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DISTORTION THEOREMS FOR CERTAIN ANALYTIC FUNCTIONS INVOLVING THE COEFFICIENT INEQUALITIES

SHIGEYOSHI OWA AND JUNICHI NISHIWAKI

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DEPARTMENT OF MATHEMATICS, KINKI UNIVERSITY, HIGASHI-OSAKA, OSAKA 577-8502, JAPAN

ABSTRACT. By virtue of the coefficient inequalities for certain analytic functions $f(z)$ in the open unit disk $\mathbb{U}$, two subclasses $\mathcal{M}_n^*(\alpha)$ and $\mathcal{N}_n^*(\alpha)$ are introduced. The object of the present paper is to discuss the distortion thorems of functions $f(z)$ belonging to the classes $\mathcal{M}_n^*(\alpha)$ and $\mathcal{N}_n^*(\alpha)$ involving the coefficient inequalities.

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1. INTRODUCTION

Let $\mathcal{A}_n$ be the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{k=n+1}^{\infty} a_n z^k \qquad (n \in \mathbb{N} = 1, 2, 3, \cdots) \tag{1.1}$$

that are analytic in the open unit disk $\mathbb{U} = \{z : z \in \mathbb{C} \; and \; |z| < 1\}$. Let $\mathcal{M}_n(\alpha)$ denote the subclass of $\mathcal{A}_n$ consisting of functions $f(z)$ which satisfy the following condition

$$\text{Re}\left\{\frac{zf'(z)}{f(z)}\right\} < \alpha \qquad (z \in \mathbb{U}) \tag{1.2}$$

for some $\alpha(\alpha > 1)$. Also, let $\mathcal{N}_n(\alpha)$ be the subclass of $\mathcal{A}$ consisting of all functions $f(z)$ which satisfy

$$\text{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} < \alpha \qquad (z \in \mathbb{U}) \tag{1.3}$$

for some $\alpha(\alpha > 1)$. By the definitions for the classes $\mathcal{M}_n(\alpha)$ and $\mathcal{N}_n(\alpha)$, we note that $f(z) \in \mathcal{N}_n(\alpha)$ if and only if $zf'(z) \in \mathcal{M}_n(\alpha)$. The classes $\mathcal{M}_1(\alpha)$ and $\mathcal{N}_1(\alpha)$ when $n = 1$ were considered by Uralegaddi, Ganigi and Sarangi [3], Nishiwaki and Owa [1], and Owa and Nishiwaki [2].

Remark 1.1. Let us consider the function $f(z)$ given by

$$f(z) = z(1 - z^n)^{\frac{2(\alpha-1)}{n}} \in \mathcal{A}_n. \tag{1.4}$$

Then, it follows that

$$\frac{zf'(z)}{f(z)} = 1 - \frac{2(\alpha-1)z^n}{1 - z^n}. \tag{1.5}$$

This implies that

$$\alpha - \frac{zf'(z)}{f(z)} = (\alpha - 1)\frac{1 + z^n}{1 - z^n}. \tag{1.6}$$

Noting that

$$\text{Re}\left\{\frac{1 + z^n}{1 - z^n}\right\} > 0 \qquad (z \in \mathbb{U}), \tag{1.7}$$

we see that

$$\text{Re}\left\{\frac{zf'(z)}{f(z)}\right\} < \alpha \qquad (z \in \mathbb{U}), \tag{1.8}$$

that is, $f(z) \in \mathcal{M}_n(\alpha)$. Furthermore, we have that

$$f(z) = \int_0^z (1 - t^n)^{\frac{2(\alpha-1)}{n}} dt \in \mathcal{N}_n(\alpha). \tag{1.9}$$

2. COEFFICIENT INEQUALITIES

Let us consider the coefficient inequalities for the classes $\mathcal{M}_n(\alpha)$ and $\mathcal{N}_n(\alpha)$.

Theorem 2.1. *If $f(z) \in \mathcal{A}_n$ satisfies*

$$\sum_{k=n+1}^{\infty} \{(k-1) + |k - 2\alpha + 1|\}|a_k| \leqq 2(\alpha - 1) \tag{2.1}$$

for some $\alpha(\alpha > 1)$, then $f(z) \in \mathcal{M}_n(\alpha)$.

