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A COEFFICIENT INEQUALITY FOR CERTAIN SUBCLASSES OF ANALYTIC FUNCTIONS RELATED TO COMPLEX ORDER

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ABSTRACT. In this present investigation, the authors obtain coefficient inequality for certain normalized analytic functions of complex order $f(z)$ defined on the open unit disk for which $1 + \frac{1}{b}\left[\frac{zf'(z)+\alpha z^2 f''(z)}{(1-\alpha)f(z)+\alpha z f'(z)} - 1\right]$ $(0 \leq \alpha \leq 1$ and $b \neq 0$ be a complex number$)$ lies in a region starlike with respect to 1 and is symmetric with respect to the real axis. Also certain applications of the main result for a class of functions of complex order defined by convolution are given. As a special case of this result, coefficient inequality for a class of functions defined through fractional derivatives is obtained. The motivation of this paper is to give a generalization of the coefficient inequalities of the subclasses of starlike and convex functions of complex order.

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1. INTRODUCTION

Let $\mathcal{A}$ denote the class of all *analytic* functions $f(z)$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (z \in \Delta := \{z \in \mathbb{C} : |z| < 1\}) \tag{1.1}$$

and $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of univalent functions. Let $\phi(z)$ be an analytic function with positive real part on Δ with $\phi(0) = 1, \phi'(0) > 0$ which maps the unit disk Δ onto a region starlike with respect to 1 which is symmetric with respect to the real axis. Let $S^*(\phi)$ be the class of functions in $f \in \mathcal{S}$ for which

$$\frac{zf'(z)}{f(z)} \prec \phi(z), \quad (z \in \Delta)$$

and $C(\phi)$ be the class of functions in $f \in \mathcal{S}$ for which

$$1 + \frac{zf''(z)}{f'(z)} \prec \phi(z), \quad (z \in \Delta),$$

where $\prec$ denotes the subordination between analytic functions. These classes were introduced and studied by Ma and Minda [5]. They have obtained the Fekete-Szegö inequality for the functions in the class $C(\phi)$. Since $f(z) \in C(\phi)$ if and only if $zf'(z) \in S^*(\phi)$, we get the Fekete-Szegö problem for functions in the class $S^*(\phi)$.

For a brief history of Fekete-Szegö problem for the class of starlike, convex and close-to-convex functions, see the paper by Srivastava et al. [9].

Very recently Ravichandran et al. [8] introduced the following classes of functions involving complex order.

Definition 1.1. Let $b \neq 0$ be a complex number. Let $\phi(z)$ be an analytic function with positive real part on Δ with $\phi(0) = 1$, $\phi'(0) > 0$ which maps the unit disk Δ onto a region starlike with respect to 1 which is symmetric with respect to the real axis. Then the class $S_b^*(\phi)$ consists of all analytic functions $f \in \mathcal{A}$ satisfying

$$1 + \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - 1\right) \prec \phi(z).$$

The class $C_b(\phi)$ consists of functions $f \in \mathcal{A}$ satisfying

$$1 + \frac{1}{b}\frac{zf''(z)}{f'(z)} \prec \phi(z).$$

They have obtained the Fekete-Szegö inequalities for functions in these classes.

Motivated by the aforementioned works, we obtain the coefficient inequality for functions of complex order in a more general class $M_{\alpha,b}(\phi)$ which we define below. Also we give applications of our results to certain functions defined through convolution (or the Hadamard product) and in particular we consider a class $M^{\lambda}_{\alpha,b}(\phi)$ of functions defined by fractional derivatives. The motivation of this paper is to give a generalization of the coefficient inequalities of the subclasses of starlike and convex functions of complex order obtained Ravichandran et al. [8].

Definition 1.2. Let $b \neq 0$ be a complex number. Let $\phi(z)$ be an analytic function with positive real part on Δ with $\phi'(0) = 1$, $\phi'(0) > 0$ which maps the unit disk Δ onto a region starlike with respect to 1 which is symmmetric with respect to the real axis. Then the class $M_{\alpha,b}(\phi)$ consists of all functions $f \in \mathcal{A}$ satisfying

$$1+\frac{1}{b}\left[\frac{zf'(z)+\alpha z^2 f''(z)}{(1-\alpha)f(z)+\alpha z f'(z)}-1\right] \prec \phi(z) \quad (0 \le \alpha \le 1).$$

For fixed $g \in \mathcal{A}$, we define the class $M^g_{\alpha,b}(\phi)$ to be the class of functions $f \in \mathcal{A}$ for which $(f * g) \in M^g_{\alpha,b}(\phi)$.

