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ON VECTOR VARIATIONAL INEQUALITY PROBLEM IN TERMS OF BIFUNCTIONS

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ABSTRACT. In this paper, we consider a generalized vector variational inequality problem expressed in terms of a bifunction and establish existence theorems for this problem by using the concepts of cone convexity and cone strong quasiconvexity and employing the celebrated Fan's Lemma. We also give two types of gap functions for this problem.

Key words and phrases: Vector variational inequality problem, Bifunctions, Cones, Convexity, Strong quasiconvexity, Gap functions.

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1. INTRODUCTION

The classical variational inequality problem is to find $x_0 \in K$ such that

$$\textbf{(VP)} \qquad \langle T(x_0), x - x_0\rangle \geq 0\,, \quad \text{for all } \ x \in K\,,$$

where K is a nonempty closed convex subset of $\mathbb{R}^n$ and $T : \mathbb{R}^n \to \mathbb{R}^n$.

Existence results for the problem (**VP**) have been studied by Harker and Pang [8] and references cited therein. Historically, the variational inequality problem was introduced by Hartman and Stampacchia in their seminal paper [12]. The early studies were set in the context of calculus of variation/optimal control theory and in connection with the solutions of boundary value problems posed in the form of differential equations. There are applications of variational inequalities in problems of engineering and physics.

If f is a real valued convex differentiable function defined on a closed convex set $K \subseteq \mathbb{R}^n$ then the following optimization problem (**P**)

$$\textbf{(P)} \ \min f(x)$$
$$\text{subject to } x \in K\,,$$

is equivalent to the variational inequality problem (**VP**) with $T = \nabla f$. However, for nondifferentiable functions there is a lack of the gradient map concept. Since a generalized derivative might be considered as a bifunction $h(x; d)$ where x refers to a point in K and d refers to a direction in $\mathbb{R}^n$, one gets motivated to associate the following variational inequality problem with the optimization problem (**P**), which is to find a vector $x_0 \in K$ such that

$$\textbf{(VIP)} \ \ h(x_0; y - x_0) \geq 0\,, \quad \forall \ \ y \in K\,,$$

where $h : K \times \mathbb{R}^n \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty, -\infty\}$ is some sort of generalized derivative of f. If we take $h(x; y - x) = \langle T(x), y - x\rangle$ then the problem (**VIP**) reduces to the problem (**VP**).

As a consequence of the extensive research carried out in the area of multiobjective optimization during the last few decades, the study of vector variational inequality problems received a great deal of attention. Giannessi [10] introduced vector variational inequality problem and since then various authors have contributed in this direction (see [1, 5, 6, 13, 14, 16, 18]).

The paper aims at establishing an existence theorem for the following generalized vector variational inequality problem which is to find $x_0 \in K$ such that

$$\textbf{(VVIP)} \ \ h(x_0; y - x_0) \notin -\mathrm{int}\, C(x_0)\,, \quad \forall\, y \in K\,,$$

where K is a nonempty closed convex subset of $\mathbb{R}^n$ and $h : K \times \mathbb{R}^n \to \overline{\mathbb{R}}^m$ is any bifunction, $C : K \to \mathbb{R}^m$ is a point to set map such that for each x in K, $C(x)$ is a closed convex pointed cone in $\mathbb{R}^m$. We assume that for each $x \in K$, $\mathrm{int}\, C(x) \neq \phi$, where $\mathrm{int}\, C(x)$ denotes the interior of the set $C(x)$.

In case $C(x) = P$, $\forall\ x \in K$ where P is a closed convex pointed cone in $\mathbb{R}^m$ then this problem reduces to the Stampacchia kind vector variational inequality problem considered by Lalitha and Mehta in [14]. In [14], the existence of this problem was studied under a weak form of pseudomonotonicity assumption and a continuity assumption on h. However, in this paper we use the concepts of cone-convexity and cone-strong quasiconvexity to establish the existence of solution for the problem (**VVIP**).

For the case when $h(x; y-x) = \langle T(x), y-x\rangle$, the problem (**VVIP**) has been investigated by many authors like Chen [5], Yu and Yao [18] etc. When $C(x) = P$, $\forall\, x \in K$ and $h(x; y-x) = \langle T(x), y - x\rangle$ then (**VVIP**) becomes (**VVIP**)$_K$ (see [5]) that is, to find $x_0 \in K$ such that

$$\langle T(x), y - x\rangle \notin -\mathrm{int}\ P\,, \quad \forall \ \ y \in K\,.$$

The existence results in all these papers were established by imposing monotonicity or some generalized monotonicity condition on T.

