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NOTE ON THE RANK OF BIRKHOFF INTERPOLATION

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ABSTRACT. The relationship between a variant of the rank of a univariate Birkhoff interpolation problem, called normal rank, and other numbers of interest associated to the interpolation problem is studied.

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1. INTRODUCTION

An $m \times d$ matrix $E = [e_{ij}]_{i=1,\, j=0}^{m \quad d-1}$ is an *incidence matrix* if its entries e_{ij} are 0 or 1. Assume that the number of ones in E, denoted by $|E|$, is equal to n and let $X \in \chi(E) = \{(x_1, \ldots, x_m) \mid x_1 < \cdots < x_m\}$. The pair E, X is called *regular* (or *poised*) if the determinant $D(E, X)$ of the matrix

$$A(E,X) = \left[\frac{x_i^{-j}}{(-j)!} \quad \frac{x_i^{1-j}}{(1-j)!} \quad \frac{x_i^{2-j}}{(2-j)!} \quad \cdots \quad \frac{x_i^{n-1-j}}{(n-1-j)!} \;;\; e_{ij} = 1 \right]$$

is nonzero; otherwise E, X is *singular*. Notice that E, X is regular if and only if for each choice of values c_{ij} (defined for $e_{ij} = 1$) there exists a unique interpolating polynomial $P(x) = \sum_{k=0}^{n-1} a_k \frac{x^k}{k!}$ that satisfies the conditions

$$P^{(j)}(x_i) = c_{ij},\ e_{ij} = 1 \tag{1.1}$$

or, equivalently, if the unique polynomial of degree at most $n - 1$ annihilated by E, X (i.e. satisfies conditions (1.1) when $c_{ij} = 0$) is the trivial polynomial $P \equiv 0$. If E, X is regular then there exists a unique monic polynomial of degree n annihilated by E, X ([5, 6]). A review on algebraic Birkhoff interpolation can be found in Lorentz, et al. [4].

We introduce the *normal rank* of the pair E, X as the order of the largest nonzero initial minor of $A(E, X)$. By initial minor of order q we mean the determinant of a matrix A_q which is obtained from any q rows and the first q columns of $A(E, X)$. Notice that the normal rank is less or equal to the rank of $A(E, X)$. Moreover, the pair E, X is regular if and only if its normal rank is equal to n.

Proposition 1.1. *Let $D = \det A_q$ be a nonzero initial minor of order q of $A(E, X)$. If all initial minors of order $q+1$ which are obtained from A_q by adding a row of $A(E, X)$ and the $(q+1)$-th column of $A(E, X)$ are zero, then the normal rank of E, X is equal to q.*

Proof. Assume that $q < p$ where p is the normal rank of E, X. The first p columns of $A(E, X)$ are linearly independent, and hence so are the first $q + 1$ columns. It follows that some initial minor of $A(E, X)$ obtained from A_q by adding a row and the $(q + 1)$-th column of $A(E, X)$ is nonzero, and this led us to a contradiction. ∎

The preceding result let us to compute the normal rank in a similar way as we compute the rank.

Our goal in this note is to study the relationship between the normal rank and other numbers of interest associated to the interpolation problem.

2. THE MAIN RESULT

Taking into account the preceding considerations, we can now state and prove our main result. We use the following notation. If $E' = \left[e'_{ij}\right]_{i=1,\, j=0}^{m \quad d-1}$ is an incidence matrix of the same size as E, we write $E' \leq E$ if $e'_{ij} \leq e_{ij}$ for all i, j. We have,

Theorem 2.1. *For a pair E, X let ℓ and s be constants defined by:*

(i) *ℓ is the lowest degree of the nontrivial polynomials annihilated by E, X.*
(ii) *s is the largest nonnegative integer number for which there exists an incidence matrix E' of the same size as E such that E', X is regular, $E' \leq E$ and $|E'| = s$.*

Then $\ell = s = r^*$ where r^* is the normal rank of E, X.

Before giving the proof of the theorem we notice that number s in (ii) has special meaning in the case when E, X is singular. In fact, every pair E', X with $E' \leq E$ defines some equations from (1.1). That is, as many equations as ones in E'. Therefore, s coincides with the greatest number of interpolatory conditions obtained from (1.1) so that the resulting interpolatory conditions determine a regular problem.

Proof. Let n be the number of ones in E. The fact that $s = r^*$ is immediate because of the initial minors of $A(E, X)$ coincide with the determinants of the form $D(E', X)$ with $E' \leq E$. In what follows, we will establish that $\ell = s$. Firstly, we prove that $\ell \geq s$. Indeed, let $P \not\equiv 0$ be a polynomial of degree l annihilated by E, X and we consider a matrix $E' \leq E$ with s ones and such that E', X is regular. Since P is annihilated by E', X then $l \geq s$ and therefore $\ell \geq s$. Secondly, we prove that $\ell \leq s$. In fact, let $E' = \left[e'_{ij}\right]_{i=1,\, j=0}^{m \quad d-1} \leq E$ be a matrix with s ones and such that E', X is regular, and let P be the unique monic polynomial of degree s annihilated by E', X.