To prove our main result, we need the following :

Lemma 1.1. [8] *If* $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$ *is a function with positive real part, then*

$$|c_2 - \mu c_1^2| \le 2\max\{1, |2\mu - 1|\}$$

and the result is sharp for the functions given by

$$p(z) = \frac{1+z^2}{1-z^2}, \quad p(z) = \frac{1+z}{1-z}.$$

2. COEFFICIENT PROBLEM

Our main result is the following :

Theorem 2.1. *Let* $\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$. *If* $f(z)$ *given by (1.1) belongs to* $M_{\alpha,b}(\phi)$*, then*

$$|a_3 - \mu a_2^2| \le \frac{B_1|b|}{2(1+2\alpha)} \max\left\{1, \left|\frac{B_2}{B_1} + \left(1 - \frac{2\mu(1+2\alpha)}{(1+\alpha)^2}\right) bB_1\right|\right\}.$$

The result is sharp.

Proof. If $f(z) \in M_{\alpha,b}(\phi)$, then there is a Schwarz function $w(z)$, analytic in Δ with $w(0) = 0$ and $|w(z)| < 1$ in Δ such that

$$1+\frac{1}{b}\left[\frac{zf'(z)+\alpha z^2 f''(z)}{(1-\alpha)f(z)+\alpha z f'(z)}-1\right] = \phi(w(z)). \tag{2.1}$$

Define $p_1(z)$ by

$$p_1(z) = \frac{1+w(z)}{1-w(z)} = 1 + c_1 z + c_2 z^2 + \cdots. \tag{2.2}$$

Since $w(z)$ is a Schwarz function, we see that $\Re(p_1(z)) > 0$ and $p_1(0) = 1$. Define the function $p(z)$ by

$$p(z) = 1+\frac{1}{b}\left[\frac{zf'(z)+\alpha z^2 f''(z)}{(1-\alpha)f(z)+\alpha z f'(z)}-1\right] = 1 + b_1 z + b_2 z^2 + \cdots. \tag{2.3}$$

In view of equations (2.1), (2.2), (2.3), we have

$$p(z) = \phi\left(\frac{p_1(z)-1}{p_1(z)+1}\right). \tag{2.4}$$

Since

$$\frac{p_1(z)-1}{p_1(z)+1} = \frac{1}{2}\left[c_1 z + \left(c_2 - \frac{c_1^2}{2}\right) z^2 + \left(c_3 - \frac{c_1^3}{4} - c_1 c_2\right) z^3 + \cdots\right]$$

and therefore

$$\phi\left(\frac{p_1(z)-1}{p_1(z)+1}\right) = 1 + \frac{1}{2}B_1 c_1 z + \left[\frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{B_2 c_1^2}{4}\right] z^2 + \cdots.$$

from this equation (2.4), we obtain

$$b_1 = \frac{B_1 c_1}{2}, \tag{2.5}$$

$$b_2 = \frac{1}{2}\left(B_1\left(c_2 - \frac{1}{2}c_1^2\right)\right) + \frac{1}{4}B_2 c_1^2. \tag{2.6}$$

from the equation (2.3), we obtain

$$a_2 = \frac{bb_1}{(1+\alpha)}, \tag{2.7}$$

$$a_3 = \frac{bb_2 + b^2 b_1^2}{2(1+2\alpha)}. \tag{2.8}$$

By applying (2.5), (2.6) in (2.7) and (2.8) we have

$$\begin{aligned} a_2 &= \frac{bB_1 c_1}{2(1+\alpha)}, \\ a_3 &= \frac{bB_1 c_2}{4(1+2\alpha)} + \frac{c_1^2}{8(1+2\alpha)}\left[b^2 B_1^2 - b(B_1 - B_2)\right]. \end{aligned}$$

Therefore we have

$$a_3 - \mu a_2^2 = \frac{bB_1}{4(1+2\alpha)}\left[c_2 - vc_1^2\right] \tag{2.9}$$

where

$$v = \frac{1}{2}\left[1 - \frac{B_2}{B_1} + \left[\frac{2\mu(1+2\alpha)}{(1+\alpha)^2} - 1\right] bB_1\right].$$

