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p-VALENT MEROMORPHIC FUNCTIONS INVOLVING HYPERGEOMETRIC AND KOEBE FUNCTIONS BY USING DIFFERENTIAL OPERATOR

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ABSTRACT. New classes of multivalent meromorphic functions involving hypergeometric and Koebe functions are introduced,we find some properties of these classes e.g. distortion bounds, radii of starlikeness and convexity, extreme points, Hadamard product and verify effect of some integral operator on members of these classes.

Key words and phrases: p-valent, meromorphic, hypergeometric and Koebe functions, distortion bound, radii of starlikeness and convexity, extreme points, convolution and integral operator.

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1. INTRODUCTION

Let $\mathcal{KH}(p)$ denote the class of functions of the form

$$f(z) = Az^{-p} + k(z) + {}_2F_1(a,b;c;z) - \sum_{k=p+1}^{2p} t_{k-p-1} z^{k-p-1} \tag{1.1}$$

where $A > 0, p \in I\!N, k(z) = \frac{z}{(1-z)^2}$ is the Koebe function and

$$2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n \left((a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}, c > b > 0, c > a+b \right) \tag{1.2}$$

is the Gaussian hypergeometric function and

$$t_q = \left(q + \frac{(a)_q (b)_q}{(c)_q q!} \right) \tag{1.3}$$

all the members of this class are analytic in the punctured disk $\mathcal{D}^* = \{z : 0 < |z| < 1\}$.

Definition 1.1. A function $f(z)$ belonging to $\mathcal{KH}(p)$ is said to be in the class $\mathcal{KH}^*(p,q,\lambda,\alpha,\beta)$ if it satisfies the condition

$$\left| \frac{z^{p+q+1} f^{(q+1)}(z) + Ab}{\lambda z^{p+q+1} f^{(q+1)}(z) - Ab + (1+\lambda)\alpha Ab} \right| < \beta \tag{1.4}$$

for some $0 < \alpha \leq 1, 0 \leq \lambda \leq 1$ and $(q = 0$ or $q = 2t), b = \frac{(p+q)!}{(p-1)!}$. For each $f(z) \in \mathcal{KH}(p)$ (here and through this paper) by direct calculation we have

$$f(z) = Az^{-p} + \sum_{n=p}^{\infty} t_n z^n \tag{1.5}$$

where t_n defined by (1.3) and

$$f^{(j)}(z) = (-1)^j A \frac{(p+j-1)!}{(p-1)! z^{p+j}} + \sum_{n=p}^{\infty} \frac{n!}{(n-j)!} t_n z^{n-j}. \tag{1.6}$$

A function $f(z) \in \mathcal{KH}(p)$ is said to be meromorphically multivalent starlike if

$$Re\left\{ -\frac{zf'}{f(z)} \right\} > 0$$

and meromorphically multivalent convex if

$$Re\left\{ -\left(1 + \frac{zf''}{f'} \right) \right\} > 0.$$

Motivated by the recent work of H. Irmak and S. Owa [1], S. Shams and S. R. Kulkarni [3] in the present paper we discuss some interesting properties of the class $\mathcal{KH}^*(p,q,\lambda,\alpha,\beta)$ such as radii of starlikeness and convexity, distortion theorem, extreme points, convolution, integral operator and etc.

2. MAIN RESULTS

For our main results we need the following lemma that has been proved in [2].

Lemma 2.1. *Let $f(z) \in \mathcal{KH}(p)$, then $f(z) \in \mathcal{KH}^*(p,q,\lambda,\alpha,\beta)$ if and only if*

$$\sum_{n=p}^{\infty} \frac{n!}{(n-q-1)!}(1+\lambda\beta)t_n \le \beta(\lambda+1)Ab(1-\alpha). \tag{2.1}$$

Now we find distortion bounds for $f^{(q+1)}(z)$ when $f(z) \in \mathcal{KH}^*(p,q,\lambda,\alpha,\beta)$.

