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ALGEBRAIC APPROACH TO THE FRACTIONAL DERIVATIVES

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ABSTRACT. In this paper we introduce an alternative definition of the fractional derivatives and also a characteristic class of so called ideal functions, which admit arbitrary fractional derivatives (also integrals). Further some ideal functions are found, which lead to representations of the Bernoulli and Euler numbers B_k and E_k for any real number k, via fractional derivatives of some functions at $x = 0$.

Key words and phrases: Fractional derivatives, Bernoulli numbers, Euler numbers, Ideal functions.

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1. SOME NEW THEORETICAL RESULTS FOR THE FRACTIONAL DERIVATIVES

Several authors have considered and introduced different methods for calculating of fractional derivatives of a given function (see [1]). An old idea for more than 170 years is to use power series and to apply the fractional derivatives to each summand. Later this method was considered by J. Liouville and B. Riemann. Recently it was developed [2, 3], such that

$$\Big(\sum_{i=-\infty}^{\infty} a_i \frac{x^i}{i!}\Big)^{(\alpha)} = \sum_{i=-\infty}^{\infty} a_i \frac{x^{i-\alpha}}{(i-\alpha)!}, \tag{1.1}$$

where $x! = \Gamma(x+1)$. Notice that $(-1)! = (-2)! = \cdots = \pm\infty$. The summands for $i \in Z^-$ play important role, although these summands are equal to zero. Indeed, assuming that the coefficients a_j $(j \geq 0)$ are known, then only for special choices of the coefficients a_{-j} $(j > 0)$ can yield to satisfactory results, called "natural" representation. For example, the natural representations for the functions e^x, $\sin x$ and $\cos x$ are the following:

$$e^x = \sum_{i=-\infty}^{\infty} \frac{x^i}{i!}, \tag{1.2}$$

$$\sin x = \sum_{i=-\infty}^{\infty} (-1)^i \frac{x^{2i+1}}{(2i+1)!}, \tag{1.3}$$

$$\cos x = \sum_{i=-\infty}^{\infty} (-1)^i \frac{x^{2i}}{(2i)!}, \tag{1.4}$$

such that we obtained [2, 3] from them the usual classical fractional derivatives. In this paper we consider the "natural" representations for the functions $x \cot x$, $\frac{xe^x}{e^x-1}$, $\frac{x}{\sin x}$ and some other functions. In this section we represent an improved version of this idea, by distinguishing a class of analytical functions which have "natural" representations for $i \in Z$.

Now let us assume that an analytical function f can be written in the form

$$f(x) = \sum_{i=-\infty}^{\infty} a_i \frac{x^{\alpha+i}}{(\alpha+i)!}, \qquad (x \neq 0 \;\text{ if }\; \alpha \notin Z), \tag{1.5}$$

which means that the sum of the right side, including the summation of divergent series, is just $f(x)$, for $x \neq 0$. The summation of any divergent series in this paper is assumed to be done via analytical continuation of functions, which is considered in more details in [2]. The formal calculation of the $(\alpha+i)$-th derivative at $x = 0$ yields that $f^{(\alpha+i)}(0) = a_i$. Hence

$$f(x) = \sum_{i=-\infty}^{\infty} f^{(\alpha+i)}(0) \frac{x^{\alpha+i}}{(\alpha+i)!}, \tag{1.6}$$

where α is an arbitrary real number. More generally

$$f(x) = \sum_{i=-\infty}^{\infty} f^{(\alpha+i)}(x_0) \frac{(x-x_0)^{\alpha+i}}{(\alpha+i)!}, \qquad (x \neq x_0 \;\text{ if }\; \alpha \notin Z), \tag{1.7}$$

which generalizes the ordinary Taylor's series.

On the other hand, if f admits fractional derivatives (integrations are also included) of arbitrary order, let $g = f^{(\alpha)}$. If we write g in the following form

$$g(x) = \sum_{i=-\infty}^{\infty} a_i \frac{x^i}{i!},$$

then

$$f(x) = g^{(-\alpha)}(x) = \sum_{i=-\infty}^{\infty} a_i \frac{x^{\alpha+i}}{(\alpha+i)!}$$

and f can be written in the required form. Namely, we proved the following theorem.

