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COINCIDENCES AND FIXED POINTS OF HYBRID MAPS IN SYMMETRIC SPACES

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ABSTRACT. The purpose of this paper is to obtain a new coincidence theorem for a single-valued and two multivalued operators in symmetric spaces. We derive fixed point theorems and discuss some special cases and applications.

Key words and phrases: Coincidence, Fixed point, Hybrid maps, Symmetric space.

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1. INTRODUCTION

$$\sup_{a\in A}\inf_{b\in B}\lim_{c\in C}$$

Fixed point theorems for multivalued contractions were first initiated by Markin [16] and Nadler, Jr. [18]. Subsequently, a number of generalizations of Nadler's multivalued contraction principle were obtained in different settings (see, for instance, [4, 12, 14, 17, 19], [22]–[26], [28, 30] and several references thereof). Fixed point theory in multivalued analysis finds applications in optimization/control theory, operating systems, disjunctive logic programs, information theory, fractals and other areas of mathematical sciences (see, for instance, [7, 11, 15, 28] and [30]). Hybrid fixed point theory for nonlinear single-valued and multivalued maps is a recent development in multivalued analysis (see, for instance, [3, 14, 19], [21]–[26] and references thereof). Recently Aamri et al. [1] and [2], Hicks and Rhoades [9, 10] and Moutawakil [17] have obtained some fixed point theorems for single-valued and multivalued maps in d-bounded symmetric spaces (see also [13]). The purpose of this paper is to present coincidence theorems for hybrid contractions on symmetric spaces (not necessarily d-bounded). The completeness requirement of the space is also relaxed. We derive fixed point theorems generalizing their results ([10] and [17]) and discuss some applications.

2. PRELIMINARIES

We will follow the notations and definitions used in [1, 2, 9, 10, 17] and [27].

Definition 2.1. A symmetric function on a nonempty set X is a nonnegative real-valued map d on $X \times X$ such that

(i) $d(x, y) = 0$ if and only if $x = y$, and
(ii) $d(x, y) = d(y, x)$ for all x, $y \in X$.

Let d be a symmetric on a set X and for $r > 0$ and any $x \in X$, let

$$B(x, r) = \{y \in X : d(x, y) < r\}.$$

A topology $t(d)$ on X is given by $U \in t(d)$ if and only if for each $x \in U$, $B(x, r) \subseteq U$, for some $r > 0$.

A symmetric d is a semi-metric if for each $r > 0$, $B(x, r)$ is a neighbourhood of x in the topology $t(d)$.

Definition 2.2. Let (X, d) be a symmetric space. Then:
A nonempty subset P of X is d-closed if and only if $\overline{P}_d = P$, where

$$\overline{P}_d = \{x \in X : d(x, P) = 0\}$$

and

$$d(x, P) = \inf \{d(x, p) : p \in P\}.$$

Definition 2.3. A nonempty set P is called d-bounded if and only if $\delta_d(P) < \infty$, where

$$\delta_d(P) = \sup \{d(x, p) : x, p \in P\}.$$

Definition 2.4. The space (X, d) is S-complete if for every d-Cauchy sequence $\{x_n\}$, there exists x in X with

$$\lim_{n\to\infty} d(x_n, x) = 0.$$

Definition 2.5. Let (X, d) be a symmetric space and let $CB(X)$ be the set of all nonempty d-closed and d-bounded subsets of X. The Hausdorff metric H induced by the symmetric d is defined in the usual way:

$$H(A, B) = \max\left\{\sup_{b\in B} d(b, A), \sup_{a\in A} d(a, B)\right\}$$

for all $A, B \in CB(X)$. As noted in [1] and [17], the hyper space $(CB(X), H)$ is a symmetric space induced by the symmetric d.

Definition 2.6. The maps $f : X \to X$ and $T : X \to CB(X)$ are (IT)-commuting at a point $x \in X$ if $fTx \subset Tfx$ [12].

This definition essentially due to Itoh and Takahashi [12] has widely been used in hybrid fixed point theory (see, for instance, [24]–[27]).

