The Australian Journal of Mathematical Analysis and Applications

AJMAA

Volume 4, Issue 1, Article 10, pp. 1-4, 2007

NECESSARY AND SUFFICIENT CONDITIONS FOR UNIFORM CONVERGENCE AND BOUNDEDNESS OF A GENERAL CLASS OF SINE SERIES

L. LEINDLER

Received 4 September, 2006; accepted 16 October, 2006; published 29 March, 2007.

BOLYAI INSTITUTE, UNIVERSITY OF SZEGED, ARADI VÉRTANÚK TERE 1, H-6720 SZEGED, HUNGARY.
leindler@math.u-szeged.hu

ABSTRACT. For all we know theorems pertaining to sine series with coefficients from the class γGBVS give only sufficient conditions. Therefore we define a subclass of γGBVS in order to produce necessary and sufficient conditions for the uniform convergence and boundedness if the coefficients of the sine series belong to this subclass; and prove two theorems of this type.

Key words and phrases: Sine series, Monotone sequences, Sequence of mean group bounded variation.

2000 *Mathematics Subject Classification.* Primary 40A05, 42A16.

ISSN (electronic): 1449-5910

The author was partially supported by the Hungarian National Foundation for Scientific Research under Grant # T042462.

1. INTRODUCTION

Several new kinds of sequences were defined for extending classical results having monotone coefficients. E.g., in [3] we defined the class of *sequences of rest bounded variation*, in brief, the class of RBVS, and showed that it is not comparable to the classical *quasi monotone sequences*, in symbol: CQMS. Utilizing this new class of sequences we generalized the classical theorem of CHAUNDY and JOLLIFFE [1]; replacing the monotone coefficients by a sequence of rest bounded variation. Soon after this LE and ZHOU [2] defined the class GBVS. They improved our theorem essentially using GBVSequences instead of RBVSequences. In their paper one can read a very nice survey of the recent results proved in this theme. Very recently we (in [5]) introduced a new class of sequences, the class $\boldsymbol{\gamma}\mathrm{RBVS}$. Its definition reads as follows:

Definition 1.1. Let $\boldsymbol{\gamma} := \{\gamma_n\}$ be a positive sequence. A null-sequence $\mathbf{c} := \{c_n\}$ $(c_n \to 0)$ of real numbers satisfying the inequalities

$$\sum_{n=m}^{\infty} |\Delta c_n| \le K(\mathbf{c})\gamma_m \qquad (\Delta c_n := c_n - c_{n+1}), \qquad m = 1, 2, \dots \tag{1.1}$$

with a positive constant $K(\mathbf{c})$ is said to be a sequence of $\boldsymbol{\gamma}$ rest bounded variation, in symbol: $\mathbf{c} \in \boldsymbol{\gamma}\mathrm{RBVS}$.

If $\boldsymbol{\gamma} \equiv \mathbf{c}$ and $c_n > 0$, then $\boldsymbol{\gamma}\mathrm{RBVS} \equiv \mathrm{RBVS}$, and if $\gamma_m := \max_{m \le n < m+N_0} |c_n|$ $(N_0 \in \mathbb{N})$, furthermore

$$\sum_{n=m}^{2m} |\Delta c_n| \le K(\mathbf{c})\gamma_m \tag{1.2}$$

is satisfied, we get the class GBVS. If $\boldsymbol{\gamma} := \{\gamma_n \ge 0\}$ is an arbitrary sequence and $\mathbf{c}$ satisfies (1.2) we get the class $\boldsymbol{\gamma}\mathrm{GBVS}$ (see [6]), which is wider than any one of the classes GBVS and $\boldsymbol{\gamma}\mathrm{RBVS}$.

It is easy to see that if $\mathbf{c} \in \mathrm{RBVS}$ then it is also *almost monotonic*, that is, for all $n \ge m$

$$c_n \le K(\mathbf{c})c_m,$$

but not if $\mathbf{c} \in \boldsymbol{\gamma}RBVS$ or $\mathbf{c} \in \boldsymbol{\gamma}GBVS$. If a sequence $\mathbf{c}$ belongs to these classes, it may have infinitely many zero terms, too. We emphasize that the sequence $\boldsymbol{\gamma}$ satisfying (1.2) may have a lot of zero terms, too; but not in (1.1), except the trivial case. It is easy to see that the condition (1.2) gives the greatest freedom for the terms of the sequences $\mathbf{c}$ and $\boldsymbol{\gamma}$. Because of this great generality, for sequences of these classes $\boldsymbol{\gamma}\mathrm{RBVS}$ and $\boldsymbol{\gamma}\mathrm{GBVS}$ we could give only conditions of sufficient type (see [4], [6]). This was one of the reasons that we (see [7]) introduced the class of *mean rest bounded variation* sequences, where $\boldsymbol{\gamma}$ is defined by a certain arithmetical mean of the coefficients, e.g.

$$\gamma_m := \frac{1}{m} \sum_{n=m}^{2m-1} |c_n|. \tag{1.3}$$

Now we define a further class of sequences as follows: If inequality (1.2) holds with γ_m given in ((1.3) then we say that $\mathbf{c}$ belongs to the class of sequences of *mean group bounded variation*, in symbol: $\mathbf{c} \in \mathrm{MGBVS}$.