Theorem 2.2. *Let $f(z) \in \mathcal{KH}^*(p,q,\lambda,\alpha,\beta)$ then for $|z| = r$*

$$\begin{aligned} \frac{A(p+q)!}{(p-1)!}\frac{1}{r^{p+q+1}} - r^{p-q-1}\frac{\beta(\lambda+1)Ab(1-\alpha)}{1+\lambda\beta} &\le \left|f^{(q+1)}(z)\right| \\ \le \frac{A(p+q)!}{(p-1)!}\frac{1}{r^{p+q+1}} + r^{p-q-1}\frac{\beta(\lambda_1(Ab(1-\alpha)}{1+\lambda\beta} & \end{aligned} \tag{2.2}$$

Proof. Let $f(z) \in \mathcal{KH}^*(p,q,\lambda,\alpha,\beta)$. In view of above lemma we have

$$\sum_{n=p}^{\infty} t_n \le \frac{\beta(\lambda+1)Ab(1-\alpha)(p-q-1)!}{p!(1+\lambda\beta)} \tag{2.3}$$

Thus

$$\begin{aligned} \left|f^{(q+1)}(z)\right| &= \left|(-1)^{q+1}A\frac{(p+q)!}{(p-1)!z^{p+q+1}} + \sum_{n=p}^{\infty}\frac{n!}{(n-q-1)!}t_n z^{n-q-1}\right| \\ &\le \frac{A}{r^{p+q+1}}\frac{(p+q)!}{(p-1)!} + r^{p-q-1}\frac{p!}{(p-q-1)!}\sum_{n=p}^{\infty} t_n. \end{aligned}$$

Now, making use of (2.3) we obtain

$$\left|f^{(q+1)}(z)\right| \le \frac{A(p+q)!}{(p-1)!}\frac{1}{r^{p+q+1}} + r^{p-q-1}\frac{\beta(\lambda+1)Ab(1-\alpha)}{(1+\lambda\beta)}.$$

Also

$$|f^{(q+1)}(z)| \ge \frac{A(p+q)!}{(p-1)!}\frac{1}{r^{p+q+1}} - r^{p-q-1}\frac{\beta(\lambda+1)Ab(1-\alpha)}{(1+\lambda\beta)}$$

which proves the assertion of Theorem 2.2. ∎

3. RADII OF STARLIKENESS AND CONVEXITY

In this section we find radii of starlikeness and convexity for the class $\mathcal{KH}^*(p,q,\lambda,\alpha,\beta)$.

Theorem 3.1. *If $f(z) \in \mathcal{KH}^*(p,q,\lambda,\alpha,\beta)$ then $f(z)$ is meromorphically multivalent starlike of order $\nu(0 \le \nu < p)$ in $|z| < r = r_1(p,q,\lambda,\alpha,\beta,\nu)$ where*

$$r_1(p,q,\lambda,\alpha,\beta) = \inf_n \left\{\frac{n!(1+\lambda\beta)(p-\nu)}{(\lambda+1)b(1-\alpha)(n-q-1)!(n+2p+\nu)}\right\}^{\frac{1}{n+p}} \tag{3.1}$$

Proof. Let $f(z) \in \mathcal{KH}^*(p,q,\lambda,\alpha,\beta)$. So by Lemma 2.1.

$$\sum_{n=p}^{\infty}\frac{n!}{(n-q-1)!}(1+\lambda\beta)t_n \le \beta(\lambda+1)Ab(1-\alpha)$$

