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A NEW FAMILY OF PERIODIC FUNCTIONS AS EXPLICIT ROOTS OF A CLASS OF POLYNOMIAL EQUATIONS

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ABSTRACT. For any positive integer n a new family of periodic functions in power series form and of period n is used to solve in closed form a class of polynomial equations of order n. The n roots are the values of the appropriate function from that family taken at $0, 1, ..., n-1$.

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1. INTRODUCTION

For centuries mathematicians have sought closed-form expressions for the roots of polynomial equations of arbitrary order n. The Abel-Ruffini theorem states that only polynomials of order four or less can be solved explicitly using rational operations and finite root extractions [1].

Several authors have proposed series solutions of algebraic (and polynomial) equations ([1], [2], [3], [2], [4], [5], [6]). These solutions, which rely on hypergeometric functions, are cumbersome to implement and have not provided feasible alternatives to standard numerical methods.

We will consider here a polynomial equation in the form

$$x^n = (a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + ... + a_1 x + a_0)t^n. \tag{1.1}$$

where the $a_k's$ are complex coefficients and t is a real parameter that may or may not be 1. In the self-contained Theorem 5.1 we will show that when t is small enough or $|a_0|$ large enough, there exists a sequence $\{\beta_m\}_{m=1,2,...}$ such that the n roots will be the values of the function

$$x(t,u) = \sum_{m=1}^{\infty} \beta_m e^{2u\pi m \times i/n} \times t^m \tag{1.2}$$

taken at $u = 0, 1, 2, ..., n-1$.

The functions $x(t,u)$ are power series in t and periodic of period n in the variable u. These functions can be thought of as "elementary" in the same way

$$cos(te^{i\theta}) = \sum_{m=0}^{\infty} (-1)^m t^{2m} e^{2m\theta i}/(2m)! \tag{1.3}$$

is an elementary function.

This will be only a first step as the class of polynomial equations solved explicitly with these power series is limited (t must be small enough or $|a_0|$ large enough). The ultimate goal is to generalize the approach proposed here to any polynomial equation.

2. PRELIMINARIES

We start off with a polynomial equation in the form

$$x^n = a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + ... + a_1 x + a_0 \tag{2.1}$$

where the a_k's are complex coefficients and $a_0 \overset{def.}{=} \rho e^{i\theta}$ $(\rho > 0, -\pi < \theta \leq \pi)$ is assumed throughout to be non-zero. (Otherwise (2.1) can trivially be reduced to an equation of degree $n-1$).

Equation (2.1) is transformed by multiplying the right-hand side by t^n where t is a real variable that we may initially think of as small but is destined to take on any real value including 1:

$$x^n = (a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + ... + a_1 x + a_0)t^n. \tag{2.2}$$

We begin by seeking the roots expressed as the infinite series

$$x(t) \overset{def.}{=} b_1 t + b_2 t^2 + b_3 t^3 + ... \tag{2.3}$$

where *a priori* we will need n different sequences $\{b_m\}_{m=1,2,...}$ to generate the n roots. We will show that in fact there exists a single sequence $\{\beta_m\}_{m=1,2,...}$ such that the n sequences $\{b_m\}_{m=1,2,...}$ are obtained through

$$b_m = \beta_m e^{2k\pi m \times i/n}, \;\; m = 1, 2, ... \tag{2.4}$$

for $k = 0, 1, ..., n-1$. The n roots $x(t)$ given in Eq. (2.3) will then be of the form

$$x(t,k) \overset{def.}{=} \sum_{m=1}^{\infty} \beta_m e^{2k\pi m \times i/n} \times t^m, \quad k = 0, 1, ..., n-1. \tag{2.5}$$

With $x(t)$ of (2.3) the equation to solve is now

$$x(t)^n = \big(a_{n-1}x(t)^{n-1} + a_{n-2}x(t)^{n-2} + ... + a_1 x(t) + a_0\big)t^n. \tag{2.6}$$

Before proceeding we need some notations and preliminary results.

We define the powers B_q^r of the partial sums of $x(t)$:

$$B_q^r \overset{def.}{=} (b_1 t + b_2 t^2 + ... + b_q t^q)^r, \quad q, r \in \mathbb{N}, \tag{2.7}$$

where for ease of notation the functional dependence of B_q^r on t is omitted. We let $K(d, B_q^r)$ denote the coefficient of t^d in B_q^r. Each b_m that we are seeking is equal to $K(m, B_m^1)$. We note that $K(d, B_q^r)$ does not depend on t and $K(d, B_q^r) = 0$ if $d < r$ or $d > rq$. With these notations

$$B_q^r = t^r K(r, B_q^r) + t^{r+1} K(r+1, B_q^r) + ... + t^{rq} K(rq, B_q^r). \tag{2.8}$$

Proposition 2.1. *When $d \geq r$ and $q - d + r - 1 \geq 0$ the coefficients $K(d, B_q^r)$ satisfy*

$$K(d, B_q^r) = K(d, B_{q-h}^r) \ , h = 0, 1, ..., q - d + r - 1. \tag{2.9}$$

For $d \geq r \geq 2$

$$K(d, B_q^r) = \sum_{m=1}^{min(d-r+1,q)} b_m K(d-m, B_q^{r-1}), \tag{2.10}$$

which for $q, s \geq 1$ is equivalent to

$$K(q+s, B_{q+s}^{s+1}) = b_1 K(q+s-1, B_{q+s-1}^s) + \sum_{m=2}^{q-1} b_m K(q+s-m, B_{q+s-m}^s) + b_q b_1^s. \tag{2.11}$$

Proof. Equation (2.9) expresses the fact that when $q \geq d - r + 1$ then only the first $d - r + 1$ b_i's enter into $K(d, B_q^r)$. Equation (2.10) is the convolution rule used to express the coefficient of t^d in B_q^r considered as the product $B_q^{r-1} \times B_q^1$. Equation (2.11) is obtained by using Eq. (2.9) to express a form of Eq. (2.10) in which all the $K(a, B_u^v)$'s have identical values for a and u. ∎

With $x(t)$ given in Eq. (2.3) both sides of Eq. (2.6) are polynomials in t with powers $\geq n$. The goal is to find recursively the b_m's so that for each $p \geq n$ the coefficients of t^p are equal on both sides of (2.6). This will be done by considering the roots $x(t)$ in the form of the gradually expanding partial sums $B_q^1 = \sum_{m=1}^q b_m t^m$ and applying Eq. (2.8) .