We remark that (IT)-commutativity of a hybrid pair T and f at a coincidence point $x \in X$ is more general than its compatibility and weak compatibility at the same point (see [25, Example 1]).

In our results we need the following axioms essentially due to Wilson [29] for the symmetric spaces (see also [1, 10] and [17]). Let (X, d) be a symmetric space. Then

(**W.3**) Given $\{x_n\}$, x and y in X,

$$\lim_{n\to\infty} d(x_n, x) = 0 \text{ and } \lim_{n\to\infty} d(x_n, y) = 0 \text{ imply } x = y.$$

(**W.4**) Given $\{x_n\}$, $\{y_n\}$ and x in X,

$$\lim_{n\to\infty} d(x_n, x) = 0 \text{ and } \lim_{n\to\infty} d(x_n, y_n) = 0 \text{ imply that } \lim_{n\to\infty} d(y_n, x) = 0.$$

If $t(d)$ is Hausdorff then (**W.3**) holds (see Hicks and Rhoades [10, p. 330]).

(iii) X is S-complete if for every d-Cauchy sequence $\{x_n\}$, there exists an x in X with

$$\lim_{n\to\infty} d(x_n, x) = 0.$$

(iv) X is d-Cauchy complete if for every d-Cauchy sequence $\{x_n\}$, there exists an x in X with $x_n \to x$ in the topology $t(d)$.

We remark that S-completeness implies d-Cauchy completeness (see [1] and [17]).

We shall need the following results:

Lemma 2.1. *Let (X, d) be a symmetric space. Let M be a d-bounded subset of X and $\{y_n\}$ be a sequence in M such that*

$$d(y_j, y_{j+1}) \le q d(y_{j-1}, y_j), j = 1, 2, 3, \ldots, \text{ where } 0 \le q < 1.$$

Then $\{y_n\}$ is a d-Cauchy sequence.

Proof. It may be completed using the relevant part of the proof of Theorem 2.2.1 [17, p. 28]. ∎

Lemma 2.2. ([17])*Let (X, d) be a symmetric space. Let A, $B \in CB(X)$ and $\lambda > 1$. For each $a \in A$, there exists $b \in B$ such that*

$$d(a, b) \le \lambda H(A, B).$$

Indeed, this result in a metric space is essentially due to Nadler, Jr. [18] and Ćirić [4].

3. MAIN RESULTS

First we give a coincidence theorem.

Theorem 3.1. *Let (X, d) be a symmetric space satisfying $($**W.4**$)$. Let $f : X \to X$ and $S, T : X \to CB(X)$ such that*

$$S(X) \cup T(X) \subseteq f(X), \tag{3.1}$$

$$H(Sx, Ty) \leq k \max\{d(fx, fy), d(fx, Sx), d(fy, Ty)\} \tag{3.2}$$

for all $x, y \in X$, where $0 < k < 1$.

If $f(X)$ is d-bounded, and one of $f(X)$ or $S(X)$ or $T(X)$ is an S-complete subspace of X, then f, S and T have a coincidence, i.e., there exists an element $z \in X$ such that

$$fz \in Sz \cap Tz.$$

Proof. Pick $x_0 \in X$. Construct sequences $\{x_n\}$ and $\{y_n\}$ in the following manner. Choose $x_1 \in X$ such that

$$y_1 = fx_1 \in Sx_0.$$

We may choose a point $x_2 \in X$ such that

$$y_2 = fx_2 \in Tx_1$$

and

$$d(y_1, y_2) = d(fx_1, fx_2) \leq \alpha H(Sx_0, Tx_1),$$

where $\alpha = \frac{1}{\sqrt{k}} > 1$, $0 < k < 1$. Similarly, we choose a point $x_3 \in X$ such that

$$y_3 = fx_3 \in Sx_2$$

and

$$d(y_2, y_3) \leq \alpha H(Tx_1, Sx_2).$$

Continuing in this fashion, we may choose

$$y_{2n} = fx_{2n} \in Tx_{2n-1}$$

and

$$y_{2n+1} = fx_{2n+1} \in Sx_{2n}$$

such that

$$d(y_{2n}, y_{2n+1}) = d(fx_{2n}, fx_{2n+1}) \leq \alpha H(Tx_{2n-1}, Sx_{2n}).$$

