The Australian Journal of Mathematical Analysis and Applications

http://ajmaa.org

Volume 3, Issue 2, Article 18, pp. 1-8, 2006

ON SOME RAMANUJAN'S SCHLÄFLI TYPE MODULAR EQUATIONS

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Received 4 August, 2005; accepted 7 November, 2005; published 5 December, 2006.

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ABSTRACT. In this paper, we give new proof of certain Ramanujan-Schläfli modular equations. We also obtain a new modular equation of degree 23.

Key words and phrases: Theta-function, Modular equations.

2000 *Mathematics Subject Classification.* 33E20, 11A55.

ISSN (electronic): 1449-5910

1. INTRODUCTION

Let, as usual

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)}.$$

A modular equation of degree n is an equation relating α and β that is induced by

$$n\frac{{}_2F_1(\frac{1}{2},\frac{1}{2};1;1-\alpha)}{{}_2F_1(\frac{1}{2},\frac{1}{2};1;\alpha)} = \frac{{}_2F_1(\frac{1}{2},\frac{1}{2};1;1-\beta)}{{}_2F_1(\frac{1}{2},\frac{1}{2};1;\beta)},$$

where

$${}_2F_1(a,b;c;x) = \sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k k!}x^k, \qquad |x|<1.$$

L. Schlafli has discovered 7 modular equations in his paper [5]. Ramanujan also recorded these modular equations in his first notebook [4] and also in second notebook [4]. These are the modular equations of degrees 3, 5, 7, 11, 13, 17 and 19. B. C. Berndt [1], has given proof of the modular equations of degrees 3, 5, 7. K. G. Ramanathan [3, p. 411] indicated a proof of the equation of degree 11, but Berndt mentioned in his book [2, p. 379] that "it is not possible to complete the proof along the lines that Ramanathan indicated". Watson [8] gave a proof of modular equation of degree 13 but erroneously remarked that Ramanujan did not discover it. These modular equations were also examined by Watson [9]. Motivated by these in this paper, we give alternative and elementary proofs of the modular equations of degrees 3, 5, 7 and 11 and also we obtain modular equation of degree 23 which is of the same nature. The identities analogues to (1.2)-(1.6) are not existing corresponding to degrees 13, 17 and 19 hence, we are unable to obtain the proof of modular equations of degrees 13, 17 and 19.

We conclude this section by recalling some definitions and certain theta function identities, which will be used in next section. After Ramanujan, for $|q|<1$, we defined

$$f(-q) := \prod_{n=1}^{\infty}(1-q^n).$$

The Ramanujan's Weber class invariants are defined by

$$G_n := 2^{-1/4}\frac{f(q)}{q^{1/24}f(-q^2)},$$

and

$$g_n := 2^{-1/4}\frac{f(-q)}{q^{1/24}f(-q^2)},$$

where $q = e^{-\pi\sqrt{n}}$. We also require the following theta function identities:

(1.1) $$i) \qquad B^{16} - A^{16}B^8 + 16A^8 = 0,$$

where

$$A = \frac{f(q)}{q^{1/24}f(-q^2)} \qquad \text{and} \qquad B = \frac{f(-q^2)}{q^{1/12}f(-q^4)}.$$

(1.2) $$ii) \qquad B^4 - A^4B^2 + 4A^2 = 0,$$

where

$$A = \frac{f(q)f(q^3)}{q^{1/6}f(-q^2)f(-q^6)} \qquad \text{and} \qquad B = \frac{f(-q^2)f(-q^6)}{q^{1/3}f(-q^4)f(-q^{12})}.$$

$$B^8 - B^4A^8 + 8(AB)^4 + 16A^4 = 0, \tag{1.3}$$
(1.3) *iii)*

where

$$A = \frac{f(q)f(q^5)}{q^{1/4}f(-q^2)f(-q^{10})} \qquad \text{and} \qquad B = \frac{f(-q^2)f(-q^{10})}{q^{1/2}f(-q^4)f(-q^{20})}.$$

(1.4) *iv)*
$$B^2 - A^2B + 2A = 0,$$

where

$$A = \frac{f(q)f(q^7)}{q^{1/3}f(-q^2)f(-q^{14})} \qquad \text{and} \qquad B = \frac{f(-q^2)f(-q^{14})}{q^{2/3}f(-q^4)f(-q^{28})}.$$

