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SOME STABILITY RESULTS FOR FIXED POINT ITERATION PROCESSES

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ABSTRACT. In this paper, we present some stability results for both the general Krasnoselskij and the Kirk's iteration processes. The method of Berinde [1] is employed but a more general contractive condition than those of Berinde [1], Harder and Hicks [5], Rhoades [11] and Osilike [9] is considered.

Key words and phrases: Complete metric space; T-stability; Iteration process.

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1. INTRODUCTION

Let (E, d) be a complete metric space and $T : E \to E$ a selfmap of E. Let $F(T) = \{ p \in E \mid Tp = p \}$ be the fixed point set of T. For $x_0 \in E$, let

$$x_{n+1} = f(T,\ x_n),\ n = 0, 1, 2, \ldots \tag{1.1}$$

denote an iteration procedure which yields a sequence of points $\{x_n\}_{n=0}^{\infty}$, for some function f. Suppose that $\{x_n\}_{n=0}^{\infty}$ converges to a fixed point p of T and $\{y_n\}_{n=0}^{\infty} \subset E$. Set $\epsilon_n = d(y_{n+1}, f(T, y_n)),\ n = 0, 1, 2, \ldots$

Then, the iteration procedure in (1.1) is said to be T-stable or stable with respect to T if and only if $\lim_{n\to\infty} \epsilon_n = 0$ implies $\lim_{n\to\infty} y_n = p$. For certain contractive definitions, the stability of some iteration procedures has been studied by several authors. See for example Harder and Hicks [5], Rhoades [11, 12, 13], Osilike [9], Jachymski [7] and Berinde [1]. Harder and Hicks [5] showed that function iteration for mappings T satisfying various contractive definitions is T-stable and similarly for several iteration processes other than function iteration. Later, Rhoades [12, 13] extended some of the results of Harder and Hicks [5] to an independent contractive definition, and also proved stability results for some additional iteration procedures. In Rhoades [11] a more general contractive definition than those of Harder and Hicks [5] and Rhoades [12, 13] was employed. This was given by:

$$d(Tx, Ty) \leq c\max\left\{d(x, y), \frac{1}{2}\left[d(x, Tx) + d(y, Ty)\right], d(x, Ty), d(y, Tx)\right\}, \tag{1.2}$$

for each $x,\ y \in E$ and a constant $c \in [0, 1)$. Using (1.2), Rhoades [11] proved several stability results which are generalizations and extensions of most of the results of Harder and Hicks [5] and Rhoades [13]. Indeed, Rhoades showed that if T satisfies (1.2) then,

$$d(Tx, Ty) \leq \frac{c}{1-c} d(x, Tx) + cd(x, y)\ . \tag{1.3}$$

Osilike [9] employed the following contractive definition: there exist constants $a \in [0, 1)$ and $L \geq 0$ such that for each $x, y \in E$,

$$d(Tx, Ty) \leq Ld(x, Tx) + ad(x, y)\ . \tag{1.4}$$

Using (1.4), he established several stability results which are generalizations of most of the results of Rhoades [11].

In this paper, we establish some stability results for a more general contractive definition than those of Rhoades [11], Osilike [9], Harder and Hicks [5] and Berinde [1]. However, in the proofs of our results, we shall employ the method of Berinde [1] which was also used by Osilike [10]. For more details and references regarding the fixed point iteration processes and their stability, we refer to the recent monograph of Berinde [4].

2. PRELIMINARIES

Let $\{x_n\}_{n=0}^{\infty}$ be the sequence generated by the iteration procedure (1.1). Then, the general Krasnoselskij (Schaefer's) iteration process is obtained from (1.1) if

$$f(T,\ x_n) = (1-a)x_n + aTx_n,\ n = 0, 1, 2, \ldots,\ a \in [0, 1]$$

while the Kirk's iteration process is obtained for

$$f(T,\ x_n) = \sum_{i=0}^{k} \alpha_i T^i x_n, n \geq 0, \alpha_i \geq 0, \alpha_1 > 0 \text{ and } \sum_{i=0}^{k} \alpha_i = 1.$$