Similarly,

$$d(y_{2n+1}, y_{2n+2}) = d(fx_{2n+1}, fx_{2n+2}) \leq \alpha H(Sx_{2n}, Tx_{2n+1}).$$

By (3.2),

$$\begin{aligned} d(y_{2n}, y_{2n+1}) &= d(fx_{2n}, fx_{2n+1}) \leq \alpha H(Tx_{2n-1}, Sx_{2n}) \\ &\leq \alpha k. \max\{d(fx_{2n-1}, fx_{2n}), d(fx_{2n}, Sx_{2n}), d(fx_{2n-1}, Tx_{2n-1})\} \\ &\leq \alpha k. \max\{d(fx_{2n-1}, fx_{2n}), d(fx_{2n}, fx_{2n+1}), d(fx_{2n-1}, fx_{2n})\}. \end{aligned}$$

This yields

$$d(y_{2n}, y_{2n+1}) \leq \sqrt{k} d(y_{2n-1}, y_{2n}).$$

Similarly,

$$d(y_{2n+1}, y_{2n+2}) \leq \sqrt{k} d(y_{2n}, y_{2n+1}).$$

Both together imply

$$d(y_n, y_{n+1}) \leq \sqrt{k}d(y_{n-1}, y_n), n = 1, 2, 3, \ldots.$$

So, by Lemma 2.1, the sequence $\{y_n\}$ is a d-Cauchy in $f(X)$. Now suppose the subspace $f(X)$ is S-complete. Then, there exists an element $u \in f(X)$ such that

$$\lim_{n\to\infty} d(u, y_n) = 0.$$

Notice that the subsequences $\{y_{2n-1}\}$ and $\{y_{2n}\}$ also converge to u. Since $u \in f(X)$, there exists an element $z \in f^{-1}u$ such that $fz = u$. From Lemma 2.2, for each $n \in \{1, 2, 3, \ldots\}$, there exists an element $fz_n \in Sz$ such that

$$d(fz_n, fx_{2n}) \leq \varepsilon H(Sz, Tx_{2n-1}),$$

where $\varepsilon = \frac{1}{\sqrt{k}} > 1$, $0 < k < 1$.

Let

$$\mu = \lim_{n\to\infty} d(fz_n, fz).$$

Then by (3.2),

$$\begin{aligned} d(fz_n, fx_{2n}) &\leq \varepsilon k. \max\{d(fz, fx_{2n-1}), d(fz, Sz), d(fx_{2n-1}, Tx_{2n-1})\} \\ &\leq \sqrt{k}\max\{d(fz, fx_{2n-1}), d(fz, fz_n), d(fx_{2n-1}, fx_{2n})\}. \end{aligned}$$

Making $n \to \infty$, we get

$$\mu \leq \sqrt{k}\max\{0, \mu, 0\}.$$

This gives $\mu = 0$.

Thus, we have

$$\lim_{n\to\infty} d(fz_n, fz) = 0$$

and

$$\lim_{n\to\infty} d(fx_{2n}, fz) = 0.$$

So, by **(W.4)**, we get

$$\lim_{n\to\infty} d(fx_{2n}, fz_n) = 0.$$

Notice that $fx_{2n} \in Tx_{2n-1}$ and $fz_n \in Sz$. So,

$$\lim_{n\to\infty} d(fx_{2n}, Sz) \leq \lim_{n\to\infty} d(fx_{2n}, fz).$$

This gives

$$d(fz, Sz) = 0 \text{ and } fz \in Sz.$$

Similar arguments give $fz \in Tz$.

Therefore,

$$fz \in Sz \cap Tz.$$

If $S(X)$ (respectively $T(X)$) is S-complete, then there exists

$$u \in S(X) \subseteq f(X)(\text{ respectively } u \in T(X) \subseteq f(X)),$$

and the above argument establishes the result. ∎

Now we apply Theorem 3.1 to obtain the following fixed point theorem.