(1.5) *v)*
$$B^4 - A^4B^2 + 4A^2B^2 + 4A^2 = 0,$$

where

$$A = \frac{f(q)f(q^{11})}{q^{1/2}f(-q^2)f(-q^{22})} \qquad \text{and} \qquad B = \frac{f(-q^2)f(-q^{22})}{qf(-q^4)f(-q^{44})}.$$

(1.6) *vi)*
$$B^2 - A^2B + 2AB + 2A = 0,$$

$$A = \frac{f(q)f(q^{23})}{qf(-q^2)f(-q^{46})} \qquad \text{and} \qquad B = \frac{f(-q^2)f(-q^{46})}{q^2f(-q^4)f(-q^{92})}.$$

For a proof of (1.1), see [7] , and for proof of (1.2) to (1.6), see [6].

2. MAIN THEOREMS

For $0 < \alpha, \beta < 1$, we set

$$P := [16\alpha\beta(1-\alpha)(1-\beta)]^{1/24} \qquad \text{and} \qquad Q := \left[\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right]^{1/24}.$$

Theorem 2.1. *If β has degree 3 over α, then*

$$Q^6 + \frac{1}{Q^6} + 2\sqrt{2}\left(P^3 - \frac{1}{P^3}\right) = 0.$$

Proof. Let

$$A := \frac{f(q)}{q^{1/24}f(-q^2)}, \qquad B := \frac{f(q^3)}{q^{1/8}f(-q^6)},$$

$$C := \frac{f(-q^2)}{q^{1/12}f(-q^4)} \qquad \text{and} \qquad D := \frac{f(-q^6)}{q^{1/4}f(-q^{12})}.$$

From (1.1), we have

$$C^{16} - A^{16}C^8 + 16A^8 = 0. \tag{2.1}$$

and also from (1.2), we have

$$(CD)^4 - (AB)^4(CD)^2 + 4(AB)^2 = 0. \tag{2.2}$$

Eliminating C between (2.1) and (2.2) using maple, we arrive at

$$A^{24}(-B^{32}D^{16} + 16B^{24}D^8 + B^{16}D^{24} - 16B^8D^{16}) + 16A^{18}(2B^{26}D^{16} - 16B^{18}D^8 - B^{10}D^{24})$$
$$+ 32A^{12}(-10B^{20}D^{16} + 16B^{12}D^8 + B^4D^{24}) + 1024A^6B^{14}D^{16}$$
$$- 16D^{32} - 4096B^{16} - 512D^{16}B^8 = 0. \tag{2.3}$$

Changing q to q^3 in (2.1), we deduce that

$$D^{16} - B^{16}D^8 + 16B^8 = 0. \tag{2.4}$$

Eliminating D between (2.3) and (2.4) using maple, we obtain

$$(B^{12} + 8A^3B^3 + A^{12} - A^9B^9)(B^{12} - 8A^3B^3 + A^{12} + A^9B^9) = 0. \tag{2.5}$$

Setting $q = e^{-\pi}$, we see that $A = 2^{1/4}G_1$ and $B = 2^{1/4}G_9$. From [2, p. 189], we have $G_1 = 1$ and $G_9 = \left[\frac{1+\sqrt{3}}{\sqrt{2}}\right]^{1/3}$. Using these values in (2.5), we see that the second factor does not vanishes. Hence the first factor is equal to zero for $q = e^{-\pi}$. Therefore first factor vanishes in some neighborhood of $e^{-\pi}$. So by the identity theorem, we have

$$B^{12} + 8A^3B^3 + A^{12} - A^9B^9 = 0.$$

Dividing this throughout by $(AB)^6$, we deduce that

$$\left(\frac{B}{A}\right)^6 + \left(\frac{A}{B}\right)^6 - (AB)^3 + \frac{8}{(AB)^3} = 0.$$

From Entry 12 (i) and (iii) of Chapter 17 [1, p. 124], we have $\left(\frac{A}{B}\right)^6 = Q^6$ and $AB = \sqrt{2}/P$. Using this in the above identity, we obtain the required result. ∎