The aim of the present paper is to show that if for all $b_n \ge 0$ and $\mathbf{b} := \{b_n\}$ belongs to MGBVS, then the series

$$\sum_{n=1}^{\infty} b_n \sin nx \tag{1.4}$$

is uniformly convergent if and only if

$$\sum_{n=m}^{2m-1} b_n \to 0. \tag{1.5}$$

We dare say that this result is a radical generalization of the classical theorem proved by Chaundy and Jolliffe, and many others proved recently.

We shall also show that the condition

$$\sum_{n=m}^{2m-1} b_n = O(1) \tag{1.6}$$

is necessary and sufficient that the partial sums of (1.4) should be uniformly bounded.

2. THEOREM

We establish the following theorem:

Theorem 2.1. *If a sequence* $\mathbf{b} := \{b_n\}$ *of nonnegative numbers belongs to the class* MGBVS*, then the conditions* (1.5) *and* (1.6)*, respectively, are both necessary and sufficient for the uniform convergence, or for the uniform boundedness of the partial sums of the series* (1.4)*.*

3. AUXILIARY RESULTS

We shall utilize the following theorems proved in [6].

Theorem 3.1. *Let* $\boldsymbol{\gamma} := \{\gamma_n\}$ *be a sequence of nonnegative numbers satisfying the condition* $\gamma_n = o(n^{-1})$*. If a sequence* $\mathbf{b} := \{b_n\} \in \boldsymbol{\gamma}\mathrm{GBVS}$*, then the series* (1.4) *is uniformly convergent, and consequently its sum function is continuous.*

Theorem 3.2. *If the sequence* $\boldsymbol{\gamma}$ *satisfies the condition* $\gamma_n = O(n^{-1})$ *and* $\mathbf{b} \in \boldsymbol{\gamma}\mathrm{GBVS}$*, then the partial sums of the series* (1.4) *are uniformly bounded.*

4. PROOF

Proof of Theorem 2.1. It is easy to see that the conditions (1.5) and (1.6), furthermore the assumption $\mathbf{b} \in \mathrm{MGBVS}$ imply that with the following sequence $\boldsymbol{\gamma} := \{\gamma_m\}$, where

$$\gamma_m := \frac{1}{m} \sum_{n=m}^{2m-1} b_n,$$

every presumption of the Theorems 3.1 and 3.2 are fulfilled, consequently the sufficiency parts of Theorem 2.1 are already proved.

Thus we have only to verify the necessity parts of our theorem, what is almost trivial.

Let $x = \pi/4m$. Then

$$\sum_{n=m}^{2m} b_n \sin nx \geq \sin\frac{\pi}{4} \sum_{n=m}^{2m} b_n \tag{4.1}$$

clearly holds. Taking into account that the series (1.4) converges uniformly, or its partial sums are uniformly bounded, the inequality (4.1) verifies the necessity of the conditions (1.5) and (1.6).

Herewith the proof is complete. ∎

Remark 4.1. We would like to call the attention to the fact that if we know only that $\mathbf{b}$ belongs to $\boldsymbol{\gamma}\mathrm{GBVS}$, then e.g. the condition $\gamma_m = o(m^{-1})$ is not necessary to the uniform convergence of (1.4); namely if $\gamma_m = m^{-1}$ and

$$b_n := \begin{cases} 2^{-m}, & \text{if } n = 2^m, \\ 0 & \text{elsewhere,} \end{cases} \tag{4.2}$$

then $\mathbf{b} := \{b_n\} \in \boldsymbol{\gamma}\mathrm{GBVS}$ and the series $\sum 2^{-m} \sin 2^m x$ converges uniformly, but $\gamma_m \neq o(m^{-1})$.

This shows that only the additional assumption $\mathbf{b} \in \mathrm{MGBVS}$ delivers that then the condition $\gamma_m = o(m^{-1})$ is already necessary, too.

Naturally the sequence $\mathbf{b}$ defined in (4.2) does not belong to MGBVS.

Remark 4.2. The series $\sum n^{-1} \sin nx$ is an example showing that the assumption $\mathbf{b} \in \mathrm{MGBVS}$ itself does not imply the uniform convergence; namely, then the condition (1.5) does not hold.

REFERENCES

[1] T. W. CHAUNDY and A. E. JOLLIFFE, The uniform convergence of a certain class of trigonometric series, *Proc. London Math. Soc.*, **15** (1916), pp. 214–216.

[2] R. J. LE and S. P. ZHOU, A new condition for uniform convergence of certain trigonometric series, *Acta Math. Hungar.*, **10** (2005) No. 1-2, pp. 161–169.

[3] L. LEINDLER, On the uniform convergence and boundedness of a certain class of sine series, *Analysis Math.*, **27** (2001), pp. 279–285.

[4] L. LEINDLER, Integrability of sine and cosine series having coefficients of a new class, *Austral. J. Math. Anal. Appl.*, **1** (2004), Iss. 1, Article 4, pp. 1–9. [`Online: http://ajmaa.org/cgi-bin/paper.pl?string=v1n1/V1I1P4.tex`].

[5] L. LEINDLER, Embedding results pertaining to strong approximation of sine series, *Acta Sci. Math. (Szeged)*, **71** (2005), pp. 91–103.

[6] L. LEINDLER, A new extension of monotone sequences and its applications, *J. Inequal. Pure Appl. Math.*, **7** (2006), No. 1, Article 39, [`Online: http://jipam.vu.edu.au/article.php?sid=656`].

[7] L. LEINDLER, Embedding results pertaining to strong approximation of Fourier series. VI, *Analysis Math.*, (submitted).