The coefficient b_1 will be found by seeking a solution of the form B_1^1 for which the coefficients of t^n on both sides of (2.6) are equal. With b_1 thus determined, b_2 is found by seeking a solution of the form B_2^1 for which the coefficients of t^{n+1} on both sides of (2.6) are equal, etc.

We begin with a candidate solution $B_1^1 = b_1 t$ by setting equal the coefficients of t^n on both side of Eq. (2.6), i.e.

$$B(n, B_1^n) = b_1^n = a_0. \tag{2.12}$$

Therefore the n possible values of b_1 are

$$b_1 = \rho^{1/n} e^{i(\theta + 2k\pi)/n}, k = 0, 1, ..., n-1 \tag{2.13}$$

which is Eq. (2.4) with $m = 1$ and

$$\beta_1 \overset{def.}{=} \rho^{1/n} e^{i\theta/n}. \tag{2.14}$$

We note that the n values $x(t) = b_1 t$ with b_1 of Eq. (2.13) are the exact trivial solutions of Eq. (2.6) when all coefficients other than a_0 are 0; otherwise they provide crude first-approximation solutions when t is small.

In order to find b_2 with a candidate solution $B_2^1 = b_1 t + b_2 t^2$, we note that the coefficient of t^{n+1} on the left side of Eq. (2.6) is $K(n+1, B_2^n)$ and on the right is $a_1 K(1, B_2^1)$. Therefore we want

$$K\left(n+1, B_2^n\right) = a_1 K\big(1, B_2^1\big). \tag{2.15}$$

Similarly, equating the coefficients of t^{n+2} with a solution B_3^1 yields

$$K\left(n+2, B_3^n\right) = a_2 K\big(2, B_3^2\big) + a_1 K\big(2, B_3^1\big). \tag{2.16}$$

When we equate the coefficients of t^{n+3}, a similar expression arises with a third term involving the coefficient a_3. In general, Eq. (2.8) shows that to equate the coefficients of t^{n+q} on both sides of Eq. (2.6) (with a candidate solution B_{q+1}^1) one needs:

$$K\big(n+q, B_{q+1}^n\big) = \sum_{m=1}^{min(n-1,q)} a_m K\big(q, B_{q+1}^m\big), q = 1, 2, ... \tag{2.17}$$

Equation (2.9) shows that for any $s \geq 1$

$$K\big(n+q, B_{q+1}^n\big) = K\big(n+q, B_{q+s}^n\big) \tag{2.18}$$

and

$$K(q, B_{q+1}^m) = K(q, B_{q+s}^m). \tag{2.19}$$

These equations show that for any $s \geq 1$ Eq. (2.17) is equivalent to

$$K\big(n+q, B_{q+s}^n\big) = \sum_{m=1}^{min(n-1,q)} a_m K\big(q, B_{q+s}^m\big). \tag{2.20}$$

Therefore if Eq. (2.17) is satisfied, the coefficients of t^{n+q} on both sides of Eq. (2.6) are also equal when any number of terms $b_k t^k$ are added to the partial sum B_{q+1}^1.

We next write the $K(a, B_u^v)$'s appearing in Eq. (2.17) in such a way that the indices a and u are equal. Equation (2.9) shows that Eq. (2.17) (with $q-1$ written instead of q) is equivalent to

$$K(n+q-1, B_{n+q-1}^n) = \sum_{m=1}^{min(n-1,q-1)} a_m K(q-1, B_{q-1}^m), q = 2, 3... \tag{2.21}$$

Given that $b_m = K(q, B_m^1)$, the task will be to find n sequences of b_m's for which the infinite systems of Eqs. (2.11) and (2.21) are satisfied for all q. Then we will give the convergence conditions for the series $\sum |\beta_m t^m| = \sum |b_m t^m|$ and show that the $x(t,k)$'s of Eq. (2.5) are indeed the n roots.

3. MATRIX FORMULATION

We define the sequence of $(n-1)$-dimensional vectors $\{W(q)\}_{q=1,2,...}$ whose $r-th$ component is

$$W(q)_r \stackrel{def.}{=} a_0 b_1^{1-q-r} K(q+r-1, B_{q+r-1}^r), r = 1, 2, ...n-1. \tag{3.1}$$

We are particularly interested in the first component of each $W(q)$ because with $r = 1$ Eq. (3.1) yields

$$b_m = W(m)_1 b_1^m / a_0. \tag{3.2}$$

We need the following definitions.

Definition 3.1. We define below the vectors A_p, u, U and the matrices P and M:

(1) For $p = 1, 2, ..., n-1$ the vector A_p is an $(n-1)$-dimensional row vector with the coefficient a_p in p-th position and zeros elsewhere.
(2) u is the $(n-1)$-dimensional row vector having 1 in the first position and zeros elsewhere.
(3) U is the $(n-1)$-dimensional column vector $\frac{1}{n}(1 \quad 2 \quad ... \quad n-1)'$.
(4) For $n \geq 3$ the $(n-1)$-dimensional square matrix P is defined as

$$P \stackrel{def.}{=} \begin{pmatrix} -2 & 1 & 0 & ... & ... & ... & 0 \\ -1 & -1 & 1 & 0 & ... & ... & 0 \\ -1 & 0 & -1 & 1 & 0 & ... & 0 \\ ... & ... & ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... & ... & ... \\ -1 & 0 & 0 & ... & ... & -1 & 1 \\ -1 & 0 & 0 & ... & ... & 0 & -1 \end{pmatrix} \tag{3.3}$$

with its inverse $M = P^{-1}$ equal to

$$M \stackrel{def.}{=} \frac{1}{n} \begin{pmatrix} -1 & -1 & ... & ... & ... & -1 \\ n-2 & -2 & -2 & ... & ... & -2 \\ n-3 & n-3 & -3 & ... & ... & -3 \\ ... & ... & ... & ... & ... & ... \\ 2 & 2 & ... & 2 & 2-n & 2-n \\ 1 & 1 & 1 & ... & 1 & 1-n \end{pmatrix}. \tag{3.4}$$

When $n = 2$ the matrices P and M reduce to $P = (-2)$ and $M = (-1/2)$.

In the next theorem we will see that the sequence $\{W(q)\}_{q=1,2,...}$ depends only on the a_k's, and not on the particular b_1. The n sequences $\{b_q\}_{q=1,2,...}$ are then obtained through Eq. (3.2) with the n values of b_1 given in Eq. (2.13).