Theorem 1.1. *If f admits fractional derivatives of arbitrary order, then f satisfies the equalities (1.6) and (1.7).*

The previous discussion naturally yields to the following definition of fractional derivatives. For the sake of simplicity we can suppose first that the considered functions are real, and the summing of divergent series is done via analytical continuation of real functions.

Definition 1.1. Assume that a real analytical mapping f can be written in the following form

$$f(x) = \sum_{i=-\infty}^{\infty} C_i \frac{(x-x_0)^{\alpha+i}}{(\alpha+i)!}, \qquad (x_0 \in R,\ x > x_0), \tag{1.8}$$

for each $\alpha \in R$. Then f together with the above representations is called ideal function and we define $f^{(\alpha+i)}(x_0) := C_i$, $(i \in Z)$.

The functions e^x, $\sin x$ and $\cos x$ are ideal and the corresponding representations for arbitrary α are the following (see [2, 3])

$$e^x = \sum_{i=-\infty}^{\infty} \frac{x^{\alpha+i}}{(\alpha+i)!},$$

$$\sin x = \sum_{i=-\infty}^{\infty} \sin\frac{(\alpha+i)\pi}{2} \cdot \frac{x^{\alpha+i}}{(\alpha+i)!},$$

$$\cos x = \sum_{i=-\infty}^{\infty} \cos\frac{(\alpha+i)\pi}{2} \cdot \frac{x^{\alpha+i}}{(\alpha+i)!}.$$

Using the new definition we come to a *natural representation* of any ideal function. Namely, let f be such an analytical function and let in Taylor series it is written as $\sum\limits_{i=0}^{\infty} a_i \frac{x^i}{i!}$. The problem is how to find the coefficients $a_{-1}, a_{-2}, \ldots$ form the "zero part" of f. Let $g = f^{(-\alpha)}$ and let b_i, $(i \in Z)$ and $\alpha \notin Z$ are such that

$$g(x) = \sum_{i=-\infty}^{\infty} b_i \frac{x^{\alpha+i}}{(\alpha+i)!}.$$

Then

$$f(x) = g^{(\alpha)}(x) = \sum_{i=-\infty}^{\infty} b_i \frac{x^i}{i!}.$$

Obviously $b_i = a_i$ for $i \in N_0 = N \cup \{0\}$ and we define $a_i = b_i$ for $i \in Z^-$. Hence f is written in the required "natural representation" $\sum\limits_{i=-\infty}^{\infty} a_i \frac{x^i}{i!}$. Moreover, the coefficients $a_{-1}, a_{-2}, \ldots$ do not depend on the choice of α. Notice that if we know the natural representation of an ideal function f, then all fractional derivatives of f are known. So the main problem in examining of an ideal function is to find its natural representation. Notice that natural representation may exists also if the function is not ideal. In that case we can use it for calculating the fractional derivatives according to the old definition of [2, 3].

Looking at this theory axiomatically, we have a class $\mathcal{I}$ (ideal functions) of analytical functions f, such that

(i) for each $f \in \mathcal{I}$ and each $\alpha \in (R \setminus Z)$ there exists the expansion

$$f(x) = \sum_{i=-\infty}^{\infty} C_i \frac{(x-x_0)^{\alpha+i}}{(\alpha+i)!}, \qquad (x > x_0) \tag{1.9}$$

which has the same meaning discussed for (1.5), such that x should be replaced also by $x - x_0$, and where we define $C_i = f^{(\alpha+i)}(x_0)$; if $\alpha \in Z$, we choose the natural representation

$$f(x) = \sum_{i=-\infty}^{\infty} C_i \frac{(x-x_0)^{i}}{i!}. \tag{1.10}$$

(ii) If $f \in \mathcal{I}$, then

$$f^{(\beta)} := \sum_{i=-\infty}^{\infty} C_i \frac{(x-x_0)^{\alpha+i-\beta}}{(\alpha+i-\beta)!} \in \mathcal{I}, \quad (\beta \in R), \tag{1.11}$$

where f is given by (1.9).