Our result now follows by the application of Lemma 1.1. The result is sharp for the function defined by

$$1 + \frac{1}{b}\left[\frac{zf'(z) + \alpha z^2 f''(z)}{(1-\alpha)f(z) + \alpha z f'(z)} - 1\right] = \phi(z^2)$$

and

$$1 + \frac{1}{b}\left[\frac{zf'(z) + \alpha z^2 f''(z)}{(1-\alpha)f(z) + \alpha z f'(z)} - 1\right] = \phi(z).$$

∎

For $\alpha = 1$, in Theorem 2.1 we get the result obtained by Ravichandran et al. [8].

Corollary 2.2. *Let $\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$. If $f(z)$ given by (1.1) belongs to $S_b^*(\phi)$, then*

$$|a_3 - \mu a_2^2| \leq \frac{B_1|b|}{2}\max\left\{1; \left|\frac{B_2}{B_1} + (1-2\mu)bB_1\right|\right\}.$$

The result is sharp.

For a special case $\alpha = 0$, Theorem 2.1 reduces to another result obtained by Ravichandran et al. [8].

Corollary 2.3. *Let $\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$. If $f(z)$ given by (1.1) belongs to $C_b(\phi)$, then*

$$|a_3 - \mu a_2^2| \leq \frac{B_1|b|}{6}\max\left\{1; \left|\frac{B_2}{B_1} + \left(1 - \frac{3\mu}{2}\right)bB_1\right|\right\}.$$

The result is sharp.

Example 2.1. *By taking $\alpha = 0$, $b = (1-\beta)e^{-i\lambda}cos\lambda$, $\phi(z) = \frac{1+z}{1-z}$, we obtain the following sharp inequality for λ-spirallike function $f(z)$ of order β;*

$$|a_3 - \mu a_2^2| \leq \frac{(1-\beta)cos\lambda}{1+2\alpha}\max\left\{1, \left|e^{i\lambda} + 2\left(1 - \frac{2\mu(1+2\alpha)}{(1+\alpha)^2}\right)(1-\beta)cos\lambda\right|\right\}.$$

This result was obtained by Keogh and Merkes [4].

3. APPLICATIONS TO FUNCTION DEFINED BY FRACTIONAL DERIVATIVES

In order to introduce the class $M^{\lambda}_{\alpha,b}(\phi)$, we need the following :

Definition 3.1. (see [6, 7]; see also [10, 11]) Let $f(z)$ be analytic in a simply connected region of the z-plane containing the origin. The fractional derivative of f of order λ is defined by

$$D^{\lambda}_z f(z) = \frac{1}{\Gamma(1-\lambda)}\frac{d}{dz}\int_0^z \frac{f(\xi)}{(z-\xi)^{\lambda}}d\xi \quad (0 \leq \lambda < 1),$$

where the multiplicity of $(z-\xi)^{\lambda}$ is removed by requiring that $\log(z-\xi)$ is real for $z - \xi > 0$.

Using the above Definition 3.1 and its known extensions involving fractional derivatives and fractional integrals, Owa and Srivastava [6] introduced the operator $\Omega^{\lambda} : \mathcal{A} \to \mathcal{A}$ defined by

$$(\Omega^{\lambda} f)(z) = \Gamma(2-\lambda)z^{\lambda}D^{\lambda}_z f(z), \quad (\lambda \neq 2, 3, 4, \dots).$$

The class $M^{\lambda}_{\alpha,b}(\phi)$ consist of functions $f \in \mathcal{A}$ for which $\Omega^{\lambda} f \in M_{\alpha,b}(\phi)$. Note that $M^{0}_{0,b}(\phi) \equiv S^*_b(\phi)$ and $M^{\lambda}_{\alpha,b}(\phi)$ is a special case of the class $M^{g}_{\alpha,b}(\phi)$ when

$$g(z) = z + \sum_{n=2}^{\infty}\frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)}z^n. \tag{3.1}$$