Theorem 3.1. *We consider the sequence $\{W(q)\}_{q=1,2,...}$ defined in Eq.* (3.1) *together with the constraints of Eqs.* (2.11) *and* (2.21)*. The $W(q)$'s are then*

$$W(1) = a_0(1 \quad 1 \quad ... \quad 1)' \tag{3.5}$$

$$W(2) = \frac{a_1}{n}(1 \quad 2 \quad ... \quad n-1)' \tag{3.6}$$

(3.7)
$$W(q) = \frac{1}{a_0}\left[\sum_{p=1}^{q-2}[u.W(p+1)]MW(q-p) + U\sum_{p=1}^{min(n-1,q-1)}[A_p.W(q-p)]\right], q = 3, 4, ...$$

Bearing in mind that $a_0 = \rho e^{i\theta}$ we define

$$\beta_m = \rho^{m/n} \times e^{i(m\theta)/n} \times W(m)_1/a_0, \ \ m = 1, 2, ... \tag{3.8}$$

The n sequences $\{b_m\}_{m=1,2,...}$ are then obtained through

$$b_m = \beta_m e^{2k\pi \times mi/n}, m = 1, 2, ... \tag{3.9}$$

for $k = 0, 1, ..., n-1$.

Proof. Equation (3.5) results from the fact that $K(r, B_r^r) = b_1^r$. To prove Eq. (3.6) we recall that $b_1^n = a_0$ and note that the s-th component of $W(2)$ is

$$W(2)_s = K(s+1, B_{s+1}^s)b_1^{n-1-s} = (s.b_2 b_1^{s-1})b_1^{n-1-s} = sb_2 a_0 b_1^{-2}. \tag{3.10}$$

We will now express b_2. The definition of $K(n+1, B_{n+1}^n)$ and Eq. (2.21) for $q = 2$ yield

$$K(n+1, B_{n+1}^n) = n.b_2 b_1^{n-1} = a_1 b_1, \tag{3.11}$$

from which $b_2 = a_1/(nb_1^{n-2})$. This expression used in Eq. (3.10) yields Eq. (3.6).

In order to prove Eq. (3.7) we multiply both sides of Eq. (2.11) by $a_0 b_1^{-q-s}$ and use Eqs. (3.1) and (3.2) to see that Eq. (2.11) then becomes

$$K(q+s, B_{q+s}^{s+1})a_0 b_1^{-q-s} = K(q+s-1, B_{q+s-1}^s)a_0 b_1^{-q-s+1}$$

$$+\sum_{m=2}^{q-1} b_m a_0 b_1^{-q-s} K(q+s-m, B_{q+s-m}^s) + b_q a_0 b_1^{-q}$$

$$= W(q)_s + \frac{1}{a_0}\sum_{m=2}^{q-1} W(m)_1 W(q-m+1)_s + W(q)_1. \tag{3.12}$$

We note that for $s = 1, 2, ..., n-2$, the left-hand side of Eq. (3.12) is

$$K(q+s, B_{q+s}^{s+1})a_0 b_1^{-q-s} = W(q)_{s+1}. \tag{3.13}$$

For $s = 1$, Eq. (3.12) is therefore

$$W(q)_2 - 2W(q)_1 = \frac{1}{a_0}\sum_{m=2}^{q-1} W(m)_1 W(q-m+1)_1. \tag{3.14}$$

Similarly, for $s = 2, 3, ..., n-2$, Eq. (3.12) is:

$$W(q)_{s+1} - W(q)_s - W(q)_1 = \frac{1}{a_0}\sum_{m=2}^{q-1} W(m)_1 W(q-m+1)_s. \tag{3.15}$$

We next use Eq. (2.21) to express $K(q+s, B_{q+s}^{s+1})a_0 b_1^{-q-s}$ of Eq. (3.12) for $s = n-1$, i.e.

$$K(n+q-1, B_{n+q-1}^n)a_0 b_1^{-q-n+1} = \sum_{m=1}^{min(n-1,q-1)} a_m K(q-1, B_{q-1}^m)a_0 b_1^{-q-n+1} =$$

$$\frac{1}{a_0}\sum_{m=1}^{min(n-1,q-1)} a_m W(q-m)_m = W(q)_{n-1} + \frac{1}{a_0}\sum_{m=2}^{q-1} W(m)_1 W(q-m+1)_{n-1} + W(q)_1 \tag{3.16}$$

and therefore

$$-W(q)_1 - W(q)_{n-1} = \frac{1}{a_0}\sum_{m=2}^{q-1} W(m)_1 W(q-m+1)_{n-1} - \frac{1}{a_0}\sum_{m=1}^{min(n-1,q-1)} a_m W(q-m)_m. \tag{3.17}$$

We recall Definition 3.1 and note that $W(m)_1$ is the scalar product $uW(m)$ and $a_m W(q-m)_m = A_m W(q-m)$. We can now express Eq. (3.14), Eq. (3.15) (for $s = 2, 3, ..., n-2$) and Eq. (3.17) compactly in matrix form as

(3.18)
$$P \times W(q) = \frac{1}{a_0}\sum_{m=2}^{q-1}[uW(m)]W(q-m+1) - \begin{pmatrix} 0 \\ 0 \\ ... \\ 0 \\ \frac{1}{a_0}\sum\limits_{m=1}^{min(n-1,q-1)}[A_m W(q-m)] \end{pmatrix}$$

After multiplying both sides of Eq. (3.18) by the inverse M of P we get
(3.19)
$$W(q) = \frac{1}{a_0}\left(\sum_{m=2}^{q-1}[uW(m)]MW(q-m+1) + U\sum_{m=1}^{min(n-1,q-1)}[A_m W(q-m)]\right), q = 3, 4, ...$$

which is Eq. (3.7).

For the n possible values of b_1 given in (2.13), the term b_m of Eq. (3.2) is now

(3.20)
$$b_m = W(m)_1 \rho^{m/n} e^{i(\theta+2k\pi)m/n}/a_0, \quad k = 0, 1, ..., n-1$$

which is Eq. (3.9) with β_m given in Eq. (3.8). ∎

4. CONVERGENCE RESULTS

4.1. **Preliminaries.** In order to establish convergence conditions for the series $\sum b_k t^k$ we need to assess the growth of the first components $W(m)_1$ of the vectors $W(m)$. We will use the ℓ_∞ norm $|V|_\infty = \underset{i}{max}|V_i|$ of a complex vector $V = (v_i)$.