To the end of this section we give some properties of the ideal functions.

1. The set of ideal functions $\mathcal{I}$ can be separated in a quotient set $\mathcal{I}/\sim$, where the equivalence relation $\sim$ is defined by $f \sim g$ iff there exists $\alpha \in R$, such that $f^{(\alpha)} = g$. Each such class determines unique sequence a_i, $(i \in Z)$, such that $\sum\limits_{i=-\infty}^{\infty} a_i \frac{x^i}{i!} \in \mathcal{I}$. Namely, then

$$\sum_{i=-\infty}^{\infty} a_i \frac{x^{\alpha+i}}{(\alpha+i)!} \sim \sum_{i=-\infty}^{\infty} a_i \frac{x^i}{i!},$$

for arbitrary $\alpha \in R$. Moreover, from the definition of ideal functions it follows that $x^n f(x)$ is an ideal function if $f(x)$ is an ideal function where n is a positive integer.

2. $\mathcal{I}$ is a nonempty set because $e^x, \sin x, \cos x \in \mathcal{I}$. The zero function is also an ideal function. The set of ideal functions is a vector space, such that if $f, g \in \mathcal{I}$, then $\lambda f, f + g \in \mathcal{I}$. Notice that also $f(\lambda x) \in \mathcal{I}$ if $\lambda \neq 0$ and $f(x + \lambda) \in \mathcal{I}$.

3. If P is a polynomial, $P(x) \sum\limits_{k=-\infty}^{n} a_k \frac{x^k}{k!}$, with known coefficients a_0, a_1, $\cdots$, a_n, then a_{-1}, a_{-2}, a_{-3},$\cdots$ are not uniquely determined such that P is an ideal function. In the remark of Section 2 is constructed a wide class of polynomials $\mathcal{P}$ which are ideal functions.

We considered until now only analytical functions of real argument. Notice that the analytical continuation on the interval (x_0, ∞) is unique, i.e. it does not yield to multivalued functions. Now we can expand the previous theory for functions of complex arguments. Let f be a complex ideal function. It means that f is an ideal function on an interval (x_0, x_1) of the real numbers. Simply, using analytical continuation we can expand it in the complex domain. Notice that an ideal function means a function together with its expansions for each $\alpha \in R$. Then its integral $f^{(-1)}$ is uniquely determined function, because the constant of integration is just the coefficient a_{-1} from the "zero part". Also the other integer integrals are uniquely determined.

Notice that the decomposition into the generalized Taylor's formula (1.6) is a rigorous condition, which any viable method of fractional derivatives should satisfy.

2. REPRESENTATIONS AND NATURAL REPRESENTATIONS OF SOME IDEAL FUNCTIONS

In this section we consider some functions whose coefficients of the Taylor series contain Bernoulli and Euler numbers.

Theorem 2.1. *The functions* $\frac{x}{e^x-1}$ *and* $\frac{xe^x}{e^x-1}$ *are ideal functions, such that*

(i)

$$\frac{x}{e^x-1} = \sum_{i=-\infty}^{\infty} (-1)^{i+\alpha} B^*_{\alpha+i} \frac{x^{\alpha+i}}{(\alpha+i)!}, \qquad \alpha \in R, \tag{2.1}$$

(ii)

$$\frac{xe^x}{e^x-1} = \sum_{i=-\infty}^{\infty} B^*_{\alpha+i} \frac{x^{\alpha+i}}{(\alpha+i)!}, \qquad \alpha \in R. \tag{2.2}$$

Proof.