The problem (**VVIP**) is a particular case of the following vector equilibrium problem. Find $x_0 \in K$ such that

$$\textbf{(VEP)}\ f(x_0; y) \notin -\text{int } C(x_0)\,, \quad \forall\ y \in K$$

where $f : K \times K \to \mathbb{R}^m$. A similar problem was considered by Ansari [1] where in two existence results were derived for the problem. Existence of some other functions, associated to the given function f, was required in order to establish existence results in this paper. Usually stringent conditions are required to establish the existence theorem because of the general nature of an equilibrium problem (see [1], [4], [15]).

The main purpose of this paper is to establish existence theorem for the problem (**VVIP**), which arises naturally via the concept of generalized derivatives, under certain continuity and generalized convexity assumptions on the bifunction h. Comparisons are also made with the corresponding results for vector equilibrium problem (**VEP**).

The paper is organized as follows. In Section 2, we present some preliminaries that will be used in the subsequent sections. In Section 3, we establish an existence theorem for (**VVIP**), when K is compact using the notions of convexity and strong quasiconvexity in terms of cones. We further give an existence theorem when K is not necessarily bounded. In the last section, we investigate two types of gap functions for the vector variational inequality problem (**VVIP**), the scalar valued type and vector valued type. We then consider a vector optimization problem (**VOP**) involving the vector valued gap function and prove that every solution of the problem (**VVIP**) is also a solution of (**VOP**). In the end, we relate the two gap functions given in the paper.

2. PRELIMINARIES

To prove the main result of this paper, we need the following concepts and results:

Let C be a closed convex pointed cone in $\mathbb{R}^m$. For any $y_1, y_2 \in \mathbb{R}^m$, let $\sup_c\{y_1, y_2\}$ denote the supremum of y_1 and y_2, that is, if $p = \sup_c\{y_1, y_2,\}$ then $y_i \in p - C, i = 1, 2$ and if $y \in \mathbb{R}^m$ with $y_i \in y - C$, $i = 1, 2$ then $p \in y - C$.

Definition 2.1. A point to set map $F : \mathbb{R}^n \to \mathbb{R}^m$ is said to be upper semicontinuous at $x \in \mathbb{R}^m$ if $x_n \to x$ in $\mathbb{R}^n$ and $y_n \to y$ in $\mathbb{R}^m$ with $y_n \in F(x_n)$ imply that $y \in F(x)$.

F is said to be upper semicontinuous on $\mathbb{R}^n$ if it is upper semicontinuous at each of its points.

Let X be a nonempty convex subset of $\mathbb{R}^n$ and C be a closed convex pointed cone in $\mathbb{R}^m$. We recall that a vector valued function $f : X \to \mathbb{R}^m$ is said to be C-convex on X if for any $y_1, y_2 \in X$ and $t \in [0, 1]$, $f(ty_1 + (1 - t)y_2) - tf(y_1) - (1 - t)f(y_2) \in -C$.

We now have the following notion of cone strong quasiconvexity analogous to the concept of cone strong quasiconcavity considered by Dong et al. [8].

Definition 2.2. Let X be a nonempty convex subset of $\mathbb{R}^n$ and C be a closed convex pointed cone in $\mathbb{R}^m$ with int $C \neq \phi$. A function $f : X \to \mathbb{R}^m$ is said to be

(i) C-quasiconvex on X if the level set

$$S(\alpha) = \{x \in X | f(x) - \alpha \in -C\}$$

is convex, for any $\alpha \in \mathbb{R}^m$ satisfying $S(\alpha) \neq \phi$;

(ii) C-strongly quasiconvex on X when f is C-quasiconvex and for any $y_1, y_2 \in X$, $y_1 \neq y_2$ and $t \in (0, 1)$, $f(ty_1 + (1 - t)y_2) \in \sup_C\{f(y_1), f(y_2)\} - \text{int}\, C$.

For the scalar case, it is well known that a strongly quasiconvex function is strictly quasi-convex (see Avriel [2]). As an extension Dong et al. [8] compared the notion of C-strong quasiconvexity with the notion of strict quasiconvexity for vector-valued function introduced by Benson and Sun [3].

The following theorem due to Fan [9] and its extension [17] will be required to establish the main result of this paper.

Several authors have followed this approach for establishing existence theorems for instance, see [1] and [15].

Lemma 2.1. ([9]) *Let K be a nonempty compact convex set in $\mathbb{R}^n$. Let L be a subset of $K \times K$ having the following properties:*

(i) *for each $x \in K, (x, x) \in L$;*
(ii) *for each fixed $y \in K$, the set $L(y) = \{x \in K | (x, y) \in L\}$ is closed in K;*
(iii) *for each $x \in K$, the set $M(x) = \{y \in K \mid (x, y) \notin L\}$ is convex or empty;*

then there exists $x_0 \in K$ such that $\{x_0\} \times K \subset L$.