Proof. Suppose that $f(z) \in \mathcal{A}_n$ satisfies the coefficient inequality. Then, if we show that

$$\left| \frac{\dfrac{zf'(z)}{f(z)} - 1}{\dfrac{zf'(z)}{f(z)} - (2\alpha - 1)} \right| < 1 \qquad (z \in \mathbb{U}), \tag{2.2}$$

we obtain that $f(z) \in \mathcal{M}_n(\alpha)$. Indeed, we have that

$$\begin{aligned} &\left| \frac{\dfrac{zf'(z)}{f(z)} - 1}{\dfrac{zf'(z)}{f(z)} - (2\alpha - 1)} \right| \\ \leqq \;& \frac{\sum_{k=n+1}^{\infty}(k-1)|a_k||z|^{k-1}}{2(\alpha-1) - \sum_{k=n+1}^{\infty}|k-2\alpha+1||a_k||z|^{k-1}} \\ < \;& \frac{\sum_{k=n+1}^{\infty}(k-1)|a_k|}{2(\alpha-1) - \sum_{k=n+1}^{\infty}|k-2\alpha+1||a_k|} \\ < \;& 1 \end{aligned} \tag{2.3}$$

for $z \in \mathbb{U}$ ∎

Noting that $f(z) \in \mathcal{N}_n(\alpha)$ if and only if $zf'(z) \in \mathcal{M}_n(\alpha)$, we have

Corollary 2.2. *If $f(z) \in \mathcal{A}_n$ satisfies*

$$\sum_{k=n+1}^{\infty} k\{(k-1) + |k - 2\alpha + 1|\}|a_k| \leqq 2(\alpha - 1) \tag{2.4}$$

for some $\alpha(\alpha > 1)$, then $f(z) \in \mathcal{N}_n(\alpha)$.

Remark 2.1. If $1 < \alpha \leqq \dfrac{n+2}{2}$, then the inequalities (2.1) and (2.4) become

$$\sum_{k=n+1}^{\infty} (k-\alpha)|a_k| \leqq \alpha - 1 \tag{2.5}$$

and

$$\sum_{k=n+1}^{\infty} k(k-\alpha)|a_k| \leqq \alpha - 1, \tag{2.6}$$

respectively.

3. DISTORTION INEQUALITIES

In view of Theorem 2.1 and Corollary 2.2, we introduce the subclasses $\mathcal{M}_n^*(\alpha)$ and $\mathcal{N}_n^*(\alpha)$ of $\mathcal{A}_n$ which satisfy the coefficient inequalities (2.1) and (2.4) for some $\alpha \geqq \frac{n+2}{2}$, respectively.

Theorem 3.1. *If* $f(z) \in \mathcal{M}_n^*(\alpha)$*, then*

$$\begin{aligned} |z| - \sum_{k=n+1}^{[2\alpha]-1} |a_k||z|^k - \frac{\alpha-1}{[2\alpha]-\alpha}\left\{1 - \sum_{k=n+1}^{[2\alpha]-1}|a_k|\right\}|z|^{[2\alpha]} \\ \leqq |f(z)| \leqq |z| + \sum_{k=n+1}^{[2\alpha]-1} |a_k||z|^k + \frac{\alpha-1}{[2\alpha]-\alpha}\left\{1 - \sum_{k=n+1}^{[2\alpha]-1}|a_k|\right\}|z|^{[2\alpha]} \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} 1 - \sum_{k=n+1}^{[2\alpha]-1} k|a_k||z|^{k-1} - \frac{[2\alpha](\alpha-1)}{[2\alpha]-\alpha}\left\{1 - \sum_{k=n+1}^{[2\alpha]-1}|a_k|\right\}|z|^{[2\alpha]-1} \\ \leqq |f'(z)| \leqq 1 + \sum_{k=n+1}^{[2\alpha]-1} k|a_k||z|^{k-1} + \frac{[2\alpha](\alpha-1)}{[2\alpha]-\alpha}\left\{1 - \sum_{k=n+1}^{[2\alpha]-1}|a_k|\right\}|z|^{[2\alpha]-1} \end{aligned} \tag{3.2}$$

for $z \in \mathbb{U}$*, where the symbol* $[\ \ ]$ *means the Gauss symbol.*

Proof. Note that

$$\begin{aligned} &\sum_{k=n+1}^{\infty} \{(k-1) + |k-2\alpha+1|\}|a_k| \\ &= 2(\alpha-1)\sum_{k=n+1}^{[2\alpha]-1}|a_k| + 2\sum_{k=[2\alpha]}^{\infty}(k-\alpha)|a_k| \\ &\leqq 2(\alpha-1). \end{aligned} \tag{3.3}$$