We also define the row-sum norm

(4.1)
$$\|X\| = \underset{i}{max}\sum_{j=1}^{j=m}|x_{ij}|.$$

of an m-dimensional square matrix $X = (x_{ij})$. We recall than $|XV|_\infty \leq \|X\| \times |V|_\infty$.

If we define the function of two variables

(4.2)
$$z(u,n) \overset{def.}{=} -2u^2 + u(2n+1) - n$$

then the norm of the $(n-1)$-dimensional matrix M of Eq. (3.4) is

(4.3)
$$\|M\| = \frac{1}{n}max\left(z[floor(n/2+1/4), n], z[ceil(n/2+1/4), n]\right).$$

where the $floor(x)$ and $ceil(x)$ functions are the largest integer smaller than x and the smallest integer larger than x.

We next define the sequence of modulii

(4.4)
$$w(q) \overset{def.}{=} |W(q)|_\infty, \quad q = 1, 2, ...$$

which are upper bounds for the modulii of the $W(q)_1$'s of interest. We also define the maximum modulus of the coefficients a_k other than a_0:

(4.5)
$$\alpha \overset{def.}{=} \max_{k=1,2,...,n-1}|a_k|.$$

We assume that $\alpha > 0$. (The trivial case $\alpha = 0$ will be considered in the Conclusion). We next define

(4.6)
$$\mu = \|M\|/|a_0|$$

and the sequence of nonnegative functions

$$\sigma_p(\alpha) = \begin{cases} \alpha/\|M\| & \text{if } p \le n; \\ 0 & \text{if } p > n. \end{cases} \quad , p = 2, 3, ... \tag{4.7}$$

We note that $w(1) = |a_0|$ and $w(2) \le \alpha$. From Eq. (3.7) we have

$$w(q) \le \frac{1}{|a_0|}\left[\sum_{p=1}^{q-2} w(p+1)\|M\|w(q-p) + \alpha \sum_{p=1}^{min(n-1,q-1)} w(q-p)\right], \quad q = 3, 4, ... \tag{4.8}$$

We now make a distinction between the cases $q \le n$ and $q > n$. In the former case the running index p in the second sum on the right-hand side of (4.8) goes to $q-1$ with a last term $w(1) = |a_0|$; (4.8) is then

$$\begin{aligned} w(q) &\le \frac{1}{|a_0|}\left[\sum_{p=1}^{q-2}[w(p+1)\|M\| + \alpha]w(q-p)\right] + \alpha \\ &= \mu\left[\sum_{p=1}^{q-2}[w(p+1) + \sigma_{p+1}(\alpha)]w(q-p)\right] + \alpha, \quad q = 3, 4, ...n. \end{aligned} \tag{4.9}$$

In the case $q > n$ the index p in the second sum on the right-hand side of (4.8) goes to $n-1$ and (4.8) can now be written

$$w(q) \le \mu\left[\sum_{p=1}^{q-2}[w(p+1) + \sigma_{p+1}(\alpha)]w(q-p)\right]$$

$$\le \mu\left[\sum_{p=1}^{q-2}[w(p+1) + \sigma_{p+1}(\alpha)][w(q-p) + \sigma_{q-p}(\alpha)]\right], q = n+1, n+2, ... \tag{4.10}$$

where the purpose of this last trivial inequality is to bound the sequence $\{w(q) + \sigma_q(\alpha)\}_{q=2,3,...}$ by the sequence $\{S_q(\alpha, |a_0|)\}_{q=2,3,...}$ defined below:

$$S_2(\alpha, |a_0|) \overset{def.}{=} \alpha + \sigma_2(\alpha). \tag{4.11}$$

For $q = 3, 4, ..., n$:

$$\begin{aligned} S_q(\alpha, |a_0|) &\overset{def.}{=} \frac{1}{|a_0|}\left(\sum_{p=1}^{q-2}[\|M\|S_{p+1}(\alpha, |a_0|) + \alpha]S_{q-p}(\alpha, |a_0|)\right) + \alpha + \sigma_q(\alpha) \\ &= \mu\left(\sum_{p=1}^{q-2}[S_{p+1}(\alpha, |a_0|) + \sigma_{p+1}(\alpha)]S_{q-p}(\alpha, |a_0|)\right) + \alpha + \sigma_q(\alpha). \end{aligned} \tag{4.12}$$

For $q \ge n+1$:

$$S_q(\alpha, |a_0|) \overset{def.}{=} \mu\left(\sum_{p=1}^{q-2} S_{p+1}(\alpha, |a_0|)S_{q-p}(\alpha, |a_0|)\right) \tag{4.13}$$

where the functional notation $S_q(\alpha, |a_0|)$ emphasizes for future reference the dependence of each function on α and $|a_0|$ (even though $S_q(\alpha, |a_0|)$ does not depend on $|a_0|$).

Proposition 4.1. *With the notations given above,*

$$w(q) \le w(q) + \sigma_q(\alpha) \le S_q(\alpha, |a_0|), q = 2, 3, ... \tag{4.14}$$

Proof. The inequality of (4.14) is true for $q = 2$ because $w(2) \leq \alpha$. In view of (4.9) and (4.12), inequality (4.14) is also true for $q = 3, 4, ..., n$. For $q = n + 1$, the inequality of (4.10) yields

$$w(n+1) = w(n+1) + \sigma_{n+1}(\alpha)$$

$$\leq \mu \left(\sum_{p=1}^{n-1} [w(p+1) + \sigma_{p+1}(\alpha)][w(n+1-p) + \sigma_{n+1-p}(\alpha)] \right)$$

$$\leq \mu \left(\sum_{p=1}^{n-1} [S_{p+1}(\alpha, |a_0|) \times S_{n+1-p}(\alpha, |a_0|)] \right) = S_{n+1}(\alpha, |a_0|). \tag{4.15}$$

To prove the result by induction for any $q \geq n + 1$ we assume (4.14) is true up to order $n + r$. For $q = n + r + 1$, inequality (4.10) and Eq. (4.13) yield

$$w(n+r+1) = w(n+r+1) + \sigma_{n+r+1}(\alpha)$$

$$\leq \mu \left(\sum_{p=1}^{n+r-1} [w(p+1) + \sigma_{p+1}(\alpha)][w(n+r+1-p) + \sigma_{n+r+1-p}(\alpha)] \right)$$

$$\leq \mu \left(\sum_{p=1}^{n+r-1} [S_{p+1}(\alpha, |a_0|) \times S_{n+r+1-p}(\alpha, |a_0|)] \right) = S_{n+r+1}(\alpha, |a_0|) \tag{4.16}$$

which completes the proof. ∎

We now provide some definitions and results pertaining to convolution-type sequences such as the S_q's of Eqs. (4.11) – (4.13).