Lemma 2.2. ([17]) *Let K be a nonempty closed convex set in $\mathbb{R}^n$ and L be a subset of $K \times K$ satisfying the following properties :*

(i) *for each $x \in K, (x, x) \in L$;*
(ii) *for each fixed $y \in K$, the set $L(y) = \{x \in K | (x, y) \in L\}$ is closed in K;*
(iii) *for each $x \in K$, the set $M(x) = \{y \in K | (x, y) \notin L\}$ is convex or empty;*
(iv) *there exists a nonempty compact convex subset D of K such that for each $x \in K \setminus D$ there exists $y \in D$ such that $(x, y) \notin L$;*

then there exists $x_0 \in K$ such that $\{x_0\} \times K \subset L$.

3. EXISTENCE THEOREM

We first establish an existence theorem for (**VVIP**) when K is nonempty compact convex subset of $\mathbb{R}^n$.

Theorem 3.1. *Let K be a nonempty compact convex subset of $\mathbb{R}^n$. Assume that*

(a) *the point to set map $W : K \to \mathbb{R}^m$ defined by $W(x) = \mathbb{R}^m \setminus (-\operatorname{int} C(x))$, for any $x \in K$ is upper semicontinuous;*
(b) *$h : K \times \mathbb{R}^n \to \bar{\mathbb{R}}^m$ is such that h is continuous in both the arguments ;*
(c) *$h(x; 0) \notin -\operatorname{int} C(x)$, for any $x \in K$*
(d) *for each $x \in K$, the function $f : K \to \mathbb{R}^m$ defined as $f(y) = h(x; y - x)$ is either $C(x)$-strongly quasiconvex or $C(x)$-convex on K;*

then there exists a solution to (**VVIP**)*.*

Proof. Let $L = \{(x, y) \in K \times K | h(x; y - x) \notin -\operatorname{int} C(x)\}$. From (c) it is clear that $L \neq \phi$ as $(x, x) \in L$, for each $x \in K$. Let $y \in K$ be arbitrary but fixed and let

$$\begin{aligned} L(y) &= \{x \in K | (x, y) \in L\} \\ &= \{x \in K | h(x; y - x) \notin -\operatorname{int} C(x)\}. \end{aligned}$$

Clearly, $L(y) \neq \phi$ as $y \in L(y)$. Let $\{x_n\}$ be a sequence in $L(y)$ such that $x_n \to x$. Since $x_n \in L(y)$ therefore, we have

$$h(x_n; y - x_n) \notin -\operatorname{int} C(x_n), \forall \ n = 1, 2, 3, \ldots$$

that is,

$$h(x_n; y - x_n) \in W(x_n) = \mathbb{R}^m \setminus (-\operatorname{int} C(x_n)), \forall \ n = 1, 2, 3, \ldots$$

Since W is upper semicontinuous so it follows that

$$\lim_{n \to \infty} h(x_n; y - x_n) \in W(x),$$

and by continuity of h it follows that

$$h(x; y - x) \in W(x),$$

which implies that $x \in L(y)$. Thus, $L(y)$ is closed for each $y \in K$. Now for each x in K, define

$$M(x) = \{y \in K | h(x; y - x) \in -\text{int}\, C(x)\}.$$

If $M(x)$ is empty the result follows from Lemma 2.1. Let $M(x)$ be a nonempty set. Let $y_1, y_2 \in M(x), y_1 \neq y_2, 0 < t < 1$ and $w = (1-t)y_1 + ty_2$. As $y_1, y_2 \in M(x)$ we have, $f(y_1) = h(x; y_1 - x) \in -\text{int}\, C(x)$ and $f(y_2) = h(x; y_2 - x) \in -\text{int}\, C(x)$. Now by condition (d) we have that either

$$f(w) \in \sup_{C(x)}\{f(y_1), f(y_2)\} - \text{ int}\, C(x)$$

or

$$f(w) \in (1-t)f(y_1) + tf(y_2) - C(x).$$

Clearly, $\sup_{C(x)}\{f(y_1), f(y_2)\} \in -\text{int}\, C(x)$, as $f(y_1), f(y_2) \in -\text{int}\, C(x)$. Since $-\text{ int}\, C(x)$ is a convex cone so in either case it follows that $f(w) \in -\text{int}\, C(x)$, that is, $w \in M(x)$. Thus, $M(x)$ is a convex set for each x in K. Hence by Lemma 2.1, there exists $x_0 \in K$ such that $\{x_0\} \times K \subset L$, that is, for each $y \in K$, $(x_0, y) \in L$. Thus, $h(x_0; y - x_0) \notin -\text{int}\, C(x_0)$, $\forall\ y \in K$. ∎

The following example illustrates the above theorem.