Theorem 3.2. *Let all the hypotheses of Theorem 3.1 hold. If $ffz = fz$ and f is (IT)- commuting with each of S and T, then the maps f, S and T have a common fixed point.*

Proof. By Theorem 3.1, there exists a $z \in X$ such that $fz \in Sz$ and $fz \in Tz$. As f is (IT)-commuting with each of S and T at z,

$$fz = ffz \in fSz \subset Sfz$$

and

$$fz = ffz \in fTz \subset Tfz.$$

Hence

$$fu = u \in Su \cap Tu,$$

where $u = fz$. ∎

We remark that the requirement $fz = ffz$ in the above theorem is essential for the existence of a common fixed point. In the absence of this requirement, the maps f, S and T need not have a common fixed point (see, for instance, [19, 22] and [23]).

Corollary 3.3. *Let (X, d) be a d-bounded symmetric space satisfying (W.4). Let $S, T : X \to CB(X)$ such that*

$$H(Sx, Ty) \leq k \max\{d(x, y), d(x, Sx), d(y, Ty)\} \tag{3.3}$$

for all x, $y \in X$, where $0 < k < 1$. If one of $S(X)$ or $T(X)$ is an S-complete subspace of X then S and T have a common fixed point.

Proof. It comes from Theorem 3.1 when f is the identity map on X. ∎

Corollary 3.4. *Let (X, d) be a symmetric space satisfying (W.4). Let $T(X) \subseteq f(X)$ and $f : X \to X$ and $T : X \to CB(X)$ such that*

$$H(Tx, Ty) \leq k \max\{d(fx, fy), d(fx, Tx), d(fy, Ty)\} \tag{3.4}$$

for all $x, y \in X$, where $0 < k < 1$.

If $f(X)$ is d-bounded, and one of $f(X)$ or $T(X)$ is an S-complete subspace of X, then f and T have a coincidence, i.e., there exists an element $z \in X$ such that $fz \in Tz$. Further, if fz is a fixed point of f, and f is (IT)-commuting with T, then fz is also a fixed point of T.

Proof. It comes from Theorems 3.1 and 3.2 when $S = T$. ∎

Corollary 3.5. *Let (X, d) be a symmetric space satisfying (W.4). Let $f : X \to X$ and $S, T : X \to CB(X)$ such that $f(X)$ is d-bounded, and one of $S(X)$ and $T(X)$ is S-complete subspace of X, and $S(X) \cup T(X) \subseteq f(X)$ and satisfying*

$$H(Sx, Ty) \leq kd(fx, fy), \tag{3.5}$$

for all $x, y \in X$, where $0 < k < 1$.

Then f, S and T have a coincidence point z (say). Further, if fz is a fixed point of f, and f is (IT)-commuting with each of S and T, then fz is a common fixed point of f, S and T.

Proof. The proof is obvious as the condition (3.5) is contained in (3.2). ∎

We remark that Corollaries 3.4 and 3.5 with $S = T$ are improved versions of Moutawakil [17, Th. 2.2.1] when f is the identity map on X.

Now following Moutawakil [17], we give an application of Corollary 3.4. First following [10, 17, 20] and [27] we give some definitions .

Definition 3.1. A function $F : R \to [0, 1]$ is a distribution function if

(v) F is non-decreasing
(vi) F is left continuous,

(vii)

$$\inf_{x \in R} F(x) = 0$$

and

$$\sup_{x \in R} F(x) = 1.$$

Definition 3.2. Let X be a set and $\Im$ a function defined on $X \times X$ such that $\Im(x, y) = F(x, y)$ is a distribution function. Consider the following conditions:

(viii) $F(x, y, 0) = 0$ for all $x, y \in X$.

(ix) $F(x, y) = f$ if and only if $x = y$, where f is the distribution function defined by $f(x) = 0$ if $x \leq 0$, and $f(x) = 1$ if $x > 0$.

(x) $F(x, y) = F(y, x)$ for all $x, y \in X$.

(xi) If $F(x, y, \alpha) = 1$ and $F(y, z, \beta) = 1$ then $F(x, z, \alpha + \beta) = 1$, for all $x, y, z \in X$.