Now it is enough to show that

$$\left|\frac{zf'}{f}+p\right|\leq p-\nu$$

But

$$\left|\frac{zf'}{f}+p\right|=\left|\frac{-pAz^{-p}+\sum\limits_{n=p}^{\infty}nt_nz^n}{Az^{-p}+\sum\limits_{n=p}^{\infty}t_nz^n}+p\right|$$

$$=\left|\frac{-pAz^{-p}+\sum\limits_{n=p}^{\infty}nt_nz^n+pAz^{-p}+\sum\limits_{n=p}^{\infty}pt_nz^n}{z^{-p}\left(A+\sum\limits_{n=p}^{\infty}t_nz^{n+p}\right)}\right|$$

$$\leq\frac{\sum\limits_{n=p}^{\infty}(n+p)t_n|z|^{n+p}}{A-\sum\limits_{n=p}^{\infty}t_nz^{n+p}}\leq p-\nu$$

or

$$\sum_{n=p}^{\infty}(n+p)t_n|z|^{n+p}\leq A(p-\nu)-(p-\nu)\sum_{n=p}^{\infty}t_nz^{n+p}$$

or

$$\sum_{n=p}^{\infty}\frac{(n+2p+\nu)}{A(p-\nu)}t_n|z|^{n+p}\leq 1.$$

Now by using (2.1) we have

$$t_n\leq\frac{\beta(\lambda+1)Ab(1-\alpha)(n-q-1)!}{n!(1+\lambda\beta)},n\geq p \tag{3.2}$$

Then

$$\sum_{n=p}^{\infty}\frac{(n+2p+\nu)}{A(p-\nu)}t_n|z|^{n+p}$$

$$\leq\sum_{n=p}^{\infty}\frac{\beta(\lambda+1)Ab(1-\alpha)(n-q-1)!}{n!(1+\lambda\beta)}|z|^{n+p}\left(\frac{n+2p+\nu}{A(p-\nu)}\right)\leq 1.$$

So it is enough to suppose

$$|z|^{n+p}\leq\frac{n!(1+\lambda\beta)(p-\nu)}{(\lambda+1)b(1-\alpha)(n-q-1)!(n+2p+\nu)}$$

and so the proof is complete. ∎

In view of Theorem 3.1 and since "f is convex if and only if zf' is starlike" we have

Corollary 3.2. *Let $f(z)\in\mathcal{KH}^*(p,q,\lambda,\alpha,\beta)$ then $f(z)$ is meromorphically convex of order $\nu(0\leq\nu<p)$ in $|z|<r=r_2(p,q,\lambda,\alpha,\beta,\nu)$ where*

$$r_2(p,q,\lambda,\alpha,\beta,\nu)=\inf_n\left\{\frac{p(p-\nu)(1+\lambda\beta)(n-1)}{(n+2p-\nu)\beta(\lambda+1)b(1-\alpha)(n-q-1)!}\right\}^{\frac{1}{n+p}} \tag{3.3}$$

4. EXTREME POINTS

We discuss the extreme points of $\mathcal{KH}^*(p,q,\lambda,\alpha,\beta)$ in the next theorem.

Theorem 4.1. *Let* $f_0(z) = az^{-p}, a > 1$ *and*

$$f_n(z) = z^{-p} + \frac{\beta(\lambda+1)Ab(1-\alpha)(n-q-1)!}{n!(1+\lambda\beta)} z^n,\ n \geq p \tag{4.1}$$

where $q \in \mathbb{N}, n \in \mathbb{N}_0, z \in \Delta^*$, *then* $f(z) \in \mathcal{KH}^*(p,q,\lambda,\alpha,\beta)$ *if and only if it can be expressed by*

$$f(z) = \sum_{n=0}^{\infty} \eta_n f_n(z)$$

where $\eta_n \geq 0, \eta_i = 0 (i = 1, 2, \cdots, p-1)$ *and* $\sum\limits_{n=0}^{\infty} \eta_n = 1.$