Example 3.1. *Let $K = [-1, 1]$ and for each x in K let*

$$C(x) = \begin{cases} C_1 = \{(y_1, y_2) \in \mathbb{R}^2 \,|\, y_1 \geq 0, y_2 \geq 0\} & \text{if } x \geq 0 \\ C_2 = \{(y_1, y_2) \in \mathbb{R}^2 \,|\, y_2 \geq -y_1, y_2 \geq 0\} & \text{if } x < 0. \end{cases}$$

Then each $C(x)$ is a closed convex pointed cone in $\mathbb{R}^2$. Let $h : K \times \mathbb{R} \to \mathbb{R}^2$ be defined as $h(x; d) = (d + x, -e^x d)$. We note that the function $f : K \to \mathbb{R}^2$ defined as $f(y) = h(x; y - x) = (y, -e^x(y - x))$ is both $C(x)$-strongly quasiconvex as well as $C(x)$-convex on K and $h(x; 0) = (x, 0) \notin -\text{int}\, C(x), \forall\ x \in K$. Thus, all the conditions of Theorem 3.1 are satisfied and it can be easily verified that each $x \in [0, 1]$ is a solution of (**VVIP**).

We now give an example of a (**VVIP**) when the function f given in condition (d) of Theorem 3.1 is $C(x)$-convex but not $C(x)$-strongly quasiconvex.

Example 3.2. *Let $K = [-1, 2]$ and let $h : K \times \mathbb{R} \to \mathbb{R}^2$ be defined as $h(x; d) = (-e^x, d)$. Let for each $x \in K$, cone $C(x)$ be same as in Example 3.1. Then $h(x; 0) = (-e^x, 0) \notin -\text{int}\, C(x)$, $\forall\ x \in K$. We note that the mapping $f : K \to \mathbb{R}^2$ defined as $f(y) = h(x; y-x) = (-e^x, y-x)$ is $C(x)$-convex on K but f is not $C(x)$-strongly quasiconvex on K because for $x = 0$, $y_1 = 2$, $y_2 = -1$ and $t = 1/2$*

$$\begin{aligned} f((1-t)y_1 + ty_2) - \sup_{C(x)}\{f(y_1), f(y_2)\} &= (-1, 1/2) - (-1, 2) \\ &= (0, -3/2) \notin -\text{int}\, C(x). \end{aligned}$$

Clearly, $x = -1$ is a solution of (**VVIP**).

Our next example is that of a vector variational inequality problem where the function f is not $C(x)$-convex but is $C(x)$-strongly quasiconvex.

Example 3.3. *Let $K = [-2, 1]$ and for each $x \in K$ define*

$$C(x) = \{(y_1, y_2) \in \mathbb{R}^2 | y_1 \geq 0, y_2 \geq 0\}.$$

Let $h : K \times \mathbb{R} \to \mathbb{R}^2$ be defined as $h(x; d) = (d^3, e^{-x} - e^d)$. Then, $h(x; 0) = (0, e^{-x} - 1) \notin -\text{int}\, C(x)$, $\forall\ x \in K$ and it can be easily verified that the function $f : K \to \mathbb{R}^2$ defined as $f(y) = h(x; y - x) = ((y - x)^3, e^{-x}(1 - e^y))$ is $C(x)$-strongly quasiconvex on K but f is not $C(x)$-convex on K because for $x = 0$, $y_1 = -1$, $y_2 = -2$ and $t = 1/2$ we have

$$f(ty_1 + (1 - t)y_2) - tf(y_1) - (1 - t)f(y_2) = (9/8, (e^{-1} + e^{-2})/2 - e^{-3/2}) \notin -C(x).$$

Here, $x = 0$ and $x = -2$ are the only solutions of (**VVIP**).

Remark 3.1. In [1], an existence result (Theorem 2) for a special case of the problem (**VEP**), when $f(x, y) = g(x, y) + h(x, y)$, $g, h : K \times K \to \bar{\mathbb{R}}^m$, has been proved assuming $g(\cdot, y)$ and $h(x, \cdot)$ to be P-convex (referred to as P-function in [1]), where $P = \bigcap_{x \in K} C(x)$, which is a more stringent condition than the cone-convexity assumption taken by us. In this paper the author has assumed that $h(x; 0) = 0$, $\forall\ x \in K$ whereas we require a milder assumption $h(x; 0) \notin -\text{int}\ C(x)$, $\forall\ x \in K$. Moreover, the author in [1] imposes certain inclusion conditions on the functions g and h in order to establish existence theorem. We now illustrate with the help of an example that the conditions (a)-(d) of Theorem 3.1 are satisfied but the conditions of Theorem 2 in Ansari [1] are not satisfied for the vector variational inequality problem (**VVIP**).