We shall employ the following contractive definition: there exist a constant $b \in [0,1)$ and a monotone increasing function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\varphi(0) = 0$, such that , for each $x, y \in E$,

$$d(Tx, Ty) \leq \varphi(d(x, Tx)) + bd(x, y) \ . \tag{2.1}$$

The contractive definition (2.1) is indeed more general in the following sense. If $\varphi(v) = Lv,\ L \geq 0$ in (2.1), then we obtain the contractive definition of Osilike [9]. If $\varphi(v) = \frac{c}{1-c}v$ in (2.1), then we have the contractive definition of Rhoades [11]. Again, if $L = 2\delta$ and $b = \delta$ in (1.4), where $\delta = \max\left\{\alpha, \frac{\beta}{1-\beta}, \frac{\gamma}{1-\gamma}\right\},\ 0 \leq \alpha < 1,\ 0 \leq \beta < 0.5, 0 \leq \gamma \leq 0.5$, then we obtain the Zamfirescu's contractive definition in Berinde [1], Harder and Hicks [5]. Furthermore, if $\varphi(u) = 0$, then (2.1) reduces to

$$d(Tx, Ty) \leq bd(x, y),\ b \in [0, 1), \tag{2.2}$$

which is the Banach's contraction condition as contained in Harder and Hicks [5], Berinde [1] and Zeidler [14].

We shall employ the following Lemmas in the sequel.

Lemma 2.1. (Berinde [1]) *: If δ is a real number such that $0 \leq \delta < 1$ and $\{\epsilon_n\}_{n=0}^{\infty}$ is a sequence of positive numbers such that $\lim_{n\to\infty} \epsilon_n = 0$, then for any sequence of positive numbers $\{u_n\}_{n=0}^{\infty}$ satisfying*

$$u_{n+1} \leq \delta u_n + \epsilon_n,\ n = 0, 1, \ldots, \tag{2.3}$$

we have

$$\lim_{n\to\infty} u_n = 0.$$

Lemma 2.2. *Let $(E,\ ||\cdot||)$ be a normed linear space and let $T : E \to E$ be a selfmap of E satisfying (2.1). Suppose that $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a subadditive, monotone increasing function such that $\varphi(0) = 0, \varphi(Lu) \leq L\varphi(u), L \geq 0$. Then, $\forall\ i \in N$, and $\forall\ x,\ y \in E$*

$$||T^i x - T^i y|| \leq \sum_{j=1}^{i} \binom{i}{j} b^{i-j} \varphi^j(||x - Tx||) + b^i ||x - y||. \tag{2.4}$$

Proof. We first establish that φ subadditive implies that each iterate φ^i of φ is also subadditive. Since φ is subadditive, we have $\varphi(x + y) \leq \varphi(x) + \varphi(y)$,$\forall\ x,\ y \in \mathbb{R}_+$. Therefore,using subadditivity of φ in φ^2 yields $\varphi^2(x + y) = \varphi(\varphi(x + y)) \leq \varphi(\varphi(x) + \varphi(y)) \leq \varphi(\varphi(x)) + \varphi(\varphi(y)) = \varphi^2(x) + \varphi^2(y)$, which implies that φ^2 is subadditive. Similarly, applying subadditivity of φ^2 in φ^3 , we get $\varphi^3(x + y) = \varphi(\varphi^2(x + y)) \leq \varphi(\varphi^2(x) + \varphi^2(y)) \leq \varphi(\varphi^2(x)) + \varphi(\varphi^2(y)) = \varphi^3(x) + \varphi^3(y)$, which implies that φ^3 is also subadditive. Hence, in general, each $\varphi^n, n = 1, 2, \ldots,$ is subadditive. The second part of the proof of the Lemma is by mathematical induction on i. If $i = 1$, then (2.4) becomes

$$\begin{aligned} ||Tx - Ty|| &\leq \sum_{j=1}^{1} \binom{1}{j} b^{1-j} \varphi^j(||x - Tx||) + b||x - y|| \ , \\ &= \varphi(||x - Tx||) + b||x - y|| \ . \end{aligned}$$

i.e. (2.4) reduces to (2.1) when $i = 1$ and hence the result holds. Assume that (2.4) holds for $i = m,\ m \in \mathbb{N}$, i.e.

$$||T^m x - T^m y|| \leq \sum_{j=1}^{m} \binom{m}{j} b^{m-j} \varphi^j(||x - Tx)||) + b^m ||x - y||.$$