4.2. μ-convolutions.

Definition 4.1. For any scalar μ, a μ-convolution of order m is an infinite sequence $\{u_k\}_{k=1,2,...}$ consisting of m initial scalars $u_1, u_2, ..., u_m$ with subsequent terms defined as

$$u_q \stackrel{def.}{=} \mu \left(\sum_{p=1}^{q-1} u_p u_{q-p} \right), \quad q = m+1, m+2, ... \tag{4.17}$$

An example of μ-convolution is provided by the Catalan numbers C_k ([7]) : $C_0 = 1$ and

$$C_s = C_0 C_{s-1} + C_1 C_{s-1} + ... + C_{s-1} C_0, \quad s = 1, 2, ... \tag{4.18}$$

The sequence $\{C_k\}$ is the 1-convolution of order 1 with initial term $C_0 = 1$. Each C_q is equal to

$$C_q = \frac{(2q)!}{(q+1)!q!}. \tag{4.19}$$

Proposition 4.2. *For a μ-convolution $\{v_k\}_{k=1,2,...}$ of order 1 and initial term v_1 we have*

$$v_{q+1} = v_1 C_q (v_1 \mu)^q, q = 0, 1, ... \tag{4.20}$$

For a μ-convolution $\{u_k\}_{k=1,2,...}$ of order m with $\mu > 0$ consisting of m initial nonnegative terms $u_1, u_2, ..., u_m$ we can define

$$v_1 \stackrel{def.}{=} \max_{k=1,2,...,m} \left(\frac{u_k}{C_{k-1} \mu^{k-1}} \right)^{1/k}. \tag{4.21}$$

The sequence $\{u_k\}$ is then bounded by the μ-convolution $\{v_k\}$ of order 1 and initial term v_1:

$$u_{q+1} \le v_{q+1} = v_1 C_q (v_1\mu)^q \sim v_1^{q+1}\mu^q \frac{4^q}{\sqrt{\pi} q^{3/2}} \text{ for } q \to \infty. \tag{4.22}$$

Proof. Equation (4.20) is easily proved by induction. The inequality in (4.22) is a direct consequence of (4.21) which states that $u_k \le v_k$ for $k = 1, 2, ..., m$. The asymptotic result in (4.22) results from the fact that C_q of (4.19) is $\sim 4^q/(\sqrt{\pi} q^{3/2})$ for $q \to \infty$ ([7]).
∎

We will now use these results to assess the radius of convergence of the power series $\sum \beta_m t^m$ with each β_m given in (3.8).

4.3. Radius of convergence of $\sum b_m t^m$.

Theorem 4.3. *With the notations used above, a lower bound for the radius of convergence of $\sum \beta_m t^m$ is*

$$LBRC(\alpha, |a_0|) \overset{def.}{=} \frac{1}{4|a_0|^{1/n}} \min_{k=1,2,...,n-1} \left(\frac{C_{k-1}|a_0|}{S_{k+1}(\alpha, |a_0|)\|M\|} \right)^{1/k}. \tag{4.23}$$

Proof. The sequence $\{S_k(\alpha, |a_0|)\}_{k=2,3,...}$ of Eqs. (4.11)-(4.13) is a μ-convolution of order $n-1$. The first $n-1$ terms are for $k = 2, 3, ..., n$ and μ is given in Eq. (4.6). We can therefore apply the results of Proposition 4.2 with $u_q \overset{def.}{=} S_{q+1}(\alpha, |a_0|)$ for $q = 1, 2, ...$ The term v_1 of Eq. (4.21) is

$$v_1(\alpha, |a_0|) = \max_{k=1,2,...,n-1} \left(\frac{S_{k+1}(\alpha, |a_0|)}{C_{k-1}\mu^{k-1}} \right)^{1/k}. \tag{4.24}$$

The results of Eqs. (4.14) and (4.22) then yield for every m

$$w(m) \le S_m(\alpha, |a_0|) \le v_1(\alpha, |a_0|)^{m-1} \times \mu^{m-2} \times C_{m-2} \tag{4.25}$$

$$\sim v_1(\alpha, |a_0|)^{m-1} \times \mu^{m-2} \times \frac{4^{m-2}}{\sqrt{\pi}(m-2)^{3/2}} \text{ for } m \to \infty. \tag{4.26}$$

We now let $RC(A)$ denote the radius of convergence of the power series $\sum \beta_m t^m$. The functional notation is to emphasize the dependence on the vector $A = (a_k)$. Given Eqs. (3.8), (4.24), (4.25) and the fact that $\rho = |a_0|$, we have

$$RC(A) \overset{def.}{=} \liminf_{m\to\infty} |\beta_m|^{-1/m} = \liminf_{m\to\infty} \rho^{-1/n+1/m}|W(m)_1|^{-1/m} \ge \tag{4.27}$$

$$\liminf_{m\to\infty} \rho^{-1/n} w(m)^{-1/m} \ge \liminf_{m\to\infty} \rho^{-1/n} \left(v_1(\alpha, |a_0|)^{m-1} \times \mu^{m-2} \times C_{m-2} \right)^{-1/m} \tag{4.28}$$

$$\sim \frac{\rho^{-1/n}}{4 v_1(\alpha, |a_0|) \times \mu} = \frac{1}{4|a_0|^{1/n}} \min_{k=1,2,...,n-1} \left(\frac{C_{k-1}|a_0|}{S_{k+1}(\alpha, |a_0|)\|M\|} \right)^{1/k} \text{for } m \to \infty \tag{4.29}$$

which yields the lower bound of (4.23).
∎

We now know that $\sum \beta_m t^m$ has a strictly positive radius of convergence. We do not have an analytical expression for $RC(A)$ but we do have the lower bound $LBRC(\alpha, |a_0|)$.