Example 3.4. *Let $K = [-1, 1]$ and for each x in K define*

$$C(x) = \begin{cases} \{(y_1, y_2) \in \mathbb{R}^2 \,|\, y_1 \geq 0,\ y_2 \geq 0\} & \textit{if } x \geq 0 \\ \{(y_1, y_2) \in \mathbb{R}^2 \,|\, y_2 \geq 0\} & \textit{if } x < 0\,. \end{cases}$$

Let $h : K \times \mathbb{R} \to \mathbb{R}^2$ be defined as $h(x; d) = (x^3(d + x)^2, (d + x)^2)$. We note that the function $f : K \to \mathbb{R}^2$ defined as $f(y) = h(x; y - x) = (x^3 y^2, y^2)$ is $C(x)$-convex but not P-convex because for $x = -1$, $y_1 = -1$, $y_2 = 1$ and $t = 1/2$ we have

$$f(ty_1 + (1 - t)y_2) - tf(y_1) - (1 - t)f(y_2) = (1, -1) \notin -P\,,$$

where $P = \bigcap_{x \in K} C(x) = \{(y_1, y_2) \in \mathbb{R}^2 \,|\, y_1 \geq 0,\ y_2 \geq 0\}$. Also, $h(x; 0) = (x^5, x^2) \notin -\text{int}\, C(x)$, $\forall\, x \in K$. In this case every $x \in K$ is a solution of (**VVIP**).

We now prove an existence theorem for (**VVIP**) for the case when K is not a compact subset of $\mathbb{R}^n$.

Theorem 3.2. *Let K be a nonempty closed convex subset of $\mathbb{R}^n$. Suppose that conditions (a)-(d) of Theorem 3.1 are satisfied. Further suppose that there exists a nonempty compact convex subset $D \subset K$ such that for each $x \in K \backslash D$ there exists $y \in D$ such that $h(x; y - x) \in -int\, C(x)$, then the problem* (**VVIP**) *has a solution.*

Proof. The result can be proved exactly on the lines of Theorem 3.1 using Lemma 2.2. ∎

4. GAP FUNCTIONS

Gap functions play a crucial role in transforming a variational inequality problem into an optimization problem. Then, the methods for solving an optimization problem can be exploited for finding the solutions of a variational inequality problem. In this section, we propose two types of gap functions for the vector variational inequality problem (**VVIP**), one is single valued and the other is set valued.

For each x in K, let $C^*(x)$ denote the positive polar cone of $C(x)$, that is,

$$C^*(x) = \{u \in \mathbb{R}^m : \langle u, v\rangle \geq 0, \forall\ v \in C(x)\}.$$

Let $B^*(x) = \{u \in C^*(x) \,|\, ||u|| = 1\}$. For each x in K, $B^*(x)$ is a compact set such that $0 \notin B^*(x)$ and $C^*(x)$=cone $B^*(x)$, where cone $B^*(x)$ is the cone generated by $B^*(x)$. Since int $C(x) \neq \phi$, we have a weak ordering in $\mathbb{R}^m$ given by

$$\xi <_{\text{int C(x)}} \eta, \text{ if and only if } \eta - \xi \in \operatorname{int} C(x),$$
$$\xi \not<_{\text{int C(x)}} \eta, \text{ if and only if } \eta - \xi \notin \operatorname{int} C(x).$$

Define $h(x; K - x) = \bigcup_{y \in K} h(x; y - x)$ and for a nonempty subset A of $\mathbb{R}^m$ we define

$$\operatorname{Min}_{\operatorname{int} C} A := \{a \in A \,|\, \text{ there is no } \ \overline{a} \in A \text{ such that } \overline{a} <_{\ \operatorname{int} C} a\},$$

where C is a cone in $\mathbb{R}^m$ with nonempty interior.

Throughout this section we assume that for each $x \in K$, $h(x; 0) = 0$.

By a scalar gap function we mean the following:

Definition 4.1. An extended real valued function $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\}$ is said to be a gap function for the problem (**VVIP**) if and only if it satisfies the following properties

(i) $\varphi(\hat{x}) = 0 \Leftrightarrow \hat{x}$ is a solution of (**VVIP**);
(ii) $\varphi(x) \leq 0, \forall\ x \in K$.

In the following theorem we have a scalar type gap functions for the vector variational inequality problem (**VVIP**).

Theorem 4.1. *The function $\varphi(x)$ defined as $\varphi(x) = \inf_{y \in K} \max_{u \in B^*(x)} \langle u, h(x; y - x)\rangle$, is a gap function for* (**VVIP**).

Proof. (i) Let $\hat{x}$ be a solution of (**VVIP**). It follows that $h(\hat{x}; y - \hat{x}) \notin -\operatorname{int} C(\hat{x}), \forall\ y \in K$ that is, for each $y \in K$ there exists $u_y \in B^*(\hat{x})$ such that

$$\langle u_y, h(\hat{x}; y - \hat{x})\rangle \geq 0.$$

This yields that

$$\max_{u \in B^*(\hat{x})} \langle u, h(\hat{x}; y - \hat{x})\rangle \geq 0,$$

and hence,

$$\inf_{y \in K} \max_{u \in B^*(\hat{x})} \langle u, h(\hat{x}; y - \hat{x})\rangle \geq 0,$$

that is, $\varphi(\hat{x}) \geq 0$. Also, $\varphi(\hat{x}) \leq \max_{u \in B^*(\hat{x})} \langle u, h(\hat{x}; 0)\rangle = 0$ implies that $\phi(\hat{x}) = 0$.

