t \geq 1$, we have $\frac{1}{\sqrt[q-1]{1+\frac{p}{r}}-1} \leq 1 \leq t$ and so $(1+\frac{1}{t})^{q-1} \leq \frac{p}{r} + 1$ or $\frac{1}{r}t^{q-1} \geq \frac{1}{p+r}(1+t)^{q-1}$.

Therefore $f(t) \geq f(1) = \frac{1}{p} + \frac{1}{r} - \frac{1}{p+r}2^q \geq 0$ for all $t \geq 1$. Note that $\frac{1}{p} + \frac{1}{r} - \frac{1}{p+r}2^q \geq 0$ holds whenever $pr2^q \leq (p+r)^2$.

Thus $\frac{1}{p} + \frac{(\frac{\|y\|}{\|x\|})^q}{r} - \frac{(1+\frac{\|y\|}{\|x\|})^q}{p+r} \geq 0$ whenever $\|x\| \leq \|y\|$. Therefore $\frac{\|x+y\|^q}{p+r} \leq \frac{(\|x\|+\|y\|)^q}{p+r} \leq \frac{\|x\|^q}{p} + \frac{\|y\|^q}{r}$ for all $x, y \in \mathcal{X}$. It follows that $\|.\|$ is a power norm. ∎

Using some ideas of [1], we prove our second result.

Theorem 2.2. *Every power norm is a usual norm.*

Proof. We shall show that $B = \{x : \|x\| \le 1\}$ is convex. Let $x, y \in B$. Then we have

$$\|x+y\|^q \le (p+r)\left(\frac{\|x\|^q}{p} + \frac{\|y\|^q}{r}\right) \le (p+r)\left(\frac{1}{p} + \frac{1}{r}\right) = 2^q,$$

whence $\|\frac{x+y}{2}\|^q \le 1$, so $\frac{1}{2}x + \left(1 - \frac{1}{2}\right) y \in B$. Thus if

$$A := \left\{\frac{k}{2^n} \mid n = 1, 2, \ldots; k = 0, 1, \ldots, n\right\},$$

then for each $\lambda \in A$ we have $\lambda x + (1-\lambda) y \in B$.

Let $0 \le \lambda \le 1$ and $z = \lambda x + (1-\lambda) y$. Since A is dense in $[0,1]$, there exists a decreasing sequence $\{r_n\}$ in A such that $\lim_n r_n = \lambda$. Put $\beta_n = \frac{1-r_n}{1-\lambda}$. Obviously $0 \le \beta_n \le 1$, $\lim_n \beta_n = 1$ and $\frac{r_n+\beta_n-1}{r_n} \le 1$. Since $\frac{r_n+\beta_n-1}{r_n} x \in B$ and $r_n \in A$ we conclude that

$$\beta_n z = \lambda \beta_n x + (1-\lambda) \beta_n y = r_n \frac{r_n + \beta_n - 1}{r_n} x + (1-r_n) y \in B.$$

Thus $\beta_n \|z\| = \|\beta_n z\| \le 1$ for all n. Tending n to infinity we get $\|z\| \le 1$, i.e. $z \in B$. ∎

3. EULER-LAGRANGE NORM

We introduce the concept of Euler-Lagrange norm by replacing the triangle inequality by an Euler-Lagrange type inequality; cf. [3].

Definition 3.1. Let $\mathcal{X}$ be a real or complex linear space, m, m_1, m_2, a_1, a_2 be non-negative fixed numbers such that $m = m_1 a_1^2 + m_2 a_2^2$. A mapping $\|\cdot\| : \mathcal{X} \to [0, \infty)$ is called an Euler-Lagrange norm on $\mathcal{X}$ if it satisfies the following conditions:

(1) $\|x\| = 0 \Leftrightarrow x = 0$,
(2) $\|\lambda x\| = \|\lambda\| \|x\|$ for all $x \in \mathcal{X}$ and all scalar λ,
(3) $\frac{\|a_1 x_1 + a_2 x_2\|^2}{m} \le \frac{\|x_1\|^2}{m_1} + \frac{\|x_2\|^2}{m_2}$, for all $x, y \in \mathcal{X}$.

We are ready to prove our next result.

Proposition 3.1. *Every usual norm is an Euler-Lagrange norm.*

Proof. Assume that $m_2 a_2 \le m_1 a_1$. Consider the function $f(t) = \frac{t^2}{m_2} + \frac{1}{m_1} - \frac{(a_1 + t a_2)^2}{m}$ having the derivative $f'(t) = \frac{2t}{m_2} - \frac{2a_2(a_1+a_2 t)}{m}$. Evidently, $f'(t) \ge 0$ if and only if $\frac{m_2 a_2}{m_1 a_1} \le t$. Hence f is monotonically increasing on $[\frac{m_2 a_2}{m_1 a_1}, \infty)$. In particular, for all $t \ge 1$, we have $f(t) \ge f(1) \ge f(\frac{m_2 a_2}{m_1 a_1}) = 0$.

Thus $f(\frac{\|y\|}{\|x\|}) = \frac{(\frac{\|y\|}{\|x\|})^2}{m_2} + \frac{1}{m_1} - \frac{(a_1 + \frac{\|y\|}{\|x\|} a_2)^2}{m} \ge 0$ whenever $\|x\| \le \|y\|$. Therefore $\frac{\|a_1 x + a_2 y\|^2}{m} \le \frac{(a_1\|x\| + a_2\|y\|)^2}{m} \le \frac{\|x\|^2}{m_1} + \frac{\|y\|^2}{m_2}$ for all $x, y \in \mathcal{X}$. It follows that $\|.\|$ is an Euler-Lagrange norm.

In the case that $m_1 a_1 \le m_2 a_2$ we can apply the same method by using the function $f(t) = \frac{t^2}{m_1} + \frac{1}{m_2} - \frac{(t a_1 + a_2)^2}{m}$. ∎

Our last result is the following.

Theorem 3.2. *Every Euler-Lagrange norm is a usual norm if* $m_1 a_1^2 = m_2 a_2^2$.

Proof. Let $B = \{x : \|x\| \leq 1\}$ and let $x, y \in B$. We have

$$\begin{aligned} \|\frac{x+y}{2}\|^2 &\leq \frac{\|a_1\frac{x}{a_1} + a_2\frac{y}{a_2}\|^2}{4} \\ &\leq \frac{(a_1\frac{\|x\|}{a_1} + a_2\frac{\|y\|}{a_2})^2}{4} \\ &\leq \frac{m}{4}\left(\frac{1}{m_1}\frac{\|x\|^2}{a_1^2} + \frac{1}{m_2}\frac{\|y\|^2}{a_2^2}\right) \\ &\leq \frac{m^2}{4m_1m_2a_1^2a_2^2} \\ &= 1, \end{aligned}$$

whence $\frac{1}{2}x + \left(1 - \frac{1}{2}\right) y \in B$.

The rest of the proof is similar to the last part of the proof of Theorem 2.2. ∎

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