If $\Im$ satisfies (viii) and (ix), then it is called a PPM-structure on X and the pair $(X, \Im)$ is called a PPM-space and $\Im$ satisfying (x) is said to be symmetric. A symmetric PPM-structure $\Im$ satisfying (xi) is a probabilistic metric structure and the pair $(X, \Im)$ is a probabilistic metric space.

Let $(X, \Im)$ be a symmetric PPM-space. For $\alpha, \gamma > 0$ and $x \in X$, let

$$N_x(\alpha, \gamma) = \{y \in X : F(x, y, \alpha) > 1 - \gamma\}.$$

A T_1 topology $t(\Im)$ on X is defined as follows:

$$t(\Im) = \{U \subseteq X : \text{ for each } x \text{ in } U, \text{ there exists } \alpha > 0, \text{ such that } N_x(\alpha, \alpha) \subseteq U\}.$$

Definition 3.3. Let $(X, \Im)$ be a symmetric PPM-space. A sequence $\{x_n\}$ in X is called a fundamental sequence if

$$\lim_{n,m \to \infty} F(x_n, x_m, t) = 1$$

for all $t > 0$. The space is called F-complete if for every fundamental sequence $\{x_n\}$ in X, there exists an $x \in X$ such that

$$\lim_{n \to \infty} F(x_n, x, t) = 1$$

for all $t > 0$.

In space $(X, \Im)$, the condition **(W.4)** is equivalent to the following:

(C.4)

$$\lim_{n \to \infty} F(x_n, x, t) = 1$$

and

$$\lim_{n \to \infty} F(x_n, y_n, t) = 1$$

imply

$$\lim_{n \to \infty} F(y_n, x, t) = 1$$

for all $t > 0$.

Definition 3.4. Let $(X, \Im)$ be a symmetric PPM-space. A nonempty subset P of X is called $\Im$-closed if and only if $\overline{P}_{\Im} = P$, where

$$\overline{P}_{\Im} = \left\{ x \in X : \sup_{a \in P} F(x, a, t) = 1 \text{ for all } t > 0 \right\}.$$

For the details of the topological preliminaries, one may refer to [6] and [20]. In all that follows we denote the set of all nonempty $\Im$-closed subsets of X by $CB_{\Im}(X)$ and the set of nonnegative real numbers by R^+.

The following is a slightly modified version of Moutawakil [17, Prop. 2.3.1].

Proposition 3.6. ([17]). *Let $(X, \Im)$ be a symmetric PPM-space. Let p a compatible symmetric function for $t(\Im)$. For A, $B \in CB(X)$, set*

$$E(A, B, \varepsilon) = \min\left\{\inf_{a\in A}\sup_{b\in B} F(a, b, \varepsilon); \inf_{b\in B}\sup_{a\in A} F(a, b, \varepsilon)\right\}, \varepsilon > 0,$$

and

$$P(A, B) = \max\left\{\sup_{a\in A}\inf_{b\in B} p(a, b); \sup_{b\in B}\inf_{a\in A} p(a, b)\right\}.$$

Let $f : X \to X$ and $T : X \to CB(X)$. Then, if

$$F(fx, fy, t) > 1 - t$$

implies

$$E(Tx, Ty, kt) > 1 - kt, 0 \le k < 1,$$

for all $t > 0$ and all x, $y \in X$, implies that

$$P(Tx, Ty) \le kp(fx, fy).$$

In a symmetric PPM space $(X, \Im)$, if p is a compatible symmetric function on $t(\Im)$ then

$$CB_{\Im}(X) = CB(X),$$

where $CB(X)$ is the set of all nonempty p-closed subsets of (X, p).

Hicks and Rhoades [10] obtained the following result showing that each symmetric PPM-space admits a compatible symmetric function.