Conversely, suppose $\varphi(\hat{x}) = 0$ for some $\hat{x} \in K$. This implies that

$$\max_{u \in B^*(\hat{x})} \langle u, h(\hat{x}; y - \hat{x})\rangle \geq 0, \ \ \forall\ y \in K.$$

Then for each $y \in K$ there exists some $u_y \in B^*(\hat{x})$ such that $\langle u_y, h(\hat{x}; y - \hat{x})\rangle \geq 0$, which implies that $h(\hat{x}; y - \hat{x}) \notin -\operatorname{int} C(\hat{x}), \forall\ y \in K$ and hence $\hat{x}$ is a solution of (**VVIP**).
(ii) Clearly, $\varphi(x) \leq \max_{u \in B^*(x)} \langle u, h(x; 0)\rangle = 0, \forall\, x \in K$.

Thus, $\varphi(x)$ is a gap function for (**VVIP**). ∎

Remark 4.1. Solving the vector variational inequality problem (**VVIP**) is thus equivalent to solving the following scalar optimization problem of maximizing the gap function $\varphi(x)$ over K:

$$\textbf{(P1)}\ \text{Max}\ \varphi(x)$$
$$\text{subject to}\ \ x \in K\,.$$

We now have the notion of a set-valued gap function.

Definition 4.2. A set valued mapping $\Phi : \mathbb{R}^n \to \mathbb{R}^m$ is said to be a gap function for the problem (**VVIP**) if and only if

(i) $0 \in \Phi(\hat{x}) \Leftrightarrow \hat{x}$ solves (**VVIP**);
(ii) $z \notin \operatorname{int} C(x), \forall\, z \in \Phi(x)$ and $\forall\ x \in K$.

The set valued gap function given in the following theorem is similar to the one considered by Chen, Goh and Yen [7].

Theorem 4.2. *The set valued mapping $\Phi : \mathbb{R}^n \to \mathbb{R}^m$ defined by*

$$\Phi(x) \;=\; \text{Min}_{\operatorname{int} C(x)}\, h(x; K - x)$$

is a gap function for (**VVIP**).

Proof. (i) Suppose $0 \in \Phi(\hat{x})$. This implies that, there does not exist any $y \in K$ such that $h(\hat{x}; y - \hat{x}) <_{\operatorname{int} C(\hat{x})} 0 = h(\hat{x}; \hat{x} - \hat{x})$, that is, $h(\hat{x}; y - \hat{x}) \notin -\operatorname{int} C(\hat{x}),\ \forall\ y \in K$. Thus, $\hat{x}$ solves (**VVIP**). Conversely, suppose that $\hat{x}$ solves (**VVIP**) then tracing back the steps we get that $0 \in \Phi(\hat{x})$.

(ii) For any $x \in K$ and $z \in \Phi(x)$ we next prove that $z \not>_{\operatorname{int} C(x)} 0$. On the contrary, suppose that there exists $y \in K$ such that $h(x; y - x) = z >_{\operatorname{int} C(x)} 0 = h(x; x - x)$. This contradicts the fact that $z \in \Phi(x)$.

Thus, $\Phi(x)$ is a gap function for (**VVIP**). ∎

Remark 4.2. When $C(x) = P$, $\forall\, x \in K$, where P is a closed convex pointed cone in $\mathbb{R}^m$ and $h(x; y - x) = \langle T(x), y - x \rangle$ then (**VVIP**) becomes the problem considered by Chen, Goh and Yang [7] and Φ reduces to the gap function considered therein.

The following example shows that the solution of the problem (**VVIP**) can also be obtained by solving the generalized equation $0 \in \Phi(x)$.

Example 4.1. *Let $K = [0, 2]$ and for each x in K let $C(x) = \mathbb{R}^2_+$. Let $h : K \times \mathbb{R} \to \mathbb{R}^2$ be defined as $h(x; d) = (xd, d(d+2x))$. Then $h(x; 0) = 0, \forall\, x \in K$ and $\Phi(x) = \text{Min}_{\operatorname{int} C(x)}\, h(x; K - x)$ is given as*

$$\Phi(x) = \begin{cases} (-x^2, -x^2), & \textit{if}\ x > 0 \\ \{(0, y^2) \,|\, y \in K\}, & \textit{if}\ x = 0\,. \end{cases}$$

Since $0 \in \Phi(x)$, if and only if, $x = 0$ therefore, $x = 0$ is the only solution of the problem (**VVIP**).