Proof. Let $f(z) = \sum\limits_{n=0}^{\infty} \eta_n f_n(z)$ so

$$\begin{aligned} f(z) &= \eta_0 z^{-p} + \sum_{n=1}^{\infty} \eta_n \left(z^{-p} + \frac{\beta(\lambda+1)Ab(1-\alpha)(n-q-1)!}{n!(1+\lambda\beta)} z^n \right) \\ &= Az^{-p} + \sum_{n=p}^{\infty} \eta_n \frac{\beta(\lambda+1)Ab(1-\alpha)(n-q-1)!}{n!(1+\lambda\beta)} z^n \end{aligned}$$

where

$$A = \eta_0 a + \sum_{n=1}^{\infty} \eta_n = \eta_0 a + 1 - \eta_0 = \eta_0(a-1) + 1 > 0$$

Therefore $f(z) \in \mathcal{KH}^*(p,q,\lambda,\alpha,\beta)$ since

$$\begin{aligned} &\sum_{n=p}^{\infty} \frac{n!(1+\lambda\beta)}{(n-q-1)!\beta(\lambda+1)Ab(1-\alpha)} \times \frac{\beta(\lambda+1)Ab(1-\alpha)(n-q-1)!}{n!(1+\lambda\beta)} \eta_n \\ &= \sum_{n=p}^{\infty} \eta_n = \sum_{n=1}^{\infty} \eta_n = 1 - \eta_0 \leq 1. \end{aligned}$$

Conversely, suppose that $f(z) \in \mathcal{KH}^*(p,q,\lambda,\alpha,\beta)$. Then by (2.1) we have

$$0 \leq t_n \leq \frac{\beta(\lambda+1)Ab(1-\alpha)(n-q-1)!}{n!(1+\lambda\beta)}$$

Setting

$$\eta_n = \frac{n!(1+\lambda\beta)}{\beta(\lambda+1)Ab(1-\alpha)(n-q-1)!} t_n,\ (n \geq p) \tag{4.2}$$

$\eta_i = 0 (i = 1, 2, \ldots, p-1)$ and $\eta_0 = 1 - \sum\limits_{n=1}^{\infty} \eta_n$. Then

$$\begin{aligned} f(z) &= Az^{-p} + \sum_{n=p}^{\infty} t_n z^n \\ &= Az^{-p} + \sum_{n=p}^{\infty} \frac{\beta(\lambda+1)Ab(1-\alpha)(n-q-1)!\eta_n}{n!(1+\lambda\beta)} z^n \\ &= Az^{-p} + \sum_{n=1}^{\infty} \eta_n (f_n(z) - z^{-p}) \\ &= \left(A - \sum_{n=1}^{\infty} \eta_n \right) z^{-p} + \sum_{n=1}^{\infty} \eta_n f_n \\ &= \eta_0 f_0(z) + \sum_{n=1}^{\infty} \eta_n f_n(z) = \sum_{n=0}^{\infty} \eta_n f_n(z). \end{aligned}$$

This gives the required result. ∎

5. SOME PROPERTIES OF $\mathcal{KH}^*(p, q, \lambda, \alpha, \beta)$

Theorem 5.1. *Let* $f_j(z)(j = 1, 2, ...m)$ *defined by*

$$f_j(z) = A_j z^{-p} + \sum_{n=p}^{\infty} t_{n,j} z^n$$

be in the class $\mathcal{KH}^*(p, q, \lambda, \alpha, \beta)$*, then the function* $F(z) = \sum\limits_{j=0}^{m} c_j f_j(z) (c_j \geq 0)$ *is also in* $\mathcal{KH}^*(p, q, \lambda, \alpha, \beta)$ *where* $\sum\limits_{j=1}^{m} c_j = 1$.