Theorem 2.2. *If β has degree 5 over α, then*

$$Q^3 + \frac{1}{Q^3} + 2\left(P^2 - \frac{1}{P^2}\right) = 0.$$

Proof. Let

$$A := \frac{f(q)}{q^{1/24}f(-q^2)}, \qquad B := \frac{f(q^5)}{q^{5/24}f(-q^{10})},$$
$$C := \frac{f(-q^2)}{q^{1/12}f(-q^4)} \quad \text{and} \quad D := \frac{f(-q^{10})}{q^{5/12}f(-q^{20})}.$$

From (1.1), we have

$$C^{16} - A^{16}C^8 + 16A^8 = 0. \tag{2.6}$$

Changing q to q^5 in the above, we have

$$D^{16} - B^{16}D^8 + 16B^8 = 0. \tag{2.7}$$

From (1.3), we have

$$(CD)^8 - (CD)^4(AB)^8 + 8(ABCD)^4 + 16(AB)^4 = 0. \tag{2.8}$$

Eliminating C between (2.6) and (2.8) using maple, we arrive at

$$A^{24}(16B^8D^{16} - B^{16}D^{24} - 16B^{24}D^8 + B^{32}D^{16}) + 16A^{20}(-2B^{28}D^{16} + 16B^{20}D^8 + B^{12}D^{24})$$
$$+ 64A^{16}(6B^{24}D^{16} - 16B^{16}D^8 - B^8D^{24}) + 32A^{12}(-66B^{20}D^{16} + 16B^{12}D^8 + B^4D^{24})$$
$$+ 5120A^8B^{16}D^{16} - 4096A^4B^{12}D^{16} + 512B^8D^{16} + 4096B^{16} + 16D^{32} = 0.$$

Now, eliminating D between the above identity and (2.7) using maple, we obtain

$$X(A,B)Y(A,B) = 0,$$

where

$$X(A,B) = B^6 + 4BA - (BA)^5 + A^6$$

and

$$Y(A,B) = (B^6 - 4BA + (BA)^5 + A^6)(B^{12} + (AB)^{10} - 10(AB)^6 + 16(BA)^2 + A^{12}).$$

Setting $q = e^{-\pi}$, we have $A = 2^{1/4}G_1$ and $B = 2^{1/4}G_{25}$. From [2, pp. 189-190], we have $G_1 = 1$ and $G_{25} = \frac{1+\sqrt{5}}{2}$. Using these values in $Y(A,B)$, we see that $Y(A,B) \neq 0$. Hence $X(A,B)$ vanishes in some neighborhood of $e^{-\pi}$. Thus,

$$X(A,B) = 0,$$

which is equivalent to

$$\frac{B^3}{A^3} + \frac{A^3}{B^3} + \frac{4}{(AB)^2} - (AB)^2 = 0.$$

Transforming the above identity through Entry 12 (i) and (iii) of Chapter 17 [1, p. 124], we obtain the required result. ∎

Theorem 2.3. *If β has degree 7 over α, then*

$$Q^4 + \frac{1}{Q^4} + 7 = 2\sqrt{2}\left(P^3 + \frac{1}{P^3}\right).$$

Proof. Let

$$A := \frac{f(q)}{q^{1/24}f(-q^2)}, \qquad B := \frac{f(q^7)}{q^{7/24}f(-q^{14})},$$

$$C := \frac{f(-q^2)}{q^{1/12}f(-q^4)} \quad \text{and} \quad D := \frac{f(-q^{14})}{q^{7/12}f(-q^{28})}.$$

From (1.1), we have

$$C^{16} - A^{16}C^8 + 16A^8 = 0. \tag{2.9}$$

Changing q to q^7 in the above, we have

$$D^{16} - B^{16}D^8 + 16B^8 = 0. \tag{2.10}$$

From (1.4), we have

$$(CD)^2 - CD(AB)^2 + 2AB = 0. \tag{2.11}$$

Eliminating C between (2.9) and (2.11) using maple, we arrive at
$$-A^{24}(B^{32}D^{16} + 16B^{24}D^8 + B^{16}D^{24} - 16B^8D^{16}) + 16A^{21}(2B^{29}D^{16} - 16B^{21}D^8 - B^{13}D^{24})$$
$$+ 16A^{18}(-26B^{26}D^{16} + 80B^{18}D^8 + 5B^{10}D^{24}) + 128A^{15}(22B^{23}D^{16} - 16B^{15}D^8 - B^7D^{24})$$
$$+ 32A^{12}(-330B^{20}D^{16} + 16B^{12}D^8 + B^4D^{24}) + 21504A^9B^{17}D^{16}$$
$$- 21504A^6B^{14}D^{16} + 8192A^3B^{11}D^{16} - 512B^8D^{16} - 4096B^{16} - 16D^{32} = 0.$$
Now, eliminating D between the above identity and (2.10) using maple, we obtain