We then show that the statement is true for $i = m+1$;

$$
\begin{aligned}
& ||T^{m+1}x - T^{m+1}y|| \\
= \; & ||T^m(Tx) - T^m(Ty)|| \\
\leq \; & \sum_{j=1}^{m} \binom{m}{j} b^{m-j}\varphi^j(||Tx - T^2x||) + b^m||Tx - Ty|| \\
\leq \; & \sum_{j=1}^{m} \binom{m}{j} b^{m-j}\varphi^j(\varphi(||x - Tx||) + b||x - Tx||) \\
+ \; & b^m(\varphi(||x - Tx||) + b||x - y|| \\
\leq \; & \sum_{j=1}^{m} \binom{m}{j} b^{m-j}\varphi^{j+1}(||x - Tx||) + \sum_{j=1}^{m} \binom{m}{j} b^{m+1-j}\varphi^j(||x - Tx||) \\
+ \; & b^m\varphi(||x - Tx||) + b^{m+1}||x - y|| \\
= \; & \binom{m+1}{m+1}\varphi^{m+1}(||x - Tx||) + \binom{m+1}{m} b\varphi^m(||x - Tx||) \\
+ \; & \binom{m+1}{m-1} b^2\varphi^{m-1}(||x - Tx||) + \ldots + \binom{m+1}{3} b^{m-2}\varphi^3(||x - Tx||) \\
+ \; & \binom{m+1}{2} b^{m-1}\varphi^2(||x - Tx||) + \binom{m+1}{1} b^m\varphi(||x - Tx||) + b^{m+1}||x - y|| \\
= \; & \sum_{j=1}^{m+1} \binom{m+1}{j} b^{m+1-j}\varphi^j(||x - Tx||) + b^{m+1}||x - y||.
\end{aligned}
$$

∎

Remark 2.1. The proof of Lemma 2.1 is contained in [1].

Remark 2.2. Lemma 2.2 above is more general than the Lemma of Osilike [9].

3. MAIN RESULTS

We now prove a stability result for the general Krasnoselskij (Schaefer's) iteration procedure.

Theorem 3.1. *Let* $(E, \, ||\cdot||)$ *be a normed linear space and* $T : E \to E$ *a selfmap of E satisfying (2.1). Suppose that T has a fixed point p. Let* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *be a monotone increasing function such that* $\varphi(0) = 0$. *Define the sequence* $\{x_n\}$ *iteratively for arbitrary* $x_0 \in E$ *by* $x_{n+1} = f(T, \, x_n) = (1-a)x_n + aTx_n, \; \forall \, n \in \mathbb{N}$ *where* $n \geq 0, \; a \in [0,1]$. *Then, the general Krasnoselskij (Schaefer's) iteration procedure above is T-stable.*

Proof. Let $\{y_n\}_{n=0}^{\infty} \subset E$ and define $\epsilon_n = ||y_{n+1} - (1-a)y_n - aTy_n||, \; n \geq 0$. Let $\lim\limits_{n\to\infty} \epsilon_n = 0$. Then , we shall prove that $\lim_{n\to\infty} y_n = p$ using (2.1) and the triangle inequality:

$$
\begin{aligned}
||y_{n+1} - p|| &\leq ||y_{n+1} - (1-a)y_n - aTy_n|| + ||(1-a)y_n + aTy_n - p|| \\
&= \epsilon_n + ||(1-a)y_n + aTy_n - [(1-a) + a]p|| \\
&\leq (1-a)||y_n - p|| + a||Ty_n - p|| + \epsilon_n \\
&= (1-a)||y_n - p|| + a||Tp - Ty_n|| + \epsilon_n \\
&\leq (1-a)||y_n - p|| + a\{\varphi(||p - Tp||) + b||p - y_n||\} + \epsilon_n \\
&= (1 - a + ab)||y_n - p|| + \epsilon_n. \qquad (3.1)
\end{aligned}
$$

Since $0 \leq 1 - a + ab < 1$, then by using Lemma 2.1 in (3.1), we have $\lim_{n\to\infty} ||y_n - p|| = 0$, which implies that,

$$\lim_{n\to\infty} y_n = p.$$

Conversely, let $\lim_{n\to\infty} y_n = p$. Then,

$$\begin{aligned}
\epsilon_n &= ||y_{n+1} - (1-a)y_n - aTy_n|| \\
&\leq ||y_{n+1} - p|| + ||p - (1-a)y_n - aTy_n|| \\
&\leq ||y_{n+1} - p|| + (1-a)||p - y_n|| + a||p - Ty_n|| \\
&= ||y_{n+1} - p|| + (1-a)||y_n - p|| + a||Tp - Ty_n|| \\
&\leq ||y_{n+1} - p|| + (1-a)||y_n - p|| + a[\varphi(||p - Tp||) + b||p - y_n||] \\
&= ||y_{n+1} - p|| + (1-a+ab)||y_n - p|| \to 0 \; as \; n \to \infty.
\end{aligned}$$

∎

We now prove a stability result for the Kirk's iteration process.