Proof. Since $f_j(z) \in \mathcal{KH}^*(p, q, \lambda, \alpha, \beta)$, by (2.1) we have

$$\sum_{n=p}^{\infty} \frac{n!}{(n-q-1)!}(1+\lambda\beta)t_{n,j} \leq \beta(\lambda+1)Ab(1-\alpha)$$

also

$$\begin{aligned} F(z) &= \sum_{j=1}^{m} c_j \left(A_j z^{-p} + \sum_{n=p}^{\infty} t_{n,j} z^n \right) \\ &= \sum_{j=0}^{m} c_j A_j z^{-p} + \sum_{n=p}^{\infty} \left(\sum_{j=0}^{n} c_j t_{n,j} \right) z^n \\ &= Bz^{-p} + \sum_{n=p}^{\infty} s_n z^n \end{aligned}$$

where

$$B = \sum_{j=0}^{m} c_j A_j, \ s_n = \sum_{j=0}^{m} c_j t_{n,j}.$$

But

$$\begin{aligned}
&\sum_{n=p}^{\infty}\left[\frac{n!}{(n-q-1)!}(1+\lambda\beta)\right]s_n \\
&=\sum_{n=p}^{\infty}\left[\frac{n!}{(n-q-1)!}(1+\lambda\beta)\right]\left[\sum_{j=0}^{\infty}c_j t_{n,j}\right] \\
&=\sum_{j=0}^{m}c_j\left\{\sum_{n=p}^{\infty}\left[\frac{n!(1+\lambda\beta)}{(n-q-1)!}\right]t_{n,j}\right\} \\
&\le\sum_{j=0}^{m}c_j(\beta(\lambda+1)Ab(1-\alpha)) \\
&=\beta(\lambda+1)Ab(1-\alpha).
\end{aligned}$$

Now the proof of Theorem 5.1 is complete. ∎

Theorem 5.2. *Let $f(z), g(z) \in \mathcal{KH}^*(p,q,\lambda,\alpha,\beta)$ and given by*

$$\begin{aligned}
f(z) &= Az^{-p}+\sum_{n=p}^{\infty}t_n z^n \\
g(z) &= Az^{-p}+\sum_{n=p}^{\infty}s_n z^n.
\end{aligned}$$

Then the function $h(z) = Az^{-p} + \sum_{n=p}^{\infty}(t_n^2 + s_n^2)z^n$ is also in $\mathcal{KH}^*(p,q,\lambda,\alpha,\gamma)$ where

$$\gamma \ge \frac{2(n-q-1)!\beta^2(\lambda+1)Ab(1-\alpha)}{n!(1+\lambda\beta)^2-2\lambda(n-q-1)!\beta^2(\lambda+1)Ab(1-\alpha)} \tag{5.1}$$

Proof. Since $f, g \in \mathcal{KH}^*(p,q,\lambda,\alpha,\beta)$ therefore we have

$$\sum_{n=p}^{\infty}\left[\frac{n!(1+\lambda\beta)}{(n-q-1)!\beta(\lambda+1)Ab(1-\alpha)}\right]^2 t_n^2 \le \left[\sum_{n=p}^{\infty}\frac{n!(1+\lambda\beta)}{(n-q-1)!\beta(\lambda+1)Ab(1-\alpha)}t_n\right]^2 < 1,$$

$$\sum_{n=p}^{\infty}\left[\frac{n!(1+\lambda\beta)}{(n-q-1)!\beta(\lambda+1)Ab(1-\alpha)}\right]^2 s_n^2 \le \left[\sum_{n=p}^{\infty}\frac{n!(1+\lambda\beta)}{(n-q-1)!\beta(\lambda+1)Ab(1-\alpha)}s_n\right]^2 < 1.$$

The above inequalities yield us

$$\sum_{n=p}^{\infty}\frac{1}{2}\left[\frac{n!(1+\lambda\beta)}{(n-q-1)!\beta(\lambda+1)Ab(1-\alpha)}\right]^2(t_n^2+s_n^2) < 1.$$

Now we must show

$$\sum_{n=p}^{\infty}\frac{n!(1+\lambda\gamma)}{(n-q-1)!\gamma(\lambda+1)Ab(1-\alpha)}(t_n^2+s_n^2) < 1.$$