For $|t| < RC(A)$ we may then define

$$c \overset{def.}{=} \frac{t/RC(A) + 1}{2} < 1. \tag{4.30}$$

There exists then $D > 0$ such that

$$|b_m t^m| = |K(m, B_m^1) t^m| = |\beta_m||t|^m \leq D c^m, \;\; m = 1, 2, ... \tag{4.31}$$

which shows that the series $x(t, k) = \sum \beta_m e^{2k\pi m \times i/n} t^m$ converges geometrically fast for $|t| < RC(A)$.

We next need a result on the growth of the $K(d, B_q^p)$'s.

Proposition 4.4. *If* $|t| < RC(A)$ *then there exist* $D > 0$ *and* c $(0 < c < 1)$ *such that for any* $p \geq 1$,

$$|K(d, B_q^p)| \leq D^p (d-1)^{p-1} (c/|t|)^d, \forall d \geq 1, \forall q \geq 1. \tag{4.32}$$

Proof. We prove the result by induction on p. With $p = 1$ Eqs. (2.9) and (4.31) show that if $d \leq q$ then

$$|K(d, B_q^1)| = |K(d, B_d^1)| = |\beta_d| \leq D(c/|t|)^d. \tag{4.33}$$

If $d > q$ then $K(d, B_q^1) = 0$ and (4.32) is trivially true. Therefore (4.32) is true for $p = 1$.

In order to prove the result by induction we assume that (4.32) is true up to order p and calculate $|K(d, B_q^{p+1})|$. If $d < p+1$ then $K(d, B_q^{p+1}) = 0$ and Eq. (4.32) is trivially true. If $d \geq p+1$ then Eq. (2.10) yields

$$|K(d, B_q^{p+1})| = \left| \sum_{m=1}^{min(d-p,q)} b_m K(d-m, B_q^p) \right| \leq \sum_{m=1}^{d-1} |b_m||K(d-m, B_q^p)| \tag{4.34}$$

$$\leq \sum_{m=1}^{d-1} D(c/|t|)^m D^p (d-1)^{p-1} (c/|t|)^{d-m} = D^{p+1}(d-1)^p (c/|t|)^d \tag{4.35}$$

which is the desired result of (4.32) at the order $p+1$. ∎

We now bring together all previous results and prove that when the series $x(t, k) = \sum \beta_m e^{2k\pi m \times i/n} t^m$ converge (i.e. for $|t| < RC(A)$) they provide for $k = 0, 1, ..., n-1$ the n roots of Eq. (2.2).

5. MAIN RESULT

Theorem 5.1. *We consider the following polynomial equation of degree* $n \geq 2$*, parameterized by* $t > 0$ *and with* $a_0 \overset{def.}{=} \rho e^{i\theta} \neq 0$*:*

$$x^n = \left(a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + ... + a_1 x + a_0\right) t^n. \tag{5.1}$$

We define $\alpha \overset{def.}{=} \max_{k=1,2,...,n-1} |a_k|$ *and recall the definition of the matrix* M *in* (3.4)*. We then define* $n-1$ *numbers* $\{S_q(\alpha, |a_0|)\}_{q=2,3,...,n}$ *recursively as follows:*

$$S_2(\alpha, |a_0|) \overset{def.}{=} \alpha + \alpha/\|M\| \tag{5.2}$$

and for $q = 3, 4, ..., n$*:*

$$S_q(\alpha, |a_0|) \overset{def.}{=} \frac{1}{|a_0|} \left(\sum_{p=1}^{q-2} [\|M\| S_{p+1}(\alpha, |a_0|) + \alpha] S_{q-p}(\alpha, |a_0|) \right) + \alpha + \alpha/\|M\|. \tag{5.3}$$

We also define recursively the infinite sequence of $(n-1)$*-dimensional vectors* $W(q)$ *starting with*

$$W(1) = a_0 (1 \;\; 1 \;\; ... \;\; 1)', \;\; W(2) = \frac{a_1}{n}(1 \;\; 2 \;\; ... \;\; n-1)'. \tag{5.4}$$

Bearing in mind u, U, *and* A_p *of Definition* 3.1, *we define the subsequent terms as*

(5.5)
$$W(q) = \frac{1}{a_0}\left[\sum_{p=1}^{q-2}[u.W(p+1)]MW(q-p) + U\sum_{p=1}^{min(n-1,q-1)}[A_p.W(q-p)]\right], q = 3, 4, ...$$

We next define the infinite sequence

(5.6)
$$\beta_m = \rho^{m/n} \times e^{i(m\theta)/n} \times W(m)_1/a_0, \ \ m = 1, 2, ...$$

The radius of convergence $RC(A)$ *of* $\sum |\beta_m t^m|$ *is*

(5.7)
$$RC(A) = \lim_{m\to\infty} inf \ \ Q(m)$$

where

(5.8)
$$Q(m) = \frac{1}{|a_0|^{1/n}|W(m)_1|^{1/m}}.$$

Furthermore, $RC(A)$ *has the lower bound*

(5.9)
$$LBRC(\alpha, |a_0|) \stackrel{def.}{=} \frac{1}{4|a_0|^{1/n}} \min_{k=1,2,...,n-1}\left(\frac{C_{k-1}|a_0|}{S_{k+1}(\alpha, |a_0|)\|M\|}\right)^{1/k},$$

which is larger than 1 for α *small enough or for* $|a_0|$ *large enough.*

For $|t| < RC(A)$ *the partial sums*

(5.10)
$$x(t, k)_q \stackrel{def.}{=} \sum_{m=1}^{q}\beta_m e^{2k\pi m\times i/n}\times t^m$$

converge to the roots of Eq. (5.1) *in the sense that for* $k = 0, 1, ..., n-1$ *the* n *differences*

(5.11)
$$\Delta(q, k) \stackrel{def.}{=} [x(t, k)_q]^n - \left(a_{n-1}[x(t, k)_q]^{n-1} + a_{n-2}[x(t, k)_q]^{n-2} + ... + a_1 x(t, k)_q + a_0\right) t^n$$

approach 0 when $q \to \infty$.

When α *is small enough or* $|a_0|$ *large enough the radius of convergence* $RC(A)$ *is larger than 1 (since* $LBRC(\alpha, |a_0|) \geq 1$) *and the functions* $x(t, k) = \sum_{m=1}^{\infty}\beta_m e^{2k\pi m\times i/n}\times t^m$ *taken at* $t = 1$ *and* $k = 0, 1, ..., n-1$ *will then provide the* n *roots of Eq.* (5.1) *when* $t = 1$.