We now associate the following vector optimization problem with the problem (**VVIP**):

$$\textbf{(VOP)}\ \ \text{Max}_{\operatorname{int} P}\, \Phi(x)\ \text{subject to}\ \ x \in K\,.$$

Solving the problem (**VOP**) is to find $\hat{x} \in K$ for which there exists $\hat{z} \in \Phi(\hat{x})$ such that $(\Phi(K) - \hat{z}) \cap \operatorname{int} P = \phi$, where $\Phi(K) = \bigcup_{y \in K} \Phi(x)$ and $P = \bigcap_{x \in K} C(x)$ and Φ is the set-valued map considered in Theorem 4.2.

Theorem 4.3. *If $\hat{x} \in K$ is a solution of the vector variational inequality problem* (**VVIP**) *then $\hat{x}$ is a solution of the problem* (**VOP**).

Proof. As $\hat{x}$ is a solution of the problem (**VVIP**) and Φ is a gap function it follows that $0 \in \Phi(\hat{x})$. For every $x \in K$ and $z \in \Phi(x)$ as $z \notin \operatorname{int} C(x)$ we have $z \notin \operatorname{int} P$, as $P = \bigcap_{x \in K} C(x)$. This implies that $(\Phi(K) - 0) \cap \operatorname{int} P = \phi$. Since $0 \in \Phi(\hat{x})$ it follows that $\hat{x}$ is also a solution of the problem (**VOP**). ∎

Remark 4.3. The converse of the above theorem is not true in general as can be seen from the following example.

Example 4.2. *Let $K = [0, 2]$ and for each x in K let $C(x) = \mathbb{R}^2_+$. Let $h : K \times \mathbb{R} \to \mathbb{R}^2$ be defined as*

$$h(x; d) = \begin{cases} (-d^2, -1), & \textit{if } d \neq 0 \\ (0, 0), & \textit{if } d = 0. \end{cases}$$

Now, $\Phi(x) = \{h(x; y - x) \mid y \in K \setminus \{x\}\} = \{(-(y - x)^2, -1) \mid y \in K \setminus \{x\}\}$. As $0 \notin \Phi(x)$ for any $x \in K$ therefore, the problem (**VVIP**) *has no solution but every element of K is a solution of the optimization problem* (**VOP**). *Also, we may note that $\varphi(x) = -1, \forall\, x \in K$.*

Next we relate the two types of gap functions proposed in the paper.

Theorem 4.4. *Let $\hat{z} \in \Phi(\hat{x})$ and $h(\hat{x}; K - \hat{x})$ be a convex set in $\mathbb{R}^m$ then the following hold:*

(i) *there exists $\hat{u} \in B^*(\hat{x})$ such that $\varphi(\hat{x}) \geq \langle \hat{u}, \hat{z} \rangle$;*
(ii) *there exists $\hat{u} \in B^*(\hat{x})$ such that $\varphi(\hat{x})e - \hat{z} \notin -\operatorname{int} C(\hat{x})$.*

Proof. Since $\hat{z} \in \Phi(\hat{x}) = \operatorname{Min}_{\operatorname{int} C(\hat{x})} h(\hat{x}; K - \hat{x})$ therefore there exists some $\hat{y} \in K$ such that $\hat{z} = h(\hat{x}; \hat{y} - \hat{x})$ and $(h(\hat{x}; K - \hat{x}) - \hat{z}) \cap (-\operatorname{int} C(\hat{x})) = \phi$. By separation theorem there exists $u \in \mathbb{R}^m \setminus \{0\}$ such that

$$\langle u, h(\hat{x}; y - \hat{x}) - \hat{z} \rangle > \langle u, c \rangle, \;\; \forall \;\, y \in K \quad \text{and} \quad \forall \;\, c \in -\operatorname{int} C(\hat{x}). \tag{4.1}$$

We assert that $u \in C^*(\hat{x}) \setminus \{0\}$. Let $d \in C(\hat{x})$ then there exists a sequence $c_n \in -\operatorname{int} C(\hat{x})$ such that $-d = \lim_{n \to \infty} c_n$. Since $tc_n \in -\operatorname{int} C(\hat{x})$ for every $t > 0$, from (4.1) it follows that

$$(1/t)\langle u, h(\hat{x}; y - \hat{x}) - \hat{z} \rangle > \langle u, c_n \rangle, \;\; \forall\, y \in K\,.$$

On taking limit $t \to +\infty$, it follows that $\langle u, c_n \rangle \leq 0$, which in turn implies $\langle u, d \rangle \geq 0$. Thus $u \in C^*(\hat{x}) \setminus \{0\}$ and $\langle u, h(\hat{x}; y - \hat{x}) \rangle \geq \langle u, \hat{z} \rangle$, $\forall\; y \in K$. Define $\hat{u} = u/\|u\| \in B^*(\hat{x})$ then it is clear that $\max_{u \in B^*(\hat{x})} \langle u, h(\hat{x}; y - \hat{x}) \rangle \geq \langle \hat{u}, h(\hat{x}; y - \hat{x}) \rangle \geq \langle \hat{u}, \hat{z} \rangle$, $\forall\; y \in K$ and hence, $\varphi(\hat{x}) = \inf_{y \in K} \max_{u \in B^*(\hat{x})} \langle u, h(\hat{x}; y - \hat{x}) \rangle \geq \langle \hat{u}, \hat{z} \rangle$.