But above inequality holds if

$$\frac{n!(1+\lambda\gamma)}{(n-q-1)!\gamma(\lambda+1)Ab(1-\alpha)} \le \frac{1}{2}\left[\frac{n!(1+\lambda\beta)}{(n-q-1)!\beta(\lambda+1)Ab(1-\alpha)}\right]^2$$

or equivalently

$$\frac{1+\lambda\gamma}{\gamma} \leq \frac{n!(1+\lambda\beta)^2}{2(n-q-1)!\beta^2(\lambda+1)Ab(1-\alpha)}$$

or

$$\gamma \geq \frac{2(n-q-1)!\beta^2(\lambda+1)Ab(1-\alpha)}{n!(1+\lambda\beta)^2 - 2\lambda(n-q-1)!\beta^2(\lambda+1)Ab(1-\alpha)}$$

and this gives the result. ∎

6. HADAMARD PRODUCT

Theorem 6.1. *If*

$$f(z) = Az^{-p} + \sum_{n=p}^{\infty} t_n z^n, \ g(z) = Bz^{-p} + \sum_{n=p}^{\infty} s_n z^n$$

be in the class $\mathcal{KH}^*(p,q,\lambda,\alpha,\beta)$*. Then Hadamard product of* f *and* g *defined by*

$$(f*g)(z) = h(z) = ABz^{-p} + \sum_{n=p}^{\infty} t_n s_n z^n$$

is in the class $\mathcal{KH}^*(p,q,\lambda,\alpha,\gamma)$ *where*

$$\gamma \geq \frac{(n-q-1)!(\lambda+1)Ab(1-\alpha)\beta^2}{n!(1+\lambda\beta)^2 - \lambda(n-q-1)!(\lambda+1)Ab(1-\alpha)\beta^2} \tag{6.1}$$

Proof. Since $f(z), g(z) \in \mathcal{KH}^*(p,q,\lambda,\alpha,\beta)$, so by (2.1) we have

$$\sum_{n=p}^{\infty} \frac{n!(1+\lambda\beta)}{(n-q-1)!\beta(\lambda+1)Ab(1-\alpha)} t_n \leq 1, \tag{6.2}$$

$$\sum_{n=p}^{\infty} \frac{n!(1+\lambda\beta)}{(n-q-1)!\beta(\lambda+1)Ab(1-\alpha)} s_n \leq 1. \tag{6.3}$$

We must find the smallest γ such that

$$\sum_{n=p}^{\infty} \frac{n!(1+\lambda\gamma)}{(n-q-1)!\gamma(\lambda+1)Ab(1-\alpha)} t_n s_n \leq 1$$

by using the Cauchy - Schwarz inequality we have

$$\sum_{n=p}^{\infty} \frac{n!(1+\lambda\beta)}{(n-q-1)!\beta(\lambda+1)Ab(1-\alpha)} \sqrt{t_n s_n} \leq 1. \tag{6.4}$$

Now it is enough to show that

$$\frac{n!(1+\lambda\gamma)}{(n-q-1)!\gamma(\lambda+1)Ab(1-\alpha)} t_n s_n \leq \frac{n!(1+\lambda\beta)}{(n-q-1)!\beta(\lambda+1)Ab(1-\alpha)} \sqrt{t_n s_n}$$

or equivalently

$$\sqrt{t_n s_n} \leq \frac{\gamma(1+\lambda\beta)}{\beta(1+\lambda\gamma)}.$$

But from (6.4) we have

$$\sqrt{t_n s_n} \leq \frac{(n-q-1)!\beta(\lambda+1)Ab(1-\alpha)}{n!(1+\lambda\beta)}$$