This gives us that

$$\sum_{k=[2\alpha]}^{\infty} |a_k| \leqq \frac{\alpha-1}{[2\alpha]-\alpha}\left\{1 - \sum_{k=n+1}^{[2\alpha]-1}|a_k|\right\}. \tag{3.4}$$

Therefore, we have that

$$\begin{aligned} |f(z) &\leqq |z| + \sum_{k=n+1}^{[2\alpha]-1}|a_k||z|^k + \sum_{k=[2\alpha]}^{\infty}|a_k||z|^k \\ &\leqq |z| + \sum_{k=n+1}^{[2\alpha]-1}|a_k||z|^k + \frac{\alpha-1}{[2\alpha]-\alpha}\left\{1 - \sum_{k=n+1}^{[2\alpha]-1}|a_k|\right\}|z|^{[2\alpha]} \end{aligned} \tag{3.5}$$

and

$$(3.6)\qquad \begin{aligned} |f(z)| &\geqq |z| - \sum_{k=n+1}^{[2\alpha]-1} |a_k||z|^k - \sum_{k=[2\alpha]}^{\infty} |a_k||z|^k \\ &\geqq |z| - \sum_{k=n+1}^{[2\alpha]-1} |a_k||z|^k - \frac{\alpha-1}{[2\alpha]-\alpha}\left\{1 - \sum_{k=n+1}^{[2\alpha]-1} |a_k|\right\}|z|^{[2\alpha]}. \end{aligned}$$

Next, we see that

$$(3.7)\qquad \begin{aligned} \frac{[2\alpha]-\alpha}{[2\alpha]} \sum_{k=[2\alpha]}^{\infty} k|a_k| &\leqq \sum_{k=[2\alpha]}^{\infty} (k-\alpha)|a_k| \\ &\leqq (\alpha-1)\left\{1 - \sum_{k=n+1}^{[2\alpha]-1} |a_k|\right\}. \end{aligned}$$

Applying (3.8), we obtain that

$$(3.8)\qquad \begin{aligned} |f'(z)| &\leqq 1 + \sum_{k=n+1}^{[2\alpha]-1} k|a_k||z|^{k-1} + \sum_{k=[2\alpha]}^{\infty} k|a_k||z|^{k-1} \\ &\leqq 1 + \sum_{k=n+1}^{[2\alpha]-1} k|a_k||z|^{k-1} + \frac{[2\alpha](\alpha-1)}{[2\alpha]-\alpha}\left\{1 - \sum_{k=n+1}^{[2\alpha]-1} |a_k|\right\}|z|^{[2\alpha]-1} \end{aligned}$$

and

$$(3.9)\qquad \begin{aligned} |f'(z)| &\geqq 1 - \sum_{k=n+1}^{[2\alpha]-1} k|a_k||z|^{k-1} - \sum_{k=[2\alpha]}^{\infty} k|a_k||z|^{k-1} \\ &\geqq 1 - \sum_{k=n+1}^{[2\alpha]-1} k|a_k||z|^{k-1} - \frac{[2\alpha](\alpha-1)}{[2\alpha]-\alpha}\left\{1 - \sum_{k=n+1}^{[2\alpha]-1} |a_k|\right\}|z|^{[2\alpha]-1}. \end{aligned}$$

This completes the proof of Theorem 3.1. ∎

Letting $\alpha = \dfrac{n+2}{2}$ in Theorem 3.1, we have

Corollary 3.2. *If* $f(z) \in \mathcal{M}_n^*\left(\dfrac{n+2}{2}\right)$*, then*

$$(3.10)\qquad \begin{aligned} &|z| - |a_{n+1}||z|^{n+1} - \frac{n}{n+2}(1-|a_{n+1}|)|z|^{n+2} \\ &\leqq |f(z)| \leqq |z| + |a_{n+1}||z|^{n+1} + \frac{n}{n+2}(1-|a_{n+1}|)|z|^{n+2} \end{aligned}$$

and

$$(3.11)\qquad \begin{aligned} &1 - (n+1)|a_{n+1}||z|^n - n(1-|a_{n+1}|)|z|^{n+1} \\ &\leqq |f'(z)| \leqq 1 + (n+1)|a_{n+1}||z|^n + n(1-|a_{n+1}|)|z|^{n+1} \end{aligned}$$

for $z \in \mathbb{U}$.