Proof. The expression of (5.7) for $RC(A)$ is a direct consequence of the definition of the β_m's in (5.6)

Each $S_q(\alpha, |a_0|)$ of Eqs. (5.2) − (5.3) is a polynomial in α with nonnegative coefficients and no constant term. This insures that $LBRC(\alpha, |a_0|)$ of Eq. (5.9) tends to infinity (and is therefore ≥ 1) for $\alpha \to 0$. For $q \geq 3$ each $S_q(\alpha, |a_0|)$ of Eq. (5.3) is a decreasing function of $|a_0|$ that approaches $\alpha + \alpha/\|M\|$ for $|a_0| \to \infty$. The lower bound $LBRC(\alpha, |a_0|)$ of Eq. (5.9) then tends to infinity (and is therefore ≥ 1) for $|a_0| \to \infty$.

The only other result that has not already been proven is the convergence of $\Delta(q, k)$ to 0 for $q \to \infty$. Equation (2.8) shows that

(5.12)
$$[x(t, k)_q]^n = b_1^n t^n + \sum_{j=1}^{n(q-1)} t^{n+j}K(n+j, B_q^n).$$

We use Eq. (2.8) and the facts that $b_1^n = a_0$ and $K(n+j, B_q^n) = K(n+j, B_{j+1}^n)$ for $j \leq q-1$ (see Eq. (2.18)) to write

$$[x(t,k)_q]^n = a_0 t^n + \sum_{j=1}^{q-1} t^{n+j} K(n+j, B_{j+1}^n) + \sum_{j=q}^{n(q-1)} t^{n+j} K(n+j, B_q^n). \tag{5.13}$$

We next turn our attention to the sum involving the a_k's on the right-hand side of Eq. (5.11). We call this sum $Y(q,k)$:

$$Y(q,k) \stackrel{def.}{=} a_0 t^n + \sum_{r=1}^{n-1} \sum_{j=r}^{rq} a_r t^{j+n} K(j, B_q^r) = a_0 t^n + \sum_{r=1}^{n-1} \sum_{j=1}^{n(q-1)} a_r t^{n+j} K(j, B_q^r). \tag{5.14}$$

Equation (2.9) shows that $K(j, B_q^r) = K(j, B_{j+1}^r)$ for $j \leq q-1$. Therefore

$$Y(q,k) = a_0 t^n + \sum_{j=1}^{q-1} t^{n+j} \sum_{r=1}^{min(n-1,j)} a_r K(j, B_{j+1}^r) + \sum_{r=1}^{n-1} \sum_{j=q}^{n(q-1)} a_r t^{n+j} K(j, B_q^r). \tag{5.15}$$

We next use Eq. (2.17) to write

$$Y(q,k) = a_0 t^n + \sum_{j=1}^{q-1} t^{n+j} K(n+j, B_{j+1}^n) + \sum_{r=1}^{n-1} \sum_{j=q}^{n(q-1)} a_r t^{n+j} K(j, B_q^r). \tag{5.16}$$

Equations (5.13) and (5.16) show that the difference $\Delta(q,k)$ of (5.11) reduces to

$$\Delta(q,k) = \sum_{j=q}^{n(q-1)} t^{n+j} K(n+j, B_q^n) - \sum_{r=1}^{n-1} a_r \sum_{j=q}^{n(q-1)} t^{n+j} K(j, B_q^r). \tag{5.17}$$

We make use of Eq. (4.32) to see that

$$\left| \sum_{j=q}^{n(q-1)} t^{n+j} K(n+j, B_q^n) \right| \leq \sum_{j=q}^{\infty} |t|^{n+j} |K(n+j, B_q^n)| \leq D^n c^n \sum_{j=q}^{\infty} (n+j-1)^{n-1} c^j. \tag{5.18}$$

The last sum on the right-hand side of (5.18) is the remainder of order q of an absolutely convergent series and therefore tends to 0 when $q \to \infty$. The first sum in (5.17) therefore tends to 0 for $q \to \infty$. The $n-1$ remainders in the double sum on the right-hand side of (5.17) similarly tend to 0, which shows that $\Delta(q,k) \to 0$ when $q \to \infty$. ∎

Remark 5.1. We make the following observations:

(1) The radius of convergence $RC(A)$ can be assessed numerically through the expression of (5.7). If $Q(m)$ asymptotically remains above 1 then $RC(A) \geq 1$ and the $x(t,k)$'s provide the n roots with $t = 1$.
(2) Regardless of the values of the a_k's, the functions $x(t,k)$ can be viewed as Taylor expansions in the variable t of the roots of Eq. (5.1). Indeed, when t is small the first few terms of the series provide approximate values for the roots (a numerical example is given below).

6. NUMERICAL ILLUSTRATIONS

As a numerical illustration we consider the polynomial equation of degree 6:

$$x^6 = (-x^5 + x^4 - 2x^3 - 3x^2 + 2x + 8)t^6 \tag{6.1}$$

with a particular interest in the case $t = 1$. The lower bound $LBRC(\alpha, |a_0|)$=$LBRC(3,8)$ for the radius of convergence is 0.094. Therefore we do not know whether $x(t,u)$ converges