Theorem 3.7. ([10]). *Let $(X, \Im)$ be a symmetric PPM-space. Let $p : X \times X \to R^+$ be a function defined as follows:*

$$p(x, y) = \begin{cases} 0 & \text{if } y \in N_x(t, t) \text{ for all } t > 0, \\ \sup\{t : y \notin N_x(t, t), 0 < t < 1\} & \text{otherwise.} \end{cases}$$

Then

(i) *$p(x, y) < t$ if and only if $F(x, y, t) > 1 - t$;*
(ii) *p is a compatible symmetric for $t(\Im)$;*
(iii) *$(X, \Im)$ is F-complete if and only if (X, p) is F-complete.*

Now we present the following result in a symmetric PPM-space.

Theorem 3.8. *Let $(X, \Im)$ be a F-complete symmetric PPM-space that satisfies* **(C.4)** *such that p is a compatible symmetric function for $t(\Im)$. Let $f : X \to X$ and $T : X \to CB_{\Im}(X)$ be maps such that*

$$F(fx, fy, t) > 1 - t$$

implies

$$E(Tx, Ty, kt) > 1 - kt, 0 \le k < 1,$$

for all x, $y \in X$.

Then there exists a $z \in X$ such that $fz \in Tz$. Further, if f and T are (IT)-commuting just at z, and if fz is a fixed point of f, then f and T have a common fixed point.

Proof. Clearly (X, p) is a bounded and S-complete symmetric space and we have

$$p(fx, fy) < t$$

if and only if

$$F(fx, fy, t) > 1 - t.$$

Given $\varepsilon > 0$, put $t = p(fx, fy) + \varepsilon$.

Then,

$$F(fx, fy, t) > 1 - t.$$

Therefore

$$F(Tx, Ty, kt) > 1 - kt, 0 \leq k < 1,$$

for all x, $y \in X$.

From Proposition 3.6, we obtain

$$P(Tx, Ty) \leq kt = kp(fx, fy) + k\varepsilon.$$

As $\varepsilon > 0$ is arbitrary, on letting $\varepsilon \to 0$,

$$P(Tx, Ty) \leq kp(fx, fy).$$

An application of Corollary 3.5 with $S = T$ completes the proof. ∎

Corollary 3.9. ([17]) *Let* $(X, \Im)$ *be a F-complete symmetric PPM-space that satisfies* (**C.4**) *such that* p *is a compatible symmetric function for* $t(\Im)$. *Let* $T : X \to CB_{\Im}(X)$ *be a multivalued mapping such that*

$$F(x, y, t) > 1 - t$$

implies

$$E(Tx, Ty, kt) > 1 - kt, 0 \leq k < 1,$$

for all x, $y \in X$ *and* $t > 0$. *Then there exists* $z \in X$ *such that* $z \in Tz$.

Proof. It comes from Theorem 3.8 when f is the identity map on X. ∎

REFERENCES

[1] M. AAMRI, A. BASSOU and D. El MOUTAWAKIL, Common fixed points for weakly compatible maps in symmetric spaces with application to probabilistic spaces, *Appl. Math. E-Notes*, **5** (2005), pp. 171–175.

[2] M. AAMRI and D. El MOUTAWAKIL, Common fixed points under contractive conditions in symmetric spaces, *Appl. Math. E-Notes*, **3** (2003), pp. 156–162.

[3] A. AHMAD and A. R. KHAN, Some common fixed point theorems for nonself hybrid contractions, *J. Math. Anal. Appl* **213** (1997), pp. 275–286.

[4] Lj. B. ĆIRIĆ, Fixed points for generalized multi-valued contractions, *Mat. Vesnik* **9** (24) (1972), pp. 265–272.

[5] Lj. B. ĆIRIĆ, A generalization of Banach's contraction principle, *Proc. Amer. Math. Soc.* **45** (2) (1974), pp. 267–273.

[6] G. CONSTANTIN and V. I. ISTRĂŢESCU, *Elements of Probabilistic Analysis*, Kluwer Academic Publishers, Dordrecht, 1989.

[7] H. W. CORLEY, Some hybrid fixed point theorems related to optimization, *J. Math. Anal. Appl.* **120** (1980), pp. 528–532.

[8] T. L. HICKS, Fixed point theory in probabilistic metric spaces II, *Math. Japon.* **44** (3) (1996), pp. 487–493.