We claim that P is annihilated by E, X. In fact, it suffices to prove that if $e_{i_0,j_0} = 1$ is an entry one of the matrix E with $e'_{i_0,j_0} = 0$, then $P^{(j_0)}(x_{i_0}) = 0$. Let E'' be the matrix obtained from E' by replacing entry $e'_{i_0,j_0} = 0$ by $e'_{i_0,j_0} = 1$. This matrix has $s+1$ ones and satisfies $E' \leq E'' \leq E$. From the definition of s we have that E'', X is a singular pair. Consider a polynomial $Q \not\equiv 0$ of degree at most s annihilated by E'', X. In particular this polynomial satisfies $Q^{(j_0)}(x_{i_0}) = 0$. Since E', X is regular and Q is also annihilated by E', X, we have $Q \equiv \lambda P$ for some constant $\lambda \neq 0$ and hence $P^{(j_0)}(x_{i_0}) = \frac{1}{\lambda} Q^{(j_0)}(x_{i_0}) = 0$. It follows that P is annihilated by E, X.

Thus, P is a polynomial of degree s annihilated by E, X and therefore $\ell \leq s$. This completes the proof. ∎

Finally, we give some examples of application of the preceding result. Let

$$E_1 = \left[\begin{array}{cc} 1 & 0 \\ \hline \multicolumn{2}{c}{\overline{E}} \\ \hline 1 & 0 \end{array}\right] \text{ and } E_2 = \left[\begin{array}{cc} 1 & 0 \\ 1 & 0 \\ \hline \multicolumn{2}{c}{\overline{E}} \\ \hline 1 & 0 \\ 1 & 0 \end{array}\right] \tag{2.1}$$

be defined by blocks, where $\overline{E} = [\bar{e}_{ij}]_{i=1,\, j=0}^{k \quad 1}$ is a matrix with zeros in the first column and ones in the second column. The number of ones in E_1 and E_2 is $n_1 = k+2$ and $n_2 = k+4$ respectively. If P is a nontrivial polynomial annihilated by E_1, X_1 or E_2, X_2, then its derivative P' has at least k and $k+2$ distinct zeros respectively (for E_2 this fact follows from Rolle's Theorem) and so the degree of P is at least $k+1$ and $k+3$ respectively. By applying Theorem 2.1 we get that the normal ranks r_1^* and r_2^* of E_1, X_1 and E_2, X_2 satisfy $r_1^* \geq k+1 = n_1 - 1$ and $r_2^* \geq k+3 = n_2 - 1$. Notice that if we apply Theorem 2.1 to E_2, X_2 again, we obtain that there exists a matrix $E_2' \leq E_2$ with $|E_2'| \geq n_2 - 1$ and being E_2', X_2 regular. It is an interesting fact since all matrices $E_2' \leq E_2$ with $|E_2'| \geq n_1 - 1$ have at least one row with exactly one odd supported sequence and they satisfy the strong Pólya condition and therefore they are order singular matrices (see [2, 4]).

For a matrix E let $r(E)$ be the lowest possible rank of $A(E, X)$ with $X \in \chi(E)$. Lower bounds for $r(E)$ have been studied in ([1, 3, 4]). The best of them can be found in [4, p. 14] and states that if E is a Pólya matrix with n ones and p odd supported sequences then

$$r(E) \geq n - \left[\frac{p+1}{2}\right] \tag{2.2}$$

where $[x]$ denotes the integer part of x. In some cases the normal rank can be used to improve (2.2). In fact, we have $r(E) \geq r^*(E)$ where $r^*(E)$ is the lowest possible normal rank of E, X with $X \in \chi(E)$. For matrices E_1 and E_2 in (2.1) we have proved that $r^*(E_1) \geq n_1 - 1$ and $r^*(E_2) \geq n_2 - 1$. It follows that $r(E_1) \geq n_1 - 1$ and $r(E_2) \geq n_2 - 1$ but, on the other hand, inequality (2.2) only assures that $r(E_1) \geq n_1 - [\frac{n_1-1}{2}] = [\frac{n_1}{2}] + 1$ and $r(E_2) \geq n_2 - [\frac{n_2-3}{2}] = [\frac{n_2}{2}] + 2$.

3. CONCLUDING REMARKS

In this note we have introduced the normal rank of a pair E, X and we have shown that it coincides with the lowest possible degree of polynomials annihilated by E, X and the greatest number of regular interpolatory conditions that can be obtained from the corresponding interpolation problem. Any of this numbers can be used to compute the normal rank. We have also seen that for some matrices the normal rank let us to improve the classical lower bounds for the rank. Finally, we notice that Theorem 2.1 remains valid when the knots x_i are distinct complex numbers, not necessarily real.

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