$$X(A,B)Y(A,B) = 0,$$

where

$$X(A,B) = A^8 - 8AB + 7(AB)^4 - (AB)^7 + B^8,$$

and

$$Y(A,B) = A^{16} + A^{15}B^7 + (AB)^{14} - 7A^{12}B^4 - 14(AB)^{11} + 8A^9B + 64(AB)^8 + A^7B^{15}$$
$$-112(AB)^5 - 7A^4B^{12} + 64(AB)^2 + 8AB^9 + B^{16}.$$

Setting $q = e^{-\pi}$, we have $A = 2^{1/4}G_1$ and $B = 2^{1/4}G_{49}$. From [2, pp. 189-191], we have $G_1 = 1$ and $G_{25} = \frac{7^{1/4}+\sqrt{4+\sqrt{7}}}{2}$. Using these values in $Y(A,B)$, we see that $Y(A,B) \neq 0$. Hence $X(A,B)$ vanishes in some neighborhood of $e^{-\pi}$. Thus,

$$X(A,B) = 0,$$

which is equivalent to

$$\frac{B^4}{A^4}+\frac{A^4}{B^4}-\frac{8}{(AB)^3}-(AB)^3+7=0.$$

Transforming the above identity through Entry 12 (i) and (iii) of Chapter 17 [1, p. 124], we obtain the required result. ∎

Theorem 2.4. *If β has degree 11 over α, then*

$$Q^6+\frac{1}{Q^6}-2\sqrt{2}\left(\frac{2}{P^5}-\frac{11}{P^3}+\frac{22}{P}-22P+11P^3-2P^5\right)=0.$$

Proof. Let

$$A:=\frac{f(q)}{q^{1/24}f(-q^2)}, \qquad B:=\frac{f(q^{11})}{q^{11/24}f(-q^{22})},$$

$$C:=\frac{f(-q^2)}{q^{1/12}f(-q^4)} \quad \text{and} \quad D:=\frac{f(-q^{22})}{q^{11/12}f(-q^{44})}.$$

From (1.1), we have

$$C^{16}-A^{16}C^8+16A^8=0. \tag{2.12}$$

Changing q to q^{11} in the above, we have

$$D^{16}-B^{16}D^8+16B^8=0. \tag{2.13}$$

From (1.5), we have

$$(CD)^4-(AB)^4(CD)^2+4(ABCD)^2+4(AB)^2=0. \tag{2.14}$$

Eliminating C between (2.12) and (2.14) using maple, we arrive at

$$\begin{aligned}&-16D^{32}+D^{24}(A^{24}B^{16}-16A^{22}B^{14}+96A^{20}B^{12}-272A^{18}B^{10}+384A^{16}B^8-256A^{14}B^6\\&+32A^{12}B^4)+D^{16}(-A^{24}B^{32}-16A^{24}B^8+32A^{22}B^{30}-448A^{20}B^{28}+3616A^{18}B^{26}-18688A^{16}B^{24}\\&+65024A^{14}B^{22}-155968A^{12}B^{20}+259072A^{10}B^{18}-292864A^8B^{16}+214016A^6B^{14}\\&-90112A^4B^{12}+16384A^2B^{10}-512B^8)+D^8(16A^{24}B^{24}-256A^{22}B^{22}+1536A^{20}B^{20}\\&-4352A^{18}B^{18}+6144A^{16}B^{16}-4096A^{14}B^{14}+512A^{12}B^{12})-4096B^{16}=0.\end{aligned}$$