Next, we derive

Theorem 3.3. *If $f(z) \in \mathcal{N}_n^*(\alpha)$, then*

$$|z| - \sum_{k=n+1}^{[2\alpha]-1} |a_k||z|^k - \frac{\alpha-1}{[2\alpha([2\alpha]-\alpha)}\left\{1 - \sum_{k=n+1}^{[2\alpha]-1} k|a_k|\right\}|z|^{[2\alpha]} \tag{3.12}$$
$$\leqq \quad |f(z)| \leqq |z| + \sum_{k=n+1}^{[2\alpha]-1} |a_k||z|^k + \frac{\alpha-1}{[\alpha]([2\alpha]-\alpha)}\left\{1 - \sum_{k=n+1}^{[2\alpha]-1} k|a_k|\right\}|z|^{[2\alpha]}$$

and

$$1 - \sum_{k=n+1}^{[2\alpha]-1} k|a_k||z|^{k-1} - \frac{\alpha-1}{[2\alpha]-\alpha}\left\{1 - \sum_{k=n+1}^{[2\alpha]-1} k|a_k|\right\}|z|^{[2\alpha]-1} \tag{3.13}$$
$$\leqq \quad |f'(z)| \leqq 1 + \sum_{k=n+1}^{[2\alpha]-1} k|a_k||z|^{k-1} + \frac{\alpha-1}{[2\alpha]-\alpha}\left\{1 - \sum_{k=n+1}^{[2\alpha]-1} k|a_k|\right\}|z|^{[2\alpha]-1}$$

for $z \in \mathbb{U}$, where the symbol $[\ \ ]$ means the Gauss symbol.

Proof. From the coefficient inequality for the class $\mathcal{N}_n^*(\alpha)$, we know that

$$\sum_{k=n+1}^{[2\alpha]-1} k(\alpha-1)|a_k| + \sum_{k=[2\alpha]}^{\infty} k(k-\alpha)|a_k| \leqq \alpha - 1 \tag{3.14}$$

which gives us that

$$\sum_{k=[2\alpha]}^{\infty} |a_k| \leqq \frac{\alpha-1}{[2\alpha]([2\alpha]-\alpha)}\left\{1 - \sum_{k=n+1}^{[2\alpha]-1} k|a_k|\right\} \tag{3.15}$$

and

$$\sum_{k=[2\alpha]}^{\infty} k|a_k| \leqq \frac{\alpha-1}{[2\alpha]-\alpha}\left\{1 - \sum_{k=n+1}^{[2\alpha]-1} k|a_k|\right\}. \tag{3.16}$$

Therefore, it is easy to derive the distortion inequalities of the theorem. ∎

If we take $\alpha = \frac{n+2}{2}$ in Theorem 3.3, then we have

Corollary 3.4. *If $f(z) \in \mathcal{N}_n^*\left(\frac{n+2}{2}\right)$, then*

$$|z| - |a_{n+1}||z|^{n+1} - \frac{n}{(n+2)^2}(1-(n+1)|a_{n+1}|)|z|^{n+2} \tag{3.17}$$
$$\leqq \quad |f(z)| \leqq |z| + |a_{n+1}||z|^{n+1} + \frac{n}{(n+2)^2}(1-(n+1)|a_{n+1}|)|z|^{n+2}$$

and

$$1 - (n+1)|a_{n+1}||z|^n - \frac{n}{n+2}(1-(n+1)|a_{n+1}|)|z|^{n+1} \tag{3.18}$$
$$\leqq \quad |f'(z)| \leqq 1 + (n+1)|a_{n+1}||z|^n + \frac{n}{n+2}(1+(n+1)|a_{n+1}|)|z|^{n+1}$$

for $z \in \mathbb{U}$.

Finally, we consider the distortoin theorem for $f''(z) \in \mathcal{N}_n^*(\alpha)$.