[9] T. L. HICKS and B. E. RHOADES, Fixed point theorems for pairs of mapping in d-complete topological spaces, *Internat. J. Math. Math. Sci.* **16** (1993), pp. 259–266.

[10] T. L. HICKS and B. E. RHOADES, Fixed point theory in symmetric spaces with applications to probabilistic spaces, *Nonlinear Anal.* **36** (1999), pp. 331–344.

[11] P. HITZLER and A. K. SEDA, *Multivalued mappings, fixed point theorems and disjunctive databases*, 3rd Irish Workshop on Formal Methods (IWFM–99).

[12] S. ITOH and W. TAKAHASHI, Single-valued mappings, multivalued mappings, and fixed point theorems, *J. Math. Anal. Appl.* **59** (1977), pp. 514–521.

[13] J. JACHYMSKI , J. MATKOWSKI and T. SWIATKOWSKI, Nonlinear contractions on semimetric spaces, *J. Appl. Anal.* **I** (1995), pp. 125–134.

[14] H. KANEKO and S. SESSA, Fixed point theorems for compatible multivalued and single-valued mappings, *Internat. J. Math. Math. Sci.* **12** (1989), pp. 257–262.

[15] M. L. LAPIDUS and M. van FRANKENHUIJSEN (eds.), *Fractal Geometry and Apllications: A Jubilee of Benoît Mandelbrot*, Proceedings of Symposia in Pure Mathematics, V. 72, Part I & II, Amer. Math. Soc., 2004.

[16] J. T. MARKIN, A fixed point theorem for set-valued mappings, *Bull. Amer. Math. Soc.* **74** (1968), pp. 639–640.

[17] D. E. MOUTAWAKIL, A fixed point theorem for multivalued mappings in symmetric spaces, *Appl. Math. E-Notes,* **4** (2004), pp. 26–32.

[18] S. B. NADLER, Jr., Multivalued contraction mappings, *Pacific J. Math.* **30** (1969), pp. 475–488.

[19] S. A. NAIMPALLY, S. L. SINGH and J. H. M. WHITFIELD, Coincidence theorems for hybrid contractions, *Math. Nachr.* **127** (1986), pp. 177–180.

[20] B. SCHWEIZER and A. SKLAR, *Probabilistic Metric Spaces*, North-Holland, Amsterdam, 1983.

[21] N. SHAHZAD, Coincidence points and R-subweakly commuting multivalued maps, *Demonstratio Math.* **36** (2003), pp. 427–431.

[22] S. L. SINGH, K. S. HA and Y. J. CHO, Coincidence and fixed points of nonlinear hybrid contractions, *Internat. J. Math. Math. Sci.* **12** (1989), pp. 247–256.

[23] S. L. SINGH and S. N. MISHRA, Nonlinear hybrid contractions, *J. Natur. Phys. Sci.* **5-8** (1991–1994), pp. 191–206.

[24] S. L. SINGH and S. N. MISHRA, On general hybrid contractions, *J. Austral. Math. Soc. Ser.* **A 66** (1999), pp. 244–254.

[25] S. L. SINGH and S. N. MISHRA, Coincidences and fixed points of nonself hybrid contractions, *J. Math. Anal. Appl.* **256** (2001), pp. 486–497.

[26] S. L. SINGH and S. N. MISHRA, Coincidence and fixed points of reciprocally continuous and compatible hybrid maps, *Internat. J. Math. Math. Sci.* **10** (2002), pp. 627–635.

[27] S. L. SINGH and B. D. PANT, Coincidence and fixed point theorems for a family of mappings on Menger spaces and extension to uniform spaces, *Math. Japon.* **33** (1988), pp. 957–973.

[28] R. WEGRZYK, *Fixed Point Theorems for Multivalued Mappings and their Applications to Functional Equations*, Dissertation Math. (Rozprawy Mat., 201), 1982.

[29] W. A. WILSON, On semi-metric spaces, *Amer. J. Math.* **53** (1931), pp. 361–373.

[30] E. ZEIDLER, *Nonlinear Functional Analysis and its Applications I: Fixed Point Theorems*, Springer-Verlag, New York, Inc. 1986.