Now, eliminating D between the above identity and (2.13) using maple, we obtain

$$X(A,B)Y(A,B)=0,$$

where

$$X(A,B)=-(AB)^{11}+11(AB)^9+88(AB)^5-88(AB)^3-44(AB)^7+A^{12}+B^{12}+32AB,$$

and

$$Y(A,B)=(AB)^{11}-11(AB)^9-88(AB)^5+88(AB)^3+44(AB)^7+A^{12}+B^{12}-32AB.$$

Setting $q=e^{-\pi}$, we have $A=2^{1/4}G_1$ and $B=2^{1/4}G_{121}$. From [2, pp. 189-191], we have $G_1=1$ and $G_{121}=\frac{1}{3\sqrt{2}}[(11+3\sqrt{11})^{1/3}((3\sqrt{11}+3\sqrt{3}+4)^{1/3}+(3\sqrt{11}-3\sqrt{3}+4)^{1/3}+2)]$. Using these values in $Y(A,B)$, we see that $Y(A,B)\neq 0$. Hence, $X(A,B)$ vanishes in some neighborhood of $e^{-\pi}$. Thus,

$$X(A,B)=0,$$

which is equivalent to

$$-(AB)^5+11(AB)^3-44AB+\frac{88}{AB}-\frac{88}{(AB)^3}+\frac{32}{(AB)^5}+\left(\frac{A}{B}\right)^6+\left(\frac{B}{A}\right)^6=0.$$

Transforming the above identity through Entry 12 (i) and (iii) of Chapter 17 [1, p. 124], we obtain the required result. ∎

Theorem 2.5. *If β has degree 23 over α, then*

$$\begin{aligned}Q^{12} &+ \frac{1}{Q^{12}} - 32\sqrt{2}\left(P^{11}+\frac{1}{P^{11}}\right) + 736\left(P^{10}+\frac{1}{P^{10}}\right) - 4048\sqrt{2}\left(P^{9}+\frac{1}{P^{9}}\right)\\ &+ 28704\left(P^{8}+\frac{1}{P^{8}}\right) - 74336\sqrt{2}\left(P^{7}+\frac{1}{P^{7}}\right) + 300840\left(P^{6}+\frac{1}{P^{6}}\right)\\ &- 495328\sqrt{2}\left(P^{5}+\frac{1}{P^{5}}\right) + 1362336\left(P^{4}+\frac{1}{P^{4}}\right) - 1592336\sqrt{2}\left(P^{3}+\frac{1}{P^{3}}\right)\\ &+ 3200864\left(P^{2}+\frac{1}{P^{2}}\right) - 2787232\sqrt{2}\left(P+\frac{1}{P}\right) + 4223122 = 0.\end{aligned}$$

Proof. Let

$$A := \frac{f(q)}{q^{1/24}f(-q^2)}, \qquad B := \frac{f(q^{23})}{q^{23/24}f(-q^{46})},$$

$$C := \frac{f(-q^2)}{q^{1/12}f(-q^4)} \qquad \text{and} \qquad D := \frac{f(-q^{46})}{q^{23/12}f(-q^{92})}.$$

From (1.1), we have

$$C^{16} - A^{16}C^8 + 16A^8 = 0. \tag{2.15}$$

Changing q to q^{23} in the above, we have

$$D^{16} - B^{16}D^8 + 16B^8 = 0. \tag{2.16}$$

From (1.6), we have

$$(CD)^2 - (AB)^2CD + 2ABCD + 2AB = 0. \tag{2.17}$$

Eliminating C between (2.15) and (2.17) and then, eliminating D between the resulting identity and (2.16) using maple, we arrive at

$$\begin{aligned}&A^{24} - (AB)^{23} + 23(AB)^{22} - 253(AB)^{21} + 1794(AB)^{20} - 9292(AB)^{19} + 37605(AB)^{18}\\ &- 123832(AB)^{17} - 123832(AB)^{17} + 340584(AB)^{16} - 796168(AB)^{15} + 1600432(AB)^{14}\\ &- 2787232(AB)^{13} + 4223122(AB)^{12} - 5574464(AB)^{11} + 6401728(AB)^{10} - 6369344(AB)^{9}\\ &+ 5449344(AB)^{8} - 3962624(AB)^{7} + 2406720(AB)^{6} - 1189376(AB)^{5} + 459264(AB)^{4}\\ &- 129536(AB)^{3} + 23552(AB)^{2} - 2048AB + B^{24} = 0.\end{aligned}$$

Dividing the above identity throughout by $(AB)^{12}$ and then transforming the resulting identity through Entry 12 (i) and (iii) of Chapter 17 [1, p. 124], we obtain the required result. ∎

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