Theorem 3.5. *If* $f(z) \in \mathcal{N}_n^*(\alpha)$, *then*

$$|f''(z)| \leqq \sum_{k=n+1}^{[2\alpha]-1} k(k-1)|a_k||z|^{k-2} + \frac{(\alpha-1)([2\alpha]-1)}{[2\alpha]-\alpha}\left\{1-\sum_{k=n+1}^{[2\alpha]-1} k|a_k|\right\}|z|^{[2\alpha]-2} \tag{3.19}$$

for $z \in \mathbb{U}$, *where the symbol* $[\ \ ]$ *means the Gauss symbol.*

Proof. Applying the coefficient inequality for the class $\mathcal{N}_n^*(\alpha)$, we see that

$$\sum_{k=[2\alpha]}^{\infty} k(k-1)|a_k| \leqq \frac{(\alpha-1)([2\alpha]-1)}{[2\alpha]-\alpha}\left\{1-\sum_{k=n+1}^{[2\alpha]-1} k|a_k|\right\}. \tag{3.20}$$

Thus, we can show that

$$\begin{aligned} &|f''(z)| \\ &\leqq \sum_{k=n+1}^{[2\alpha]-1} k(k-1)|a_k||z|^{k-2} + \sum_{k=[2\alpha]}^{\infty} k(k-1)|a_k||z|^{k-2} \\ &\leqq \sum_{k=n+1}^{[2\alpha]-1} k(k-1)|a_k||z|^{k-2} + \frac{(\alpha-1)([2\alpha]-1)}{[2\alpha]-\alpha}\left\{1-\sum_{k=n+1}^{[2\alpha]-1} k|a_k|\right\}|z|^{[2\alpha]-2} \end{aligned} \tag{3.21}$$

∎

Letting $\alpha = \dfrac{n+2}{2}$ in Theorem 3.5, we have

Corollary 3.6. *If* $f(z) \in \mathcal{N}_n^*\left(\dfrac{n+2}{2}\right)$, *then*

$$|f''(z)| \leqq n(n+1)|a^{n+1}||z|^{n-1} + \frac{n(n+1)}{n+2}(1-(n+1)|a_{n+1}|)|z|^n \tag{3.22}$$

for $z \in \mathbb{U}$.

4. SUBORDINATIONS

Let $f(z)$ and $g(z)$ be analytic in $\mathbb{U}$ with $f(0) = g(0)$. Then $f(z)$ is said to be subordinate to $g(z)$, written by $f(z) \prec g(z)$, if there exists an analytic function $w(z)$ in $\mathbb{U}$ such that $w(0) = 0$, $|w(z)| < 1$ $(z \in \mathbb{U})$, and $f(z) = g(w(z))$. In particular, if $g(z)$ is univalent in $\mathbb{U}$, then the subordination $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$.
Now, we show

Theorem 4.1. *If* $f(z) \in \mathcal{M}_n(\alpha)$ $(\alpha > 1)$, *then*

$$\frac{zf'(z)}{f(z)} \prec \frac{1-(2\alpha-1)z^n}{1-z^n} \qquad (z \in \mathbb{U}). \tag{4.1}$$

Proof. By means of Remark 1.1, let us define the function $w(z)$ by

$$\frac{zf'(z)}{f(z)} = \frac{1-(2\alpha-1)w(z)}{1-w(z)} \qquad (z \in \mathbb{U}). \tag{4.2}$$

Then $w(z)$ is analytic in $\mathbb{U}$ and $w(z) = z^n + b_1 z^{n+1} + \cdots$. It follows from (4.2) that

$$w(z) = \frac{\dfrac{zf'(z)}{f(z)} - 1}{\dfrac{zf'(z)}{f(z)} - (2\alpha - 1)}. \tag{4.3}$$

Noting that $f(z) \in \mathcal{M}_n(\alpha)$ is equivalent to

$$\left| \frac{\dfrac{zf'(z)}{f(z)} - 1}{\dfrac{zf'(z)}{f(z)} - (2\alpha - 1)} \right| < 1 \qquad (z \in U), \tag{4.4}$$

we know that $|w(z)| < 1$ $(z \in \mathbb{U})$. Therefore, by the difinition of subordinations, we see the subordination (4.1). ∎

Finally, for the class $\mathcal{N}_n(\alpha)$, we have

Theorem 4.2. *If* $f(z) \in \mathcal{N}_n(\alpha)$ $(\alpha > 1)$, *then*

$$\frac{zf''(z)}{f'(z)} \prec \frac{2(1-\alpha)z^n}{1-z^n} \qquad (z \in \mathbb{U}). \tag{4.5}$$

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