Let

$$g(z) = z + \sum_{n=2}^{\infty} g_n z^n \quad (g_n > 0).$$

Since

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in M^{g}_{\alpha,b}(\phi)$$

iff

$$f(z) * g(z) = z + \sum_{n=2}^{\infty} g_n a_n z^n \in M_{\alpha,b}(\phi),$$

we obtain the coefficient estimate for functions in the class $M^{g}_{\alpha,b}(\phi)$, from the corresponding estimate for functions in the class $M_{\alpha,b}(\phi)$. Applying Theorem 2.1 for function

$$f * g(z) = z + g_2 a_2 z^2 + g_3 a_3 z^3 + \dots,$$

we get the following theorem after an obvious change of the parameter μ :

Theorem 3.1. *Let the function $\phi(z)$ be given by $\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$. If $f(z)$ given by (1.1) belongs to $M^{g}_{\alpha,b}(\phi)$, then*

$$|a_3 - \mu a_2^2| = \frac{B_1|b|}{2(1+2\alpha)g_3}\max\left\{1, \left|\frac{B_2}{B_1} + \left(1 - \frac{2\mu(1+2\alpha)g_3}{(1+\alpha)^2 g_2^2}\right)bB_1\right|\right\}.$$

The result is sharp.

Since

$$(\Omega^\lambda f)(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} a_n z^n,$$

we have

$$g_2 := \frac{\Gamma(3)\Gamma(2-\lambda)}{\Gamma(3-\lambda)} = \frac{2}{(2-\lambda)}, \tag{3.2}$$

$$g_3 := \frac{\Gamma(4)\Gamma(2-\lambda)}{\Gamma(4-\lambda)} = \frac{6}{(2-\lambda)(3-\lambda)}. \tag{3.3}$$

for g_2 and g_3 given by (3.2) and (3.3), Theorem 3.1 reduces to the following :

Theorem 3.2. *Let the function $\phi(z)$ be given by $\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$. If $f(z)$ given by (1.1) belongs to $M^\lambda_{\alpha,b}(\phi)$ then*

$$|a_3 - \mu a_2^2| \leq \frac{(2-\lambda)(3-\lambda)B_1|b|}{12(1+2\alpha)} \max\left\{1, \left|\frac{B_2}{B_1} + \left(1 - \frac{3\mu(1+2\alpha)(2-\lambda)}{(3-\lambda)(1+\alpha)^2}\right) bB_1\right|\right\}.$$

The result is sharp.

REFERENCES

[1] B. C. CARLSON and D. B. SHAFFER, Starlike and prestarlike hypergeometric functions, *SIAM J. Math. Anal.*, **15** (1984), no.4, pp. 737–745.

[2] P. L. DUREN, *Univalent Functions*, Springer, New York, 1983.

[3] A. W. GOODMAN, *Univalent Functions. Vol. I, II*, Mariner, Tampa, FL, 1983.

[4] F. R. KEOGH and E. P. MERKES, A coefficient inequality for certain classes of analytic functions, *Proc. Amer. Math. Soc.*, **20** (1969), pp. 8–12.

[5] W. C. MA and D. MINDA, A unified treatment of some special classes of univalent functions, *Proceedings of the Conference on Complex Analysis (Tianjin, 1992)*, pp. 157–169, Conf. Proc. Lecture Notes Anal., I, Int. Press, Cambridge, MA, 1994.

[6] S. OWA and H. M. SRIVASTAVA, Univalent and starlike generalized hypergeometric functions, *Canad. J. Math.*, **39** (1987), no.5, pp. 1057–1077.

[7] S. OWA, On the distortion theorems I. *Kyungpook Math. J.*, **18** (1978), no.1, pp. 53–59.

[8] V. RAVICHANDRAN, Y. POLATOGLU, M. BOLCAL and A. SEN, Certain subclasses of starlike and convex functions of complex order, *Hact. J. Math. Stat.*, **34** (2005), pp. 9-15.

[9] H. M. SRIVASTAVA, A. K. MISHRA and M. K. DAS, The Fekete-Szegö problem for a subclass of close-to-convex functions, *Complex Variables Theory Appl.*, **44** (2001), no.2, pp. 145–163.

[10] H. M. SRIVASTAVA and S. OWA, An application of the fractional derivative, *Math. Japon.*, **29** (1984), no.3, pp. 383–389.

[11] H. M. SRIVASTAVA and S. OWA, *Univalent Functions, Fractional Calculus, and their Applications*, Halsted Press / John Wiley and Sons, Chichester / New York, (1989).