Theorem 3.2. *Let $(E, ||\cdot||)$ is a normed linear space and $T : E \to E$ a selfmap of E satisfying (2.1). Let $k \geq 1$ be a fixed integer, $x_0 \in E$, and let*

$$x_{n+1} = f(T, x_n) = \sum_{i=0}^{k} \alpha_i T^i x_n, \;\; n \geq 0, \; \alpha_i \geq 0, \; \alpha_1 > 0 \text{ and } \sum_{\mathrm{i}=0}^{\mathrm{k}} \alpha_{\mathrm{i}} = 1.$$

Suppose that T has a fixed point p. Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a subadditive, monotone increasing function such that $\varphi(0) = 0, \; \varphi(Lu) \leq L\varphi(u), \; L \geq 0$. Then, the Kirk's iteration process is T-stable.

Proof. Let $\{y_n\}_{n=0}^{\infty} \subset E$ and $\epsilon_n = ||y_{n+1} - \sum_{i=0}^{k} \alpha_i T^i y_n||$.

Let $\lim_{n\to\infty} \epsilon_n = 0$. Then, we shall prove that $\lim_{n\to\infty} y_n = p$, using Lemma 2.2 and the triangle inequality:

$$\begin{aligned}
||y_{n+1} - p|| &\leq ||y_{n+1} - \sum_{i=0}^{k} \alpha_i T^i y_n|| + ||\sum_{i=0}^{k} \alpha_i T^i y_n - p|| \\
&= \epsilon_n + ||\sum_{i=0}^{k} \alpha_i T^i y_n - \sum_{i=0}^{k} \alpha_i T^i p|| \\
&\leq \sum_{i=0}^{k} \alpha_i ||T^i y_n - T^i p|| + \epsilon_n \\
&= \alpha_0 ||p - y_n|| + \sum_{i=1}^{k} \alpha_i ||T^i p - T^i y_n|| + \epsilon_n \\
&\leq \sum_{i=1}^{k} \alpha_i \left\{ \sum_{j=1}^{i} \binom{i}{j} b^{i-j} \varphi^j(||p - Tp||) + b^i ||p - y_n|| \right\} + \epsilon_n + \alpha_0 ||y_n - p|| \\
&= \sum_{i=0}^{k} \alpha_i b^i ||y_n - p|| + \epsilon_n, \qquad (3.2)
\end{aligned}$$

since $\varphi^j(0) = \varphi(0) = 0$.

Since $0 \leq \sum_{i=0}^{k} \alpha_i b^i < 1$, then using Lemma 2.1 in (3.2) yields

$$\lim_{n\to\infty} ||y_n - p|| = 0,$$

that is,

$$\lim_{n\to\infty} y_n = p.$$

Conversely, let $\lim_{n\to\infty} y_n = p$. Then,

$$\begin{aligned}
&\epsilon_n \\
&= ||y_{n+1} - \sum_{i=o}^{k} \alpha^i T^i y_n|| \\
&\leq ||y_{n+1} - p|| + ||p - \sum_{i=0}^{k} \alpha_i T^i y_n|| \\
&\leq ||y_{n+1} - p|| + \sum_{i=0}^{k} \alpha_i ||T^i p - T^i y_n|| \\
&= ||y_{n+1} - p|| + \alpha_0 ||p - y_n|| + \sum_{i=1}^{k} ||T^i p - T^i y_n|| \\
&\leq ||y_{n+1} - p|| + \sum_{i=1}^{k} \alpha_i \left\{ \sum_{j=1}^{i} \binom{i}{j} b^{i-j} \varphi^j(||p - Tp||) + b^i ||p - y_n|| \right\} + \alpha_0 ||y_n - p|| \\
&= ||y_{n+1} - p|| + \left[\sum_{i=0}^{k} \alpha_i b^i \right] ||y_n - p|| \to 0 \text{ as n} \to \infty,
\end{aligned}$$

since $\varphi^j(0) = 0$.
This completes the proof . ∎

Remark 3.1. Theorem 3.1 is a generalization of Theorem 3.1 of Imoru and Olatinwo [6], since we obtain Picard iteration with $a = 1$.

Remark 3.2. Theorem 3.2 in this paper is a generalization of Theorem 3 of Osilike [9] which is itself a generalization of Theorem 3 of Rhoades [11]. Theorem 3 of Rhoades [11] is also a generalization of both Theorem 4 of Harder and Hicks [5] and Theorem 3 of Rhoades [13].

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