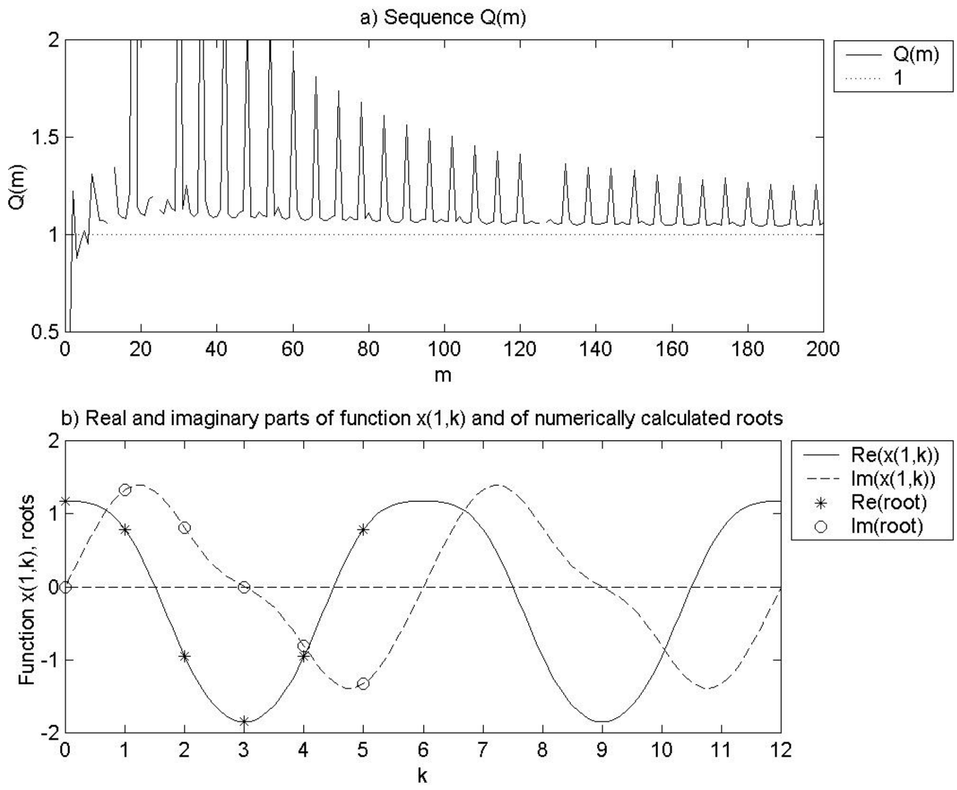

Figure 1: a) Sequence Q(m) of Eq. (5.8)*; b) Real/imaginary parts of function x(1,k) together with numerically calculated roots*

with $t = 1$. However $LBRC(\alpha, |a_0|)$ is an extremely conservative bound calculated only with $\alpha, |a_0|$, and $\|M\|$. (The main usefulness of the sufficient convergence condition $t < LBRC(\alpha, |a_0|)$ lies in the qualitative fact that with $t = 1$ the function $x(t, u)$ does provide the roots when α is small enough or $|a_0|$ large enough).

The values $Q(m)$ are plotted in Figure 1a. These values appear to remain above approximately 1.05 for $m \to \infty$. This suggests that $RC(A) \geq 1$ and that with $t = 1$ the $x(1, k)$'s will indeed converge and provide the roots of Eq. (6.1).

To verify this we plotted in Figure 1b the real and imaginary parts of each $x(1, k)$ over two periods (i.e. k from 0 to $2n = 12$). (The first 200 terms are used in the series expansion). On the same figure we also plotted the values found using Matlab's built-in polynomial equation routine (stars and circles). These values coincide with the values $Re(x(1, k)) + i \times Im(x(1, k))$ for $k = 0, 1, ..., n - 1 = 5$.

With real coefficients for the polynomial equation the complex roots come in conjugate pairs. There are two such pairs. One for $k = 1, k = 5$ and the other for $k = 2, k = 4$. In addition there are two real roots at $k = 0$ and at $k = 3$. In this and other cases these real roots appear to be at local minima or maxima of the $Re(x(t, k))$ function. These special behaviors of and relationships between the real and imaginary parts no doubt arise from particular patterns in the sequence $\{W(q)_1\}_{q=1,2,...}$ which have yet to be explored. These structures disappear with complex coefficients since in this case there are no more complex conjugate roots.

The first five terms of the series $x(t, u)$ are

$$x(t, u) \approx \sqrt{2}e^{2u\pi\times i/6}t + 0.08333e^{4u\pi\times i/6}t^2 - 0.18414e^{6u\pi\times i/6}t^3 - 0.14506e^{8u\pi\times i/6}t^4 + 0.11441e^{10u\pi\times i/6}t^5 \tag{6.2}$$

There are not enough terms to calculate the roots when $t = 1$. With $t = 0.4$ however, the values provided by (6.2) are very close to those calculated with Matlab (Table 1).

	k=0	k=1	k=2	k=3	k=4	k=5
x(0.4,k)	0.565	0.290+0.504i	-0.300+0.474i	-0.545	-0.300-0.474i	0.290-0.504i
Matlab	0.564	0.290+0.504i	-0.301+0.474i	-0.546	-0.301-0.474i	0.290-0.504i

Table 1: Six values *x(0.4,k)* and numerically calculated roots of Eq. (6.1) with *t=0.4*

It was found numerically that when the constant term $a_0 = 8$ in Eq. (6.1) falls below approximately 4, the radius of convergence of the series $x(t, u)$ drops below 1. The method can no longer be used to solve Eq. (6.1) with $t = 1$.

Several approaches have been tried in order to extend the method to the situation in which $|a_0|$ is small, meaning that at least one root is close to 0. One possibility would be to transform the unknown in such a way that small roots are moved away from 0. For example one could write the polynomial equation in terms of a changed unknown $y = 1/x$: if a root x is small then the corresponding y is large. However this approach did not change the problem. Another possibility would be to inject the parameter t differently into the equation. For example one could use a similar approach after multiplying the left rather than the right side of Eq. (2.1) by t^n. Or one could multiply each term of the equation by a well-chosen (and different) power t^p. To date such attempts have proved largely inconclusive.

7. CONCLUSION

We end with a note on terminology inspired by the trivial polynomial equation (5.1) in which each coefficient a_k is 0 for $k \geq 1$ (i.e. α of Eq. (4.5) is 0). The equation is then

$$x^n = a_0 t^n = \rho e^{i\theta} t^n. \tag{7.1}$$

As one might expect the solution series $x(t, k)$ reduces to the single exponential term

$$x(t, k) = \rho^{1/n} e^{(2k\pi+\theta)i/n} t, \quad k = 0, 1, ..., n-1. \tag{7.2}$$

This special case suggests the following definition.

Definition 7.1. Let $\{\beta_m\}_{m=1,2,...}$ be an infinite sequence of complex numbers such that the powers series $\sum \beta_m t^m$ has a radius of convergence $RC > 0$. (The β_m's may or may not be generated through the process of Theorem 5.1). For any integer n the complex functions

$$x(t, u) = \sum_{m=1}^{\infty} \beta_m e^{2u\pi m\times i/n} \times t^m \tag{7.3}$$

of the two real variables u and t are of period n in the variable u and have a radius of convergence RC in the variable t. Such functions may be called *generalized exponential functions*.

Ours is only a first step, which shows that the roots of a particular class of polynomial equations can be expressed explicitly with an infinite number of rational operations and root extractions. It is to be hoped that some variant or adaptation of the family of generalized exponential functions $x(t, u)$ will eventually emerge to provide closed-form expressions for the roots of arbitrary polynomial equations.

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