(i) Using the identity (e2.3) in [4] we obtain

$$\sum_{i=-\infty}^{\infty} (-1)^{i+\alpha} B^*_{\alpha+i} \frac{x^{\alpha+i}}{(\alpha+i)!} = \sum_{i=-\infty}^{\infty} (-1)^{i+\alpha} \Big[\sum_{n=1}^{\infty} -(\alpha+i) n^{\alpha+i-1}\Big] \frac{x^{\alpha+i}}{(\alpha+i)!}$$

$$= x \sum_{n=1}^{\infty} \sum_{i=-\infty}^{\infty} \frac{(-nx)^{\alpha+i-1}}{(\alpha+i-1)!} = x \sum_{n=1}^{\infty} e^{-nx} = x \frac{e^{-x}}{1-e^{-x}} = \frac{x}{e^x-1}.$$

(ii) $\frac{xe^x}{e^x-1}$ is an ideal function because $\frac{xe^x}{e^x-1} = f(-x)$, where $f(x) = \frac{x}{e^x-1}$ is an ideal function.

From (2.2) it follows that for each $\alpha \in R$, $\frac{d^\alpha}{dx^\alpha}\left(\frac{xe^x}{e^x-1}\right)\Big|_{x=0} = B^*_\alpha$.

∎

Remark. In Theorem 2.1 were obtained the natural representations for $\frac{x}{e^x-1}$ and $\frac{xe^x}{e^x-1}$:

$$\frac{x}{e^x-1} = \sum_{i=-\infty}^{\infty} (-1)^i B^*_i \frac{x^i}{i!}, \qquad \frac{xe^x}{e^x-1} = \sum_{i=-\infty}^{\infty} B^*_i \frac{x^i}{i!}.$$

Both functions $\frac{xe^x}{e^x-1}$ and $\frac{x}{e^x-1}$ with the corresponding representations are ideal and hence their difference

$$h(x) = \frac{xe^x}{e^x-1} - \frac{x}{e^x-1} = x + 2B^*_{-1}\frac{x^{-1}}{(-1)!} + 2B^*_{-3}\frac{x^{-3}}{(-3)!} + 2B^*_{-5}\frac{x^{-5}}{(-5)!} + \cdots \tag{2.3}$$

is also an ideal function.

This function h generates a family of ideal polynomials $\mathcal{P}$, using the following generator transformations:

(i) if $P \in \mathcal{P}$, then $P^{(\alpha)} \in \mathcal{P}$ for each $\alpha \in Z$,
(ii) if $P \in \mathcal{P}$, then $x^n P \in \mathcal{P}$ for each nonnegative integer n,
(iii) if $P, Q \in \mathcal{P}$, then $\lambda P + \mu Q \in \mathcal{P}$ for each scalars λ, μ.

The proof of the following theorem is analogous to the Theorem 2.1.

Theorem 2.2. The functions $x \cdot \cot x$, $\frac{x}{\sin x}$, $\frac{xe^x}{e^x+1}$, and $x \tanh x$ are ideal functions, such that

(i)

$$x \cdot \cot x = \sum_{j=-\infty}^{\infty} 2^{\alpha+j} \cos\frac{(\alpha+j)\pi}{2} \cdot B^*_{\alpha+j} \frac{x^{\alpha+j}}{(\alpha+j)!}, \quad \alpha \in R, \tag{2.4}$$

(ii)

$$\frac{x}{\sin x} = \sum_{i=-\infty}^{\infty} B^*_{\alpha+i}(2-2^{\alpha+i}) \cos\frac{(\alpha+i)\pi}{2} \cdot \frac{x^{\alpha+i}}{(\alpha+i)!}, \quad \alpha \in R, \tag{2.5}$$

(iii)

$$\frac{xe^x}{e^x+1} = \sum_{i=-\infty}^{\infty} (2^{\alpha+i}-1)B^*_{\alpha+i}\frac{x^{\alpha+i}}{(\alpha+i)!}, \quad \alpha \in R, \tag{2.6}$$

(iv)

$$x\tanh x = \sum_{i=-\infty}^{\infty} (2^{\alpha+i}-1)B^*_{\alpha+i}\cdot 2^{\alpha+i}\frac{x^{\alpha+i}}{(\alpha+i)!}, \quad \alpha \in R. \tag{2.7}$$

From (2.4-2.7) it just follows that for each $\alpha \in R$,

$$\frac{d^\alpha}{dx^\alpha}(x\cdot\cot x)\Big|_{x=0} = 2^\alpha\cos\frac{\alpha\pi}{2}\cdot B^*_\alpha, \tag{2.8}$$

$$\frac{d^\alpha}{dx^\alpha}\Big(\frac{x}{\sin x}\Big)\Big|_{x=0} = (2-2^\alpha)\cos\frac{\alpha\pi}{2}\cdot B^*_\alpha. \tag{2.9}$$

$$\frac{d^\alpha}{dx^\alpha}\Big(\frac{xe^x}{e^x+1}\Big)\Big|_{x=0} = (2^\alpha-1)B^*_\alpha. \tag{2.10}$$

$$\frac{d^\alpha}{dx^\alpha}(x\tanh x)\Big|_{x=0} = (2^\alpha-1)B^*_\alpha\cdot 2^\alpha. \tag{2.11}$$

Theorem 2.3. *The functions $\frac{1}{\cos x}$ and $\frac{1}{\cosh x}$ are ideal functions, such that*

$$\frac{1}{\cos x} = \sum_{j=-\infty}^{\infty} E_{\alpha+j}\frac{x^{\alpha+j}}{(\alpha+j)!}, \quad \alpha \in R, \tag{2.12}$$

$$\frac{1}{\cosh x} = \sum_{i=-\infty}^{\infty} \frac{(-1)^{\alpha+i}E_{\alpha+i}}{(\alpha+i)!}x^{\alpha+i}. \tag{2.13}$$

Proof. Using the formula $E_\alpha = 2\cos\frac{\alpha\pi}{2}\sum_{n=1}^{\infty} n^\alpha\cdot\cos(n-1)\frac{\pi}{2}$ (see [4], sec. 3) we obtain

$$\sum_{j=-\infty}^{\infty} E_{\alpha+j}\frac{x^{\alpha+j}}{(\alpha+j)!}$$

$$= \sum_{j=-\infty}^{\infty} 2\cos\frac{(\alpha+j)\pi}{2}\Big(\sum_{n=1}^{\infty} n^{\alpha+j}\cdot\cos(n-1)\frac{\pi}{2}\Big)\frac{x^{\alpha+j}}{(\alpha+j)!}$$

$$= \sum_{n=1}^{\infty}\cos(n-1)\frac{\pi}{2}\sum_{j=-\infty}^{\infty} 2\cos\frac{(\alpha+j)\pi}{2}n^{\alpha+j}\cdot\frac{x^{\alpha+j}}{(\alpha+j)!}$$

$$= \sum_{n=1}^{\infty}\cos(n-1)\frac{\pi}{2}\sum_{j=-\infty}^{\infty}\Big[e^{i\frac{(\alpha+j)\pi}{2}} + e^{-i\frac{(\alpha+j)\pi}{2}}\Big]n^{\alpha+j}\cdot\frac{x^{\alpha+j}}{(\alpha+j)!}$$

$$= \sum_{n=1}^{\infty}\cos(n-1)\frac{\pi}{2}\sum_{j=-\infty}^{\infty}\Big[\frac{(e^{i\pi/2}\cdot nx)^{\alpha+j}}{(\alpha+j)!} + \frac{(e^{-i\pi/2}\cdot nx)^{\alpha+j}}{(\alpha+j)!}\Big]$$

$$= \sum_{n=1}^{\infty}\cos(n-1)\frac{\pi}{2}(e^{inx}+e^{-inx}) = 2\sum_{n=1}^{\infty}\cos(n-1)\frac{\pi}{2}\cos nx = \frac{1}{\cos x}.$$

From (2.12) it just follows that for each $\alpha \in R$, $E_\alpha = \frac{d^\alpha}{dx^\alpha}\Big(\frac{1}{\cos x}\Big)\Big|_{x=0}$.

The function $\frac{1}{\cosh x}$ is an ideal function, because $\frac{1}{\cosh x} = f(ix)$, where $f(x) = \frac{1}{\cos x}$, and moreover (2.13) is satisfied. ∎

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