So it is enough that

$$\frac{(n-q-1)!\beta(\lambda+1)Ab(1-\alpha)}{n!(1+\lambda\beta)} \leq \frac{\gamma(1+\lambda\beta)}{\beta(1+\lambda\gamma)}$$

or

$$\frac{(n-q-1)!(\lambda+1)Ab(1-\alpha)}{n!}\frac{\beta^2}{(1+\lambda\beta)^2} \leq \frac{\gamma}{1+\lambda\gamma}$$

or

$$\frac{1}{\gamma}+\lambda \leq \frac{n!(1+\lambda\beta)^2}{(n-q-1)!(\lambda+1)Ab(1-\alpha)\beta^2}$$

or

$$\gamma \geq \frac{(n-q-1)!(\lambda+1)Ab(1-\alpha)\beta^2}{n!(1+\lambda\beta)^2-\lambda(n-q-1)!(\lambda+1)Ab(1-\alpha)\beta^2}.$$

∎

7. INTEGRAL OPERATORS ON $\mathcal{KH}^*(p,q,\lambda,\alpha,\beta)$

Theorem 7.1. *If $f(z) \in \mathcal{KH}^*(p,q,\lambda,\alpha,\beta)$ then the function $F_c(z)$ defined by*

(7.1) $$F_c(z) = (c+1-p)\int_0^1 v^c f(vz)dv, \ c \geq 1$$

is also in $\mathcal{KH}^*(p,q,\lambda,\alpha,\beta)$.

Since $f(z) \in \mathcal{KH}^*(p,q,\lambda,\alpha,\beta)$ so

$$f(z) = Az^{-p} + \sum_{n=p}^{\infty} t_n z^n$$

Therefore

$$\begin{aligned} F_c(z) &= (c+1-p)\int_0^1 v^c\left[A(vz)^{-p}+\sum_{n=p}^{\infty} t_n (vz)^n\right]dv \\ &= (c+1-p)\int_0^1 \left[Av^{c-p}z^{-p}+\sum_{n=p}^{\infty} t_n v^{c+n} z^n\right]dv \\ &= (c+1-p)\left[\frac{A}{c-p+1}v^{c-p+1}z^{-p}+\sum_{n=p}^{\infty} t_n z^n\left(\frac{1}{c+n+1}v^{c+n+1}\right)\right]_0^1 \\ &= (c+1-p)\left(\frac{A}{c-p+1}z^{-p}+\sum_{n=p}^{\infty}\frac{1}{c+n+1}t_n z^n\right) \\ &= Az^{-p}+\sum_{n=p}^{\infty}\frac{c+1-p}{c+1+n}t_n z^n \end{aligned}$$

Since $\frac{c+1-p}{c+1+n} < 1$ we have

$$\sum_{n=p}^{\infty} \frac{n!}{(n-q-1)!}(1+\lambda\beta)\frac{c+1-p}{c+1+n}t_n$$
$$< \sum_{n=p}^{\infty} \frac{n!}{(n-q-1)!}(1+\lambda\beta)t_n \quad \text{by(2.1)} < \beta(\lambda+1)Ab(1-\alpha).$$

Therefore, by using Lemma 2.1

$$F_c(z) \in \mathcal{KH}^*(p,q,\lambda,\alpha,\beta).$$

Remark 7.1. With the same way that used in Theorem 3.1, Corollary 3.2 we can prove $F_c(z)$ is multivalently starlike and multivalently convex of order $\eta (0 \le \eta < p)$ in

$$|z| < R_1(p,q,\lambda,\alpha,\beta,c,\eta) = \inf_n \left\{ \frac{n!(1+\lambda\beta)(p-\eta)(c+1-n)}{(\lambda+1)b(1-\alpha)(n-q-1)!(n+2p+\eta)(c+1-p)} \right\}^{\frac{1}{n+p}}$$

and

$$|z| < R_2(p,q,\lambda,\alpha,\beta,c,\eta) = \inf_n \left\{ \frac{p(n-1)(1+\lambda\beta)(p-\eta)(c+1-n)}{(\lambda+1)b(1-\alpha)(n-q-1)!(n+2p+\eta)(c+1-p)} \right\}^{\frac{1}{n+p}}$$

respectively.

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