(ii) Choose $e \in \operatorname{int} C(\hat{x})$ such that $\langle \hat{u}, e \rangle = 1$. Then $\langle \hat{u}, \varphi(\hat{x})e \rangle \geq \langle \hat{u}, \hat{z} \rangle$ which implies that $\varphi(\hat{x})e - \hat{z} \notin -\operatorname{int} C(\hat{x})$. ∎

Theorem 4.5. *Suppose that infimum in φ is attained at some $\hat{y} \in K$ that is, $\varphi(\hat{x}) = \max_{u \in B^*(\hat{x})} \langle u, h(\hat{x}; \hat{y} - \hat{x}) \rangle$ then $h(\hat{x}; \hat{y} - \hat{x}) \in \Phi(\hat{x})$.*

Proof. Observe that

$$\begin{aligned} \varphi(\hat{x}) &= \max_{u \in B^*(\hat{x})} \langle u, h(\hat{x}, \hat{y} - \hat{x}) \rangle \\ &= \inf_{y \in K} \max_{u \in B^*(\hat{x})} \langle u, h(\hat{x}; y - \hat{x}) \rangle \\ &\leq \max_{u \in B^*(\hat{x})} \langle u, h(\hat{x}; y - \hat{x}) \rangle, \quad \forall \;\, y \in K\,. \end{aligned}$$

Let $\max_{u\in B^*(\hat{x})} \langle u, h(\hat{x}, y-\hat{x})\rangle = \langle u_y, h(\hat{x}, y-\hat{x})\rangle$ for some $u_y \in B^*(\hat{x})$ and for each $y \in K$. Then we have

$$\max_{u\in B^*(\hat{x})} \langle u, h(\hat{x}, \hat{y}-\hat{x})\rangle \leq \langle u_y, h(\hat{x}, y-\hat{x})\rangle , \quad \forall\, y \in K\,.$$

As $\langle u_y, h(\hat{x}, \hat{y}-\hat{x})\rangle \leq \max_{u\in B^*(\hat{x})} \langle u, h(\hat{x}; \hat{y}-\hat{x})\rangle$ from the above relation we get

$$\langle u_y, h(\hat{x}; \hat{y}-\hat{x}) - h(\hat{x}, y-\hat{x})\rangle \leq 0\,.$$

Thus, $h(\hat{x}; \hat{y}-\hat{x}) - h(\hat{x}; y-\hat{x}) \notin \operatorname{int} C(\hat{x})$, $\forall\, y \in K$ that is, $(h(\hat{x}; K-\hat{x}) - h(\hat{x}; \hat{y}-\hat{x})) \cap (-\operatorname{int} C(\hat{x})) = \phi$. Thus, $h(\hat{x}; \hat{y}-\hat{x}) \in \Phi(\hat{x})$.

We now prove using the above two theorems that every solution of the problem (**P1**) is also a solution of the problem (**VOP**). This result can also be deduced from Theorem 4.3 due to the observation made in Remark 4.1.

Theorem 4.6. *If $\hat{x} \in K$ is a solution of the problem* (**P1**) *and $h(\hat{x}; K-\hat{x})$ is a convex set in $\mathbb{R}^m$, then $\hat{x}$ is also a solution of the problem* (**VOP**)*.*

Proof. Since $\hat{x}$ solves (**P1**) therefore, $\varphi(\hat{x}) = 0 = \max_{u\in B^*(\hat{x})} \langle u, h(\hat{x}; \hat{x}-\hat{x})\rangle$. Then from Theorem 4.5 it follows that $0 = h(\hat{x}; \hat{x}-\hat{x}) \in \Phi(\hat{x})$. From Theorem 4.4 we have that for each $x \in K$ and for each $z \in \Phi(x)$ there exists $u_z \in B^*(x)$ such that $\varphi(x) \geq \langle u_z, z\rangle$. As $\varphi(x) \leq 0$, $\forall\, x \in K$ we get that $\langle u_z, z\rangle \leq 0$, $\forall\, z \in \Phi(x)$. This implies that $z \notin \operatorname{int} P$, $\forall\, z \in \Phi(x)$ and $\forall\, x \in K$ that is, $(\Phi(K) - 0) \cap \operatorname{int} P = \phi$. Since $0 \in \Phi(\hat{x})$ it follows that $\hat{x}$ solves (